Title	Invariant subspace theorems for subdiagonal algebras
Author(s)	Nakazi, T.; Watatani, Y.
Citation	Hokkaido University Preprint Series in Mathematics, 351, 1-21
Issue Date	1996-10-1
DOI	10.14943/83497
Doc URL	http://hdl.handle.net/2115/69101
Type	bulletin (article)
File Information	pre351.pdf

Instructions for use

INVARIANT SUBSPACE
THEOREMS FOR
SUBDIAGONAL ALGEBRAS

T. Nakazi and Y. Watatani

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#326   D. Lehmann and T. Suwa, Residues of holomorphic vector fields on singular varieties, 21 pages. 1995.

#327   K. Tsutaya, Local regularity of non-resonant nonlinear wave equations, 23 pages. 1996.

#328   T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, 25 pages. 1996.

#329   O. Ogurisu, Anticommutativity and spin 1/2 Schrödinger operators with magnetic fields, 12 pages. 1996.

#330   Y. Kurokawa, Singularities for projections of contour lines of surfaces onto planes, 24 pages. 1996.

#331   M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, 94 pages. 1996.

#332   M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the p-laplace diffusion equation, 20 pages. 1996.

#333   T. Nakazi, The spectra of Toeplitz operators with unimodular symbols, 9 pages. 1996.

#334   B. Khanedani and T. Suwa, First variation of horomorphic forms and some applications, 11 pages. 1996.

#335   J. Seade and T. Suwa, Residues and topological invariants of singular holomorphic foliations[1], 28 pages. 1996.

#336   Y. Giga, M.E. Gurtin and J. Matias, On the dynamics of crystalline motions, 67 pages. 1996.

#337   I. Tsuda, A new type of self-organization associated with chaotic dynamics in neural networks, 22 pages. 1996.

#338   F. Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, 34 pages. 1996.

#339   N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators, 29 pages. 1996.

#340   A. Inoue, Abel-Tauber theorems for Fourier-Stieltjes coefficients, 17 pages. 1996.

#341   G. Ishikawa, Topological classification of the tangent developables of space curves, 19 pages. 1996.

#342   Y. Shimizu, A remark on estimates of bilinear forms of gradients in Hardy space, 8 pages. 1996.

#343   N. kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, 11 pages. 1996.

#344   M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, 18 pages. 1996.

#345   A. Douai and H. Terao, The determinant of a hypergeometric period matrix, 20 pages. 1996.

#346   H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in even space dimensions, 66 pages. 1996.

#347   T. Nakazi and T. Yamamoto, Weighted Norm Inequalities For Some Singular Integral Operators, 17 pages. 1996.

#348   Y. Ito and I. Nakamura, Hilbert schemes and simple singularities A_n and D_n, 22 pages. 1996

#349   R. Agemi and K. Yokoyama, The null condition and global existence of solutions to systems of wave equations with different speeds, 42 pages. 1996

#350   F. Hiroshima, Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field, 21 pages. 1996.

INVARIANT SUBSPACE THEOREMS FOR SUBDIAGONAL ALGEBRAS

TAKAHIKO NAKAZI AND YASUO WATATANI

ABSTRACT. We investigate a certain class of invariant subspaces of subdiagonal algebras which contains the both cases of (extended) weak-*Dirichle algebras and analytic crossed products. We show a version of the Beurling-Lax-Halmos theorem.

1. INTRODUCTION

Classical Beurling-Lax theorem on invariant subspaces of H^2 in the unit disc (with an easy geometric proof due to Halmos) has several applications and has been extended in many directions. The aim of the paper is to provide another version of Beurling- Lax-Halmos theorem for general subdiagonal algebras.

Srinivasan and Wang [12] introduced weak-*Dirichlet algebras as an abstract function theory. It is known that any simply invariant subspace $\mathfrak{M}$ has the form $\mathfrak{M} = qH^2$ for $|q| = 1$ a.e. Most of important theorems for weak-*Dirichlet algebras are generalized to extended weak-*Dirichlet algebras by the first-named author in [9] and [10]. On the other hand Arveson [1] introduced the notion of subdiagonal algeras to unify several aspects of non-selfadjoint operator algebras. Subdiagonal algebras are regarded as the noncommutative analogue of weak-*Dirichlet algebras.

After the study of Kawamura-Tomiyama [5] and Loeble-Muhly [6] on subdiagonal algebras determined by flows on von Neumann algebras, McAsey-Muhly-Saito [7], [8] concentrated the case of analytic crossed products to attack the invariant space

problem. They finally showed that if the action is trivial on the center,then the Berling-Lax-Halmos theorem is valid. Furthermore they proved a strong converse: If a (strong) form of the Berling-Lax-Halmos theorem is valid, then it must be an analitic crossed product. Solel [11] compaired two invariant subspaces. But their study excludes certain commutative cases of (extended) weak-*Dirichlet algebras.

In this paper we investigate invariant subspaces for general subdiagonal algebras to include both analytic crossed products and (extended) weak-*Dirichlet algebras. To avoid the storng converse mentioned above due to McAsey-Muhly-Saito, we introduce an invariant subspace of $\mathfrak{A}$-type I, which is a generalization of both a simply invariant subspace in a weak-*Dirichlet algebra and a pure invariant subspace in an analytic crossed product. We show a version of Beurling-Lax-Halmos theorem for invariant subspaces of $\mathfrak{A}$-type I. The notion of $\mathfrak{A}$-type I is not so restrictive because we also have a decomposition theorem into a part of $\mathfrak{A}$-type I in Theorem 10.

2. DECOMPOSITION

Let B be a finite von Neumann algebra with a (faithful normal normalized) trace τ. We recall the definition of a subdiagonal algebra by Arveson [1]. Let $\mathfrak{A}$ be a σ-weakly closed unital subalgebra of B, and let Φ be a faithful and normal conditional expectation from B onto $D = \mathfrak{A} \cap \mathfrak{A}^*$ such that $\tau(\Phi(x)) = \tau(x)$ for $x \in B$. Then $\mathfrak{A}$ is called a maximal subdiagonal algebra of B with respect to Φ if the following conditions are satisfied:

(1) $\mathfrak{A} + \mathfrak{A}^*$ is σ-weakly dense in B.

(2) $\Phi(xy) = \Phi(x)\Phi(y)$ for $x, y \in \mathfrak{A}$.

(3) $\mathfrak{A}$ is maximal among those subalgebras of B satisfying (1) and (2).

By a result [2] of Exel, a σ-weakly closed finite subdiagonal algebra is automatically maximal. So we may omit the condition (3) in our setting.

If B is abelian and $B = L^\infty(X, \mu)$, then a subdiagonal algebra $\mathfrak{A}$ is an exactly extended weak-*Dirichlet algebra introduced by the first named author in [10].

Let π (resp. ρ) be a left (resp. right) multiplication of B on $L^2(B, \tau)$ defined by

$$\pi(b)\eta(x) = \eta(bx) \quad \text{and} \quad \rho(b)\eta(x) = \eta(xb) \quad \text{for} \quad b, x \in B$$

where $\eta : B \to L^2(B, \tau)$ is the cannonical emmbedding. We sometimes omit the symbol η and π. The closure of $\mathfrak{A}$ in $L^2(B, \tau)$ is denoted by $H^2 = \overline{\eta(\mathfrak{A})}$ and the closure of $\mathfrak{A}_0 = \{a \in \mathfrak{A} | \Phi(a) = 0\}$ is denoted by $H_0^2 = \overline{\eta(\mathfrak{A}_0)}$. A closed subspace $\mathfrak{M}$ of $L^2(B, \tau)$ is (left) $\mathfrak{A}$-*invariant* if $\pi(\mathfrak{A})\mathfrak{M} \subset \mathfrak{M}$, $\mathfrak{A}$-*reducing* if $\mathfrak{A}$ and $\mathfrak{A}^*$-invariant, $\mathfrak{A}$-*pure* if $\mathfrak{M}$ contains no non-trivial $\mathfrak{A}$- reducing subspace, $\mathfrak{A}$-*full* if the smallest $\mathfrak{A}$-reducing subspace containig $\mathfrak{M}$ is $L^2(B, \tau)$.

Definition. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$. Then $\mathfrak{M}$ is called *simply invariant* if $[\pi(\mathfrak{A}_0)\mathfrak{M}]_2 \subsetneq \mathfrak{M}$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$. Then $\mathfrak{M}$ is called of $\mathfrak{A}$-*type I* if $[\pi(B)S]_2 = [\pi(B)\mathfrak{M}]_2$. Let $\mathfrak{M}^\wedge = [\pi(B)\mathfrak{M}]_2 \ominus \mathfrak{M}$. Then $\mathfrak{M}^\wedge$ is an $\mathfrak{A}^*$-invariant subspace. If $\mathfrak{M}^\wedge$ is of $\mathfrak{A}^*$-type *I*, then $\mathfrak{M}$ is called of $\mathfrak{A}$-*type II*. And $\mathfrak{A}$ is called of $\mathfrak{A}$-*type III* if $\mathfrak{M} = [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $\mathfrak{M}^\wedge = [\pi(\mathfrak{A}_0^*)\mathfrak{M}^\wedge]_2$.

Example. In the above situation, H^2 is of $\mathfrak{A}$-type *I*. Many H_0^2 are of $\mathfrak{A}$-type *II* . For instance, if H_0^2 is $\mathfrak{A}$-full (in particular, if $D \subset \mathfrak{A}_0^*\mathfrak{A}_0$), then H_0^2 is of $\mathfrak{A}$-type *II*.

Definition. Let W be a subspace of $L^2(B, \tau)$. Then W is called *left-wandering* if W and $\pi(\mathfrak{A}_0)W$ are orthogonal. In particular a vector $\xi \in L^2(B, \tau)$ is left wandering in the sense of [1] if and only if the one dimensional subspace $[\xi]$ spanned by ξ is left-wandering in the above sense. If $\mathfrak{A}$ is the analytic crossed product in $B = D \rtimes_\alpha \mathbb{Z}$

as in [MMS], then W is left-wandering in our sense if and only if W is left-wandering in the sense of [MMS] or [So], i.e., $L_\delta^n W$ and $L_\delta^m W$ are orthogonal when $n \neq m$, where L_δ is the unitary in $B = D \rtimes_\alpha \mathbb{Z}$ implimenting the automorphism α as in [7] and [11]. A left-wandering subspace W is *complete* if $L^2(B, \tau) = [\pi(B)W]_2$.

Lemma 1. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2$. Then we have the following:*

(1) *S is a left-wandering subspace.*

(2) *$[\pi(B)S]_2$ and K are orthogonal.*

(3) *The projection p of $L^2(B, \tau)$ onto S is in $\pi(D)'$.*

Proof. (1) Since $S \subset \mathfrak{M}$, it is trivial that $S \perp \pi(\mathfrak{A}_0)S$.

(2) Since K and $[\pi(\mathfrak{A})S]_2$ are orthogonal and $\mathfrak{A}_0^* + \mathfrak{A}$ is σ-weakly dense in B, it is enough to show K and $\pi(\mathfrak{A}_0^*)S$ are orthogonal. For any $k \in K$, $a \in \mathfrak{A}_0$ and $s \in S$,

$$(k | \pi(a^*)s) = (\pi(a)k | s) = 0$$

since $\pi(a)k \in \pi(\mathfrak{A}_0)K \subset \pi(\mathfrak{A}_0)\mathfrak{M}$.

(3) For $d \in D$, $s \in S$, $a \in \mathfrak{A}_0$ and $m \in \mathfrak{M}$,

$$(\pi(d)s | \pi(a)m) = (s | \pi(d^*a)m) = 0$$

Since $\mathfrak{M}$ is D-invariant, S is D-invariant. $\square$

Definition. In the above setting, we shall call that $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ is *the wandering subspace* of $\mathfrak{M}$.

Lemma 2. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ be the wandering subspace of $\mathfrak{M}$. Then the following conditions are equivalent:*

(1) *$\mathfrak{M}$ is $\mathfrak{A}$-type I (i.e., $[\pi(B)\mathfrak{M}]_2 = [\pi(B)S]_2$)*

(2) $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$

Proof. (2)$\Rightarrow$(1): Suppose that $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$. Then

$$[\pi(B)\mathfrak{M}]_2 = [\pi(B)\pi(\mathfrak{A})S]_2 = [\pi(B)S]_2$$

(1)$\Rightarrow$(2): Suppose that $\mathfrak{M}$ is $\mathfrak{A}$-type I and $\mathfrak{M} \neq [\pi(\mathfrak{A})S]_2$. Then $K = \mathfrak{M}\ominus[\pi(\mathfrak{A})S]_2 \neq 0$. By Lemma 1, K and $[\pi(B)S]_2$ are orthogonal. Thus $[\pi(B)K]_2$ and $[\pi(B)S]_2$ are also orthogonal. Therefore

$$[\pi(B)\mathfrak{M}]_2 \ominus [\pi(B)S]_2 \supset [\pi(B)K]_2 \neq 0$$

Thus $[\pi(B)\mathfrak{M}]_2 \neq [\pi(B)S]_2$. This is a contradiction. $\square$

Corollary 3. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. If $\mathfrak{M}$ is $\mathfrak{A}$-type I and non-zero, then $\mathfrak{M}$ is simply invariant.*

Proof. On the contrary suppose that $\mathfrak{M}$ is not simply invariant. Then the wandering subspace S of $\mathfrak{M}$ is zero. Then by Lemma 2, $\mathfrak{M} = [\pi(\mathfrak{A})S]_2 = 0$. This is a contradiction. $\square$

Proposition 4. *Let D be a finite von Neumann algebra with a trace τ and α an automorphism on D with $\tau \circ \alpha = \tau$. Let $B = D \rtimes_\alpha \mathbb{Z}$ be the crossed product with the canonically extended trace τ and $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ be the analytic crossed product. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Let L_δ be the unitary in B which impliments α. Then the following conditions are equivalent:*

(1) *$\mathfrak{M}$ is pure.*

(2) *$\cap_{n\geq 0}L_\delta{}^n\mathfrak{M} = \{0\}$.*

(3) *$\mathfrak{M}$ is $\mathfrak{A}$-type I .*

6

Proof. It is shown in [7] that (1) and (2) are equivalent. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}] = \mathfrak{M} \ominus L_\delta \mathfrak{M}$ be the wandering subspace of $\mathfrak{M}$. Then we have that

$$\mathfrak{M} = \cap_{n \geq 0} L_\delta{}^n \mathfrak{M} \oplus \sum_{n=0}^{\infty} \oplus L_\delta{}^n S$$

Therefore $\cap_{n \geq 0} L_\delta{}^n \mathfrak{M} = \{0\}$ if and only if $\mathfrak{M} = \sum_{n=0}^{\infty}{}^{\oplus} L_\delta{}^n S$ if and only if $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$. By Lemma 2, $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$ is equivalent to that $\mathfrak{M}$ is $\mathfrak{A}$-type I . $\square$

Proposition 5. *Let $\mathfrak{A}$ be a weak-*Dirichlet algebra of $L^{\infty}(X, \mu)$ and $\mathfrak{M}$ an $\mathfrak{A}$-invariant subspace of $L^2(X, \mu)$. Then the following conditions are equivalent:*

(1) *$\mathfrak{M}$ is simply invariant.*

(2) *$\mathfrak{M}$ is of $\mathfrak{A}$-type I and non-zero.*

(3) *There exists a unimodular function $q \in L^{\infty}(X, \mu)$ such that*

$$\mathfrak{M} = qH^2$$

Proof. It is clear that (3) implies (2). Corollary 3 shows that (2) imples (1). It is a classical result that (1) imples (3). $\square$

Lemma 6. *Let W be a left-wandering subspace of $L^2(B, \tau)$. Let W_1 and W_2 be the subspaces of W. Suppose that W_1 and W_2 are orthogonal and W_1 is D-invariant. then $\pi(B)W_1$ and $\pi(B)W_2$ are orthogonal.*

Proof. For $d \in D, x, y \in \mathfrak{A}_0, \xi_1 \in W_1, \xi_2 \in W_2$ we have

$$(\pi(x^* + d + y)\xi_1 | \xi_2) = (\xi_1 | \pi(x)(\xi)_2) + (\pi(d)\xi_1 | \xi_2) + (\pi(y)\xi_1 | \xi_2) = 0$$

Since $\mathfrak{A}_0{}^* + D + \mathfrak{A}_0$ is σ-weakly dense in B ,

$$(\pi(b)\xi_1 | \xi_2) = 0 \qquad \text{for} \quad b \in B, \xi_1 \in W_1, \xi_2 \in W_2 \quad .$$

Therefore for $b_1, b_2 \in B$

$$(\pi(b_1)\xi_1 | \pi(b_2)\xi_2) = (\pi(b_2{}^*b_1)\xi_1|\xi_2) = 0$$

Thus $\pi(B)W_1$ and $\pi(B)W_2$ are orthogonal. $\square$

Proposition 7. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2$. Let p_1 (resp. $p_{\mathfrak{M}}$) be the projection of $L^2(B, \tau)$ onto $[\pi(B)S]$ (resp. $\mathfrak{M}$) . Then we have the following:*

(1) $p_1 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_1$ and $p_1 \in \pi(B)'$

(2) $p_1(\mathfrak{M}) = [\pi(\mathfrak{A})S]_2$

(3) $(1 - p_1)\mathfrak{M} = K$

(4) $\mathfrak{M} = p_1 \mathfrak{M} \oplus (1 - p_1)\mathfrak{M}$

(5) $p_1 \mathfrak{M}$ is an $\mathfrak{A}$-invariant subspace and of $\mathfrak{A}$-type I .

(6) $(1 - p_1)\mathfrak{M} = K$ is an $\mathfrak{A}$-invarant subspace such that $K = [\pi(\mathfrak{A}_0)K]_2$

Proof. Put $C = [\pi(B)S]_2 \ominus [\pi(\mathfrak{A})S]_2$. Then we have $[\pi(B)S]_2 = C \oplus [\pi(\mathfrak{A})S]_2$ and $\mathfrak{M} = K \oplus [\pi(\mathfrak{A})S]_2$. Since K and $C \subset [\pi(B)S]_2$ are orhogonal by Lemma 1, we have

$$p_1 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_1, \quad p_1 \mathfrak{M} = [\pi(\mathfrak{A})S]_2 \text{and} \quad (1 - p_1)\mathfrak{M} = K$$

Furthermore $\mathfrak{M} = p_1 \mathfrak{M} \oplus (1 - p_1)\mathfrak{M}$. It is clear that $p_1 \mathfrak{M} = [\pi(\mathfrak{A})S]_2$ is $\mathfrak{A}$-invariant and p_1 is in $\pi(B)'$. For $a \in \mathfrak{A}$

$$\pi(\mathfrak{A})K = \pi(a)(1 - p_1)\mathfrak{M} = (1 - p_1)\pi(a)\mathfrak{M} \subset (1 - p_1)\mathfrak{M} = K$$

Thus K is $\mathfrak{A}$-invariant. In particular $[\pi(\mathfrak{A}_0)K]_2 \subset K$. Since $\mathfrak{M} = [\pi(\mathfrak{A})S]_2 \oplus K = S \oplus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $[\pi(\mathfrak{A})S]_2 \supset S$, we have $K \subset [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$. Since $K = (1 - p_1)\mathfrak{M}$,

we have $(1 - p_1)K = (1 - p_1)\mathfrak{M}$. Therefore

$$K = (1 - p_1)\mathfrak{M} = (1 - p_1)K \subset (1 - p_1)[\pi(\mathfrak{A}_0)\mathfrak{M}]_2$$

$$\subset [(1 - p_1)\pi(A_0)\mathfrak{M}]_2 = [\pi(\mathfrak{A}_0)(1 - p_1)\mathfrak{M}]_2 = [\pi(\mathfrak{A}_0)K]_2$$

Thus $[\pi(\mathfrak{A}_0)K]_2 = K$. Finally we shall show that $p_1\mathfrak{M}$ is of $\mathfrak{A}$-type I . Since

$$p_1\mathfrak{M} = [\pi(\mathfrak{A})S]_2 = [S]_2 \oplus [\pi(\mathfrak{A}_0)S]_2 \qquad (\because S \perp [\pi(\mathfrak{A}_0)S]_2)$$

and

$$[\pi(\mathfrak{A}_0)p_1\mathfrak{M}]_2 = [\pi(\mathfrak{A}_0)\pi(\mathfrak{A})S]_2 = [\pi(\mathfrak{A}_0)S]_2$$

We have $S^{\sim} \equiv p_1\mathfrak{M} \ominus [\pi(\mathfrak{A}_0)p_1\mathfrak{M}]_2 = S$. Therefore $[\pi(\mathfrak{A})S^{\sim}]_2 = [\pi(\mathfrak{A})S]_2 = p_1\mathfrak{M}$. This shows that $p_1\mathfrak{M}$ is of $\mathfrak{A}$-type I by Lemma 2. $\square$

Lemma 8. *Let $\mathcal{E}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$ and $\mathcal{F}$ a B-invariant subspace of $L^2(B, \tau)$. Let e (resp. f) be the projection of $L^2(B, \tau)$ onto $\mathcal{E}$ (resp. $\mathcal{F}$). If $ef = fe$, then the following hold:*

(1) *If $[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = \mathcal{E}$, then $[\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2 = \mathcal{E} \cap \mathcal{F}$.*

(2) *If $\mathcal{E}$ is of $\mathfrak{A}$-type I, then $\mathcal{E} \cap \mathcal{F}$ is also of $\mathfrak{A}$-type I .*

(3) *$\mathcal{E} \cap \mathcal{F} = f\mathcal{E}$ and $(\mathcal{E} \cap \mathcal{F}) \ominus [(\pi(\mathfrak{A}_0))(\mathcal{E} \cap \mathcal{F})]_2 = f(\mathcal{E} \ominus [\pi(\mathfrak{A}_0)\mathcal{E}]_2)$.*

Proof. Since $ef = fe$, $\mathcal{E} = (\mathcal{E} \cap \mathcal{F}) \oplus (\mathcal{E} \cap \mathcal{F}^\perp)$ and $\mathcal{E} \cap \mathcal{F} = f\mathcal{E}$. By assumption $\mathcal{E}$ is $\mathfrak{A}$-inavariant and $\mathcal{F}$ is $\mathfrak{A}$-reducing. Therefore we have

$$[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2 \oplus [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F}^\perp)]_2 \quad ,$$

$$[\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2 \subset \mathcal{E} \cap \mathcal{F} \qquad \text{and} \qquad [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F}^\perp)]_2 \subset \mathcal{E} \cap \mathcal{F}^\perp \quad .$$

Thus we have

$$\mathcal{E} \ominus [\pi(\mathfrak{A}_0)\mathcal{E}]_2 = ((\mathcal{E} \cap \mathcal{F}) \ominus [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2) \oplus ((\mathcal{E} \cap \mathcal{F}^\perp) \ominus [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F}^\perp)]_2)$$

Hence we have

$$[\pi(\mathfrak{A})(\mathcal{E} \ominus [\pi(\mathfrak{A}_0)\mathcal{E}]_2)]_2$$

$$= [\pi(\mathfrak{A})((\mathcal{E} \cap \mathcal{F}) \ominus [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2)]_2 \oplus (\pi(\mathfrak{A})((\mathcal{E} \cap \mathcal{F}^\perp) \ominus [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F}^\perp)]_2))$$

By these consideration we have the following:

(1): If $[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = \mathcal{E}$, then $[\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2 = \mathcal{E} \cap \mathcal{F}$.

(2): Suppose that $\mathcal{E}$ is $\mathfrak{A}$-type I, then by Lemma 2 we have

$$\mathcal{E} = [\pi(\mathfrak{A})(\mathcal{E} \ominus [\pi(\mathfrak{A}_0)\mathcal{E}]_2)]_2$$

Therefore

$$\mathcal{E} \cap \mathcal{F} = [\pi(\mathfrak{A})((\mathcal{E} \cap \mathcal{F}) \ominus [\pi(\mathfrak{A}_0)(\mathcal{E} \cap \mathcal{F})]_2)]_2$$

This shows that $\mathcal{E} \cap \mathcal{F}$ is also $\mathfrak{A}$-type I .

(3): It is easily observed. □

Proposition 9. *Let $\mathfrak{A}$ be an extended weak-*Dirichlet algebra of $B = L^\infty(X, \mu)$ and $\mathfrak{M}$ an $\mathfrak{A}$-invariant subspace of $L^2(X, \mu)$. Then the following conditions are equivalent:*

(1) $\mathfrak{M}$ is of type I in the sense of [10], i.e., for every nonzero projection $\chi_E \in D$ with $\chi_E \mathfrak{M} \neq 0$

$$\chi_E \mathfrak{M} \supsetneq \chi_E [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$$

(2) $\mathfrak{M}$ is nonzero and of $\mathfrak{A}$-type I in the sense of this paper.

(3) There exist a unimodular function $q \in B = L^\infty(X, \mu)$ and a nonzero projection $\chi_E \in D$ such that

$$\mathfrak{M} = \chi_E q H^2$$

Proof. (3)$\Rightarrow$(2): Since qH^2 is of $\mathfrak{A}$-type I, $\mathfrak{M} = \chi_E q H^2$ is also of $\mathfrak{A}$-type I by applying the above Lemma 8 for $\mathfrak{F} = \chi_E L^2(X, \mu)$. Since $\chi_E \in \chi_E H^2$, $\mathfrak{M} = q\chi_E H^2 \neq 0$.

$(2) \Rightarrow (1)$: Let $\mathfrak{M}$ be of $\mathfrak{A}$-type I. Then for every non-zero projection $\chi_E \in D$ with $\chi_E \mathfrak{M} \neq 0$, $\chi_E \mathfrak{M}$ is of $\mathfrak{A}$-type I by the above Lemma 8. Hence $\chi_E \mathfrak{M}$ is simply invariatnt by Corollary 3. Thus

$$\chi_E \mathfrak{M} \supsetneq [\pi(\mathfrak{A}_0)\chi_E \mathfrak{M}]_2 = \chi_E [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$$

$(1) \Rightarrow (3)$:This is shown in [10]. $\square$

The following Theorem 8 is a generalization of a decomposition theorem of an invariant subspace for (extended) weak-*Dirichlet algebras studied in [9], [10].

Theorem 10. *Let B be a finite von Neumann algebra with a finite trace τ and $\mathfrak{A}$ a maximal subdiagonal algebra. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$ and $p_{\mathfrak{M}}$ the projection of $L^2(B, \tau)$ onto $\mathfrak{M}$. Then there esist projections $p_1, p_2, p_3 \in \pi(B)'$ with $p_1 + p_2 + p_3 = 1$ and $p_i p_{\mathfrak{M}} = p_{\mathfrak{M}} p_i$ for $i = 1, 2, 3$ such that*

(1) $\mathfrak{M} = p_1 \mathfrak{M} \oplus p_2 \mathfrak{M} \oplus p_3 \mathfrak{M}$

(2) $p_1 \mathfrak{M}$ *is an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type I .*

(3) $p_2 \mathfrak{M}$ *is an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type II .*

(4) $p_3 \mathfrak{M}$ *is an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type III .*

(5) $p_2 \mathfrak{M}$ *contains no nonzero invariant subspace $\mathfrak{V}$ of $\mathfrak{A}$-type I of the form $\mathfrak{V} = g\mathfrak{M}$ for some projection $g \in \pi(B)'$ with $p_{\mathfrak{M}} g = g p_{\mathfrak{M}}$.*

Proof. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2$. Let p_1 be the projection of $L^2(B, \tau)$ onto $[\pi(B)S]_2$. Then $p_1 \in \pi(B)'$, $p_1 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_1$ and $p_1 \mathfrak{M} = [\pi(\mathfrak{A})S]_2$ is an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type I by Proposition 7. Let

$$\mathfrak{N} = [\pi(B)\mathfrak{M}]_2 \ominus ([\pi(B)S]_2 \oplus K) \quad .$$

Since $K = (1 - p_1)\mathfrak{M}$, $[\pi(\mathfrak{A})S]_2 = p_1\mathfrak{M}$ and $\mathfrak{M} = p_1\mathfrak{M} \oplus (1 - p_1)\mathfrak{M}$ by Proposition 7, we have

$$\mathfrak{N} = ((1 - p_1)[\pi(B)\mathfrak{M}]_2) \ominus K = (1 - p_1)([\pi(B)\mathfrak{M}]_2 \ominus \mathfrak{M})$$

Then $\mathfrak{N}$ is $\mathfrak{A}^*$-invariant because $[\pi(B)\mathfrak{M}]_2$ is B-invariant and $[\pi(B)S]_2 \oplus K$ is $\mathfrak{A}$-invariant by Proposition 7. Let $T = \mathfrak{N} \ominus [\pi(\mathfrak{A}_0{}^*)\mathfrak{N}]_2$ and $L = \mathfrak{N} \ominus [\pi(\mathfrak{A}^*)T]_2$. See Figure 1.

———————— Figure 1 ————————

We shall show that three subspaces L, $[\pi(B)T]_2$ and $[\pi(B)S]_2$ are orthogonal each other. Applying Lemma 1 to $\mathfrak{N}$ and $\mathfrak{A}^*$ instead of $\mathfrak{M}$ and $\mathfrak{A}$, we have that $[\pi(B)T]_2$ and L are orthogonal. Since $\mathfrak{N}$ and $[\pi(B)S]_2$ are orthogonal and $L \subset \mathfrak{N}$, L and $[\pi(B)S]_2$ are orthogonal. For $x \in S, t \in T, x \in B$ and $y \in B$, we have

$$(\pi(x)s|\pi(y)t) = (\pi(y^*x)s|t) = 0$$

because $\pi(y^*x)s \in [\pi(B)S]_2$, $t \in T \subset \mathfrak{N}$, $[\pi(B)S]$ and T are orthogonal. Thus $[\pi(B)S]_2$ and $[\pi(B)T]_2$ are orthogonal.

Let p_2 (resp. $p_\mathfrak{N}$) be the projection of $L^2(B,\tau)$ onto $[\pi(B)T]_2$ (resp. $\mathfrak{N}$). Then applying Proposition 7 to $\mathfrak{N}$ and $\mathfrak{A}^*$ instead of $\mathfrak{M}$ and $\mathfrak{A}$, we have that $p_2 \in \pi(B)'$,

$$p_2 p_\mathfrak{N} = p_\mathfrak{N} p_2, \quad p_2\mathfrak{N} = [\pi(\mathfrak{A}^*)T]_2 = [\pi(B)T] \cap \mathfrak{N},$$

$(1 - p_2)\mathfrak{N} = L$, $[\pi(\mathfrak{A}^*)T]_2$ is an $\mathfrak{A}^*$-invariant subspace of $\mathfrak{A}^*$-type I, and L is an $\mathfrak{A}^*$-invariant subspace such that $L = [\pi(\mathfrak{A}_0^*)L]_2$. Put $C = [\pi(B)S]_2 \ominus [\pi(\mathfrak{A})S]_2$. Let p_C (resp. r) be the projection of $L^2(B,\tau)$ onto C (resp. $[\pi(B)\mathfrak{M}]_2$). Since

$$[\pi(B)\mathfrak{M}]_2 = [\pi(B)S]_2 \oplus K \oplus \mathfrak{N} = C \oplus [\pi(\mathfrak{A})S]_2 \oplus K \oplus \mathfrak{N} = C \oplus \mathfrak{M} \oplus \mathfrak{N}$$

12

we have $r = p_C + p_{\mathfrak{M}} + p_{\mathfrak{N}}$. Since $[\pi(B)T]_2 \subset [\pi(B)\mathfrak{M}]_2$, $p_2 r = r p_2 = p_2$. Because $[\pi(B)S]_2$ and $[\pi(B)T]_2$ are are orthogonal and $C \subset [\pi(B)S]_2$, C and $[\pi(B)T]_2$ are orthogonal and $p_C p_2 = p_2 p_C = 0$. Therefore p_2 commutes with $p_{\mathfrak{M}} = r - p_C - p_{\mathfrak{N}}$. Thus

$$p_2 \mathfrak{M} = \mathfrak{M} \cap [\pi(B)T]_2$$

Since $\mathfrak{M} = [\pi(\mathfrak{A})S]_2 \oplus K$ and that $[\pi(B)S]_2$ $(\supset [\pi(\mathfrak{A})S]_2)$ and $[\pi(B)T]_2$ are orthogonal, we have $p_2 \mathfrak{M} \subset K$. It is clear that $p_2 \mathfrak{M} = \mathfrak{M} \cap [\pi(B)T]_2$ is an $\mathfrak{A}$-invariant subspace. We shall show that $p_2 \mathfrak{M}$ is of $\mathfrak{A}$-type II. Put $\mathfrak{M}_2 = p_2 \mathfrak{M} \subset [\pi(B)T]_2$ and $\mathfrak{M}_2^\wedge = [\pi(B)\mathfrak{M}_2] \ominus \mathfrak{M}_2$. We have

$$p_2 = p_2 r = p_2(p_C + p_{\mathfrak{M}} + p_{\mathfrak{N}}) = p_2 p_{\mathfrak{M}} + p_2 p_{\mathfrak{N}}$$

Therefore $[\pi(B)T]_2 = \mathfrak{M}_2 \oplus [\pi(\mathfrak{A}^*)T]_2$. Thus $[\pi(B)\mathfrak{M}_2]_2 \subset [\pi(B)T]_2$. Then

$$\mathfrak{M}_2^\wedge = [\pi(B)\mathfrak{M}_2]_2 \ominus \mathfrak{M}_2 = [\pi(B)\mathfrak{M}_2]_2 \cap \mathfrak{M}_2^\perp$$

$$= [\pi(B)\mathfrak{M}_2]_2 \cap ([\pi(B)T]_2 \cap \mathfrak{M}_2^\perp) = [\pi(B)\mathfrak{M}_2]_2 \cap [\pi(\mathfrak{A}^*)T]_2$$

Thus

$$\mathfrak{M}_2^\wedge = [\pi(B)\mathfrak{M}_2]_2 \cap [\pi(\mathfrak{A}^*)T]_2$$

and

$$[\pi(B)\mathfrak{M}_2]_2 = \mathfrak{M}_2 \oplus \mathfrak{M}_2^\wedge \subset \mathfrak{M}_2 \oplus [\pi(\mathfrak{A}^*)T]_2$$

Recall that $[\pi(\mathfrak{A}^*)T]_2$ is of $\mathfrak{A}^*$-type I. The projection of $L^2(B, \tau)$ onto $[\pi(\mathfrak{A}^*)T]_2$ is $p_2 p_{\mathfrak{N}}$. Let $\mathfrak{F} = [\pi(B)\mathfrak{M}_2]_2$ and f be the projection of $L^2(B, \tau)$ onto $\mathfrak{F}$. Then $(p_2 p_{\mathfrak{N}})f = f(p_2 p_{\mathfrak{N}})$. Applying Lemma 8, we have that $\mathfrak{M}_2^\wedge = [\pi(\mathfrak{A}^*)T]_2 \cap [\pi(B)\mathfrak{M}_2]$ is $\mathfrak{A}^*$-type I.

Finally put $p_3 = I - p_1 - p_2$. Then $p_3 \in \pi(B)'$ and $p_3 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_3$. Put $\mathfrak{M}_3 = p_3\mathfrak{M} = \mathfrak{M} \cap p_3 L^2(B, \tau)$. Then

$$\mathfrak{M}_3 = \mathfrak{M} \cap (p_1\mathfrak{M})^\perp \cap (p_2\mathfrak{M})^\perp = \mathfrak{M} \cap [\pi(\mathfrak{A})S]_2^\perp \cap \mathfrak{M}_2^\perp = K \cap \mathfrak{M}_2^\perp$$

Since $\mathfrak{M}_2 \subset K$, we have $\mathfrak{M}_3 = K \ominus \mathfrak{M}_2$. Let p_K be the projection of $L^2(B, \tau)$ onto K. Then $p_K = p_2 + p_3 p_{\mathfrak{M}}$. Let $\mathfrak{F}' = [\pi(B)\mathfrak{M}]_2 \ominus [\pi(B)T]_2$ and f' be the projection of $L^2(B, \tau)$ onto $\mathfrak{F}'$. Let ℓ be the projection of $L^2(B, \tau)$ onto L. Then by Figure 1 we have

$$\pi(B)' \ni f' = r - p_2 = p_1 + p_2 + p_3 p_{\mathfrak{M}} + \ell - p_2 = p_1 + p_3 p_{\mathfrak{M}} + \ell \quad ,$$

$p_K f' = f' p_K = p_3 p_{\mathfrak{M}}$ and $\mathfrak{M}_3 = K \cap \mathfrak{F}'$. Since $K = [\pi(\mathfrak{A}_0)K]_2$ by Proposition 7, we have

$$\mathfrak{M}_3 = [\pi(\mathfrak{A}_0)\mathfrak{M}_3]_2$$

by Lemma 8. Let $\mathfrak{M}_3^\wedge = [\pi(B)\mathfrak{M}_3]_2 \ominus \mathfrak{M}_3$. Let $\mathfrak{F}^b = [\pi(B)\mathfrak{M}_3]_2$ and f^b be the projection of $L^2(B, \tau)$ onto $\mathfrak{F}^b$. Recall that $L = \mathfrak{N} \ominus [\pi(\mathfrak{A}^*)T]_2$ and $L = [\pi(\mathfrak{A}_0^*)L]_2$. We have that $[\pi(B)\mathfrak{M}_3]_2 \subset \mathfrak{M}_3 \oplus L$, because $[\pi(B)\mathfrak{M}]_2 = [\pi(B)S] \oplus \mathfrak{M}_3 \oplus [\pi(B)T]_2 \oplus L$. Thus

$$\mathfrak{M}_3^\wedge = [\pi(B)\mathfrak{M}_3]_2 \ominus \mathfrak{M}_3 = L \cap [\pi(B)\mathfrak{M}_3]_2 = L \cap \mathfrak{F}^b$$

Since $f^b \in \pi(B)'$ and $f^b \ell = \ell f^b$, we have

$$\mathfrak{M}_3^\wedge = [\pi(\mathfrak{A}_0^*)\mathfrak{M}_3^\wedge]_2$$

by Lemma 8. Thus we have shown that $\mathfrak{M}_3 = p_3\mathfrak{M}$ is of $\mathfrak{A}$-type III.

Let $\mathfrak{V}$ be an invariant subspace of $p_2\mathfrak{M}$ of $\mathfrak{A}$-type I of the form $\mathfrak{V} = g\mathfrak{M}$ for some projection $g \in \pi(B)'$ with $p_{\mathfrak{M}} g = g p_{\mathfrak{M}}$. Let $\mathfrak{G}$ be the range of g. By Lemma 8,

14

$$\mathfrak{V} = g\mathfrak{M} = \mathfrak{M} \cap \mathfrak{G} \text{ and}$$

$$\mathfrak{V} \ominus [\pi(\mathfrak{A}_0)\mathfrak{V}]_2 = g(\mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2) = gS$$

Since $g\mathfrak{M} \subset \mathfrak{M}$ and $g\pi(\mathfrak{A}_0)\mathfrak{M} \subset \pi(\mathfrak{A}_0)g\mathfrak{M} \subset \pi(\mathfrak{A}_0)\mathfrak{M}$, we have $gS \subset S$. Therefore

$$\mathfrak{V} = [\pi(\mathfrak{A})gS]_2 \subset [\pi(\mathfrak{A})S]_2 = p_1\mathfrak{M}$$

Hence $\mathfrak{V} \subset p_1\mathfrak{M} \cap p_2\mathfrak{M}$ and $\mathfrak{V} = 0$. $\square$

Example. Let D be a finite von Neumann algebra with a trace τ and α an automorphism on D with $\tau \circ \alpha = \tau$. Let $B = D \rtimes_\alpha \mathbb{Z}$ be the crossed product with the canonically extended trace τ and $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ be the analytic crossed product. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$. Let L_δ be the unitary in B which impliments α. Then we have that

$$\mathfrak{M} = \cap_{n \geq 0} L_\delta{}^n \mathfrak{M} \oplus \sum_{n=0}^{\infty} \oplus L_\delta{}^n S$$

Using the notation of the above decomposition theorem, we have that $p_1\mathfrak{M} = \sum_{n=0}^{\infty} \oplus L_\delta{}^n S$, $p_2\mathfrak{M} = 0$ and $p_3\mathfrak{M} = \cap_{n \geq 0} L_\delta{}^n S$.

Example. Let D be a type II_1 factor and $\alpha : \mathbb{Z}^2 \to AutD$ an outer action. Let $B = D \rtimes_\alpha \mathbb{Z}^2$ be the crossed product. Fix an irrational positive number θ and cosider the positive cones

$$P_\theta = \{(m, n) \in \mathbb{Z}^2; \theta m + n \geqq 0\}.$$

Let λ_g be the unitary B which impliments α_g for $g \in \mathbb{Z}^2$. Let $\mathfrak{A}$ be the σ-weak closure of the set of all finite sums $\sum_{g \in P_\theta} x_g \lambda_g$, where $x_g \in D$ and $x_g = 0$ except for finitely many $g \in P_\theta$. Let $\mathfrak{M} = H_0^2$. Then H_0^2 is of $\mathfrak{A}$-type II. Since the wandering subspace $S = 0$, we have $p_1\mathfrak{M} = 0, p_2\mathfrak{M} = H_0^2$ and $p_3\mathfrak{M} = 0$.

Example. We can also construct several examples such that none of $p_i\mathfrak{M}(i = 1, 2, 3)$ are zero in the decomposition of the above Theorem 10. For example consider the direct sum of the subdiagonal algebras and invariant subspaces of the just above two examples. Helson-Lowdenslanger [4] give an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type *III* which is not $\mathfrak{A}$-reducing.

3. INVARIANT SUBSPACES

Loebl-Muhly [6] and Kawamura -Tomiyama [5] investigate maximal subdiagonal algebras determined by flows in von Neumann algebras. Later the study on invariant subspaces is focussed on analytic crossed product $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ and Mcasey, Muhly and Saito [7] determined when a version of the Beurling-Lax-Halmos theorem (abbreviated the BLH theorm) is valid for $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$. But their situation excludes certain commutative cases as in [10], [12].

The aim of the section is to unify the both cases of analytic crossed products and (extended) weak-*Dirichlet algebras.

The following Lemma 11 is a generalization of Lemma 4.2.2. in Arveson [1] on wandering vectors and is a key lemma to prove our main theorem.

Lemma 11. *For $i = 1, 2$, let K_i be a D-invariant subspace of $L^2(B, \tau)$ and q_i be the projection of $L^2(B, \tau)$ onto K_i. Assume that K_1 and K_2 are left wandering. Suppose that there exists a partial isometry $w \in \pi(D)'$ such that $w^*w = q_1$ and $ww^* = q_2$. Then there exists a partial isometry $v \in \pi(B)'$ satisfying the following:*

(1) *v^*v is the projection onto $[\pi(B)K_1]_2$ and vv^* is a projection onto $[\pi(B)K_2]_2$.*

(2) *$v\pi(b)\xi = \pi(b)w\xi$ for all $\xi \in K_1$ and $b \in B$*

(3) *$v([\pi(\mathfrak{A})K_1]_2) = [\pi(\mathfrak{A})K_2]_2$*

Furthermore there exists a unitary $u \in \pi(B)'$ such that $u\xi = v\xi$ for $\xi \in [\pi(B)K_1]_2$.

Proof. First we shall show that

$$(\natural) \qquad (\pi(b)\xi_1|\xi_2) = (\pi(b)w\xi_1|w\xi_2) \qquad \text{for} \qquad b \in B, \xi_1, \xi_2 \in K_1$$

Since $\mathfrak{A}_0^* + D + \mathfrak{A}_0$ is σ-weakly dense in B, it is enough to consider three cases (i) $b \in D$, (ii) $b \in \mathfrak{A}_0$, and (iii) $b \in \mathfrak{A}_0^*$.

(i) Suppose that $b \in D$. Since $\pi(b)K_1 \subset K_1$, $w \in \pi(D)'$ and w is a partial isometry whose support is K_1, we have

$$(\pi(b)w\xi_1|w\xi_2) = (w\pi(b)\xi_1|w\xi_2) = (\pi(b)\xi_1|\xi_2).$$

(ii) Suppose that $b \in \mathfrak{A}_0$. Since K_1 and K_2 are left wondering, K_i and $\pi(\mathfrak{A}_0)K_i$ are orthogonal for $i = 1, 2$. We have $\xi_1, \xi_2 \in K_1$ and $w\xi_1, w\xi_2 \in K_2$. Therefore

$$(\pi(b)\xi_1|\xi_2) = (\pi(b)w\xi_1|w\xi_2)$$

(iii) Suppose that $b \in \mathfrak{A}_0^*$. Since K_i and $\pi(\mathfrak{A}_0)K_i$ are orthogonal, $\pi(\mathfrak{A}_0^*)K_i$ and K_i are orthogonal for $i = 1, 2$. Therefore

$$(\pi(b)\xi_1|\xi_2) = 0 = (\pi(b)w\xi_1|w\xi_2)$$

Thus we have proved $(\natural)$.

For $b_i \in B$ and $\xi_i \in K_1$ $\quad (i = 1, 2, .., n)$, using $(\natural)$, we have

$$\| \sum_{i=1}^{n} \pi(b_i)\xi_i \|_2^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (\pi(b_i)\xi_i|\pi(b_j)\xi_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\pi(b_j^* b_i)\xi_i|\xi_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (\pi(b_j^* b_i)w\xi_i|w\xi_j) = \| \sum_{i=1}^{n} \pi(b_i)w\xi_i \|_2^2$$

Therefore there exists a partial isometry $v \in \mathcal{L}(L^2(B), \tau)$ such that

$$v\pi(b)\xi = \pi(b)w\xi \qquad \text{for} \qquad b \in B, \xi \in K_1$$

and $v\eta = 0$ for $\eta \in [\pi(B)K_1]_2^\perp$. Thus (1) and (2) are proved.

We shall show that $v \in \pi(B)'$. Take $x \in B$. For $b \in B, \xi \in K_1$,

$$v\pi(x)\pi(b)\xi = v\pi(xb)\xi = \pi(xb)w\xi = \pi(x)\pi(b)w\xi = \pi(x)v\pi(b)\xi$$

For $b \in B, \eta \in [\pi(B)K_1]_2^\perp$, we have $\pi(b)\eta \in [\pi(B)K_1]_2^\perp$. Terefore

$$v\pi(x)\eta = 0 = \pi(x)v\eta$$

Thus $v\pi(x) = \pi(x)v$ for any $x \in B$, that is, $v \in \pi(B)'$. Since $wK_1 = K_2$, we have

$$v[\pi(\mathfrak{A})K_1]_2 = [\pi(\mathfrak{A})wK_1]_2 = [\pi(\mathfrak{A})K_2]_2$$

Furthermore, since $\pi(B)'$ is also a finite von Neumann algebra, $v \in \pi(B)'$ can be extended to a unitary u in $\pi(B)'$

$\square$

The following Lemma 12 is an extension of Theorem 3.2 in [7].

Lemma 12. *Let* $\mathfrak{M}_1$ *amd* $\mathfrak{M}_2$ *be* $\mathfrak{A}$-*invariant subspace of* $\mathfrak{A}$-*type I. Let* $S_i = \mathfrak{M}_i \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}_i]_2$ *be the left wandering subspaces of* $\mathfrak{M}_i$ *and* $p_i \in \pi(D)'$ *be the projections of* $L^2(B, \tau)$ *onto* S_i *for* $i = 1, 2$. *If* $p_2 \preceq p_1$ *in* $\pi(D)'$, *then there exists a partial isometry* $v \in \pi(B)'$ *such that* $v\mathfrak{M}_1 = \mathfrak{M}_2$.

Proof. Since $p_2 \preceq p_1$, there exists a partial isometry $w \in \pi(D)'$ such that $ww^* = p_2$ and $w^*w \leq p_1$. Put $p_1' = w^*w \in \pi(D)'$ and let S_1' be the range of p_1'. Then

$$\pi(D)S_1' = \pi(D)p_1'S_1 = p_1'\pi(D)S_1 \subset p_1'S_1 = S_1'.$$

18

Since $\pi(\mathfrak{A}_0)S_1' \subset \pi(\mathfrak{A}_0)S_1$, $S_1' \subset S_1$ and $\pi(\mathfrak{A}_0)S_1$ is orthogonal to S_1, we have that $\pi(\mathfrak{A}_0)S_1'$ and S_1' are orthogonal. Thus S_1' is also D-invariant and left-wandering. By Lemma 11, there exists a partial isometry $v \in \pi(B)'$ such that $v([\pi(\mathfrak{A})S_1']_2) = [\pi(\mathfrak{A})S_2]_2$. Furthermore the support of v is $[\pi(B)S_1']_2$ and the range of v is $[\pi(B)S_2]_2$. Let $S_1^\circ = S_1 \ominus S_1'$. then S_1° is also a D-invariant and left-wandering. Therefore $\pi(B)S_1'$ and $\pi(B)S_1^\circ$ are orthogonal by Lemma 6 . Thus $[\pi(B)S_1]_2 = [\pi(B)S_1']_2 \oplus [\pi(B)S_1^\circ]$. Hence

$$v[(\pi(\mathfrak{A})S_1)]_2 = v[\pi(\mathfrak{A})S_1']_2$$

Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are of $\mathfrak{A}$-type I,

$$v\mathfrak{M}_1 = v[\pi(\mathfrak{A})S_1]_2 = v[\pi(\mathfrak{A})S_1']_2 = [\pi(\mathfrak{A})S_2]_2 = \mathfrak{M}_2$$

$\square$

Lemma 13. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace and $u \in \pi(B)'$ a partial isometry such that $\mathfrak{M} = uH^2$ Then the following are equivalent:*

(1) $\mathfrak{M}$ *is full.*

(2) u *is a unitary.*

Proof. $(1) \Rightarrow (2)$: Suppose that $\mathfrak{M}$ is full. Then

$$uL^2(B) = u[\pi(B)H^2]_2 = [\pi(B)uH^2]_2 = [\pi(B)\mathfrak{M}]_2 = L^2(B)$$

Hence u is a co-isometry in $\pi(B)'$. Since $\pi(B)'$ is finite, u is in fact a unitary.

$(2) \Rightarrow (1)$: Suppose that u is a unitary. Then

$$[\pi(B)\mathfrak{M}]_2 = [\pi(B)uH^2]_2 = [u\pi(B)H^2]_2 = u[\pi(B)H^2]_2 = uL^2(B) = L^2(B)$$

Thus $\mathfrak{M}$ is full $\square$

We have a version of the Beurling-Lax-Halmos theorem for subdiagonal algebras.

Theorem 14. *Let B be a finite von Neumann algebra with a trace τ, D a von Neumann subalgebra of B and $\Phi : B \to D$ a (faithful normal) conditional expectation with $\tau \circ \Phi = \tau$. Let $\mathfrak{A}$ be a maximal subdiagonal algebras with respect to Φ. Suppose that the center $Z(B)$ of B contains the center $Z(D)$ of D. Let $\mathfrak{M}$ be a $\mathfrak{A}$-invariant subspace of $L^2(B, \tau)$ and of $\mathfrak{A}$-type I. Then there exists a partial isometry $v \in \pi(B)'$ such that*

$$\mathfrak{M} = vH^2$$

Proof. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ be the left wandering subspace of $\mathfrak{M}$. Recall that $L^2(D)$ is the left wandering subspace of H^2. Let e_D (resp. e_S) be the projection of $L^2(B)$ onto $L^2(D)$ (resp. S). Then e_D and e_S are in $\pi(D)'$. By the comparability theorem, there exists a projection $q \in Z(D)$ such that

$$\pi(q)e_D \preceq \pi(q)e_S \quad \text{and} \quad \pi(1-q)e_S \preceq \pi(1-q)e_D$$

By the assumption, we have $q \in Z(D) \subset Z(B)$. Since $\mathfrak{M}$ and H^2 is of $\mathfrak{A}$-type I, $\pi(q)\mathfrak{M} = \mathfrak{M} \cap (\pi(q)L^2(B))$ and $\pi(q)H^2 = H^2 \cap (\pi(q)L^2(B))$ are $\mathfrak{A}$-invariant subspaces of $\mathfrak{A}$-type I by Lemma 8. And the projection of $L^2(B)$ onto the left wandering subspace of $\pi(q)\mathfrak{M}$ (resp. $\pi(q)H^2$) is $\pi(q)e_S$ (resp. $\pi(q)e_D$) by Lemma 8. Since $\pi(q)e_D \preceq \pi(q)e_S$ in $\pi(D)'$, there exists a partial isometry $v_1 \in \pi(B)'$ such that $\pi(q)H^2 = v_1\pi(q)\mathfrak{M}$ by Lemma 12. Similarly there exists a partial isometry $v_2 \in \pi(B)'$ such that $\pi(1-q)\mathfrak{M} = v_2\pi(1-q)H^2$. We shall show that $v_1\pi(q)$ is a co-isometry on $\pi(q)L^2(B)$. In fact

$$v_1\pi(q)(\pi(q)L^2(B)) \supset [v_1\pi(q)\pi(B)\mathfrak{M}]_2 = [\pi(B)v_1\pi(q)\mathfrak{M}]_2$$
$$= [\pi(q)\pi(B)H^2]_2 = \pi(q)L^2(B)$$

Thus $v_1\pi(q)$ is a co-isometry in a finite von Neumann algebra $\pi(q)\pi(B)'$, so that $v_1\pi(q)$ is a unitary on $\pi(q)L^2(B)$. Therefore $(v_1\pi(q))(\pi(q)\mathfrak{M}) = \pi(q)H^2$ implies that

$$\pi(q)\mathfrak{M} = \pi(q)v_1^*\pi(q)H^2 = \pi(q)v_1^*H^2$$

We also have $\pi(1-q)\mathfrak{M} = \pi(1-q)v_2H^2$. Let

$$v = \pi(q)v_1^* + \pi(1-q)v_2$$

Then v is a partial isometry in $\pi(B)'$ and we have

$$\mathfrak{M} = \pi(q)\mathfrak{M} \oplus \pi(1-q)\mathfrak{M} = \pi(q)v_1^*H^2 \oplus \pi(1-q)v_2H^2 = vH^2$$

$\square$

Remark. The above proof itself is a generalization of that of Halmos [3] and McAsey, Muhly and Saito [7].

Corollary 15. (*McAsey, Muhly and Saito* [7]). *Let D be a finite von Neumann algebra with a finite trace τ and α an automorphism of D with $\tau \circ \alpha = \tau$. Let $B = D \rtimes_\alpha \mathbb{Z}$ be the crossd product and $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ the analytic crossed product. Suppose that α fixes the center $Z(D)$ of D elementwise. Then every pure invariant subspace $\mathfrak{M}$ of $L^2(B)$ has the form vH^2 for some partial isometry $v \in \pi(B)'$.*

Proof. The condition that α fixes $Z(D)$ elementwise is equivalent to that $Z(D) \subset Z(B)$. Proposition 4 shows that $\mathfrak{M}$ is pure if and only if $\mathfrak{M}$ is $\mathfrak{A}$-type I. Therefore we can apply Theorem 13. $\square$

Remark. The assumption that $Z(D) \subset Z(B)$ in Theorem 13 is necessary. For example let $B = M_2(\mathbb{C})$ be the algebra of 2 by 2 matrices and D the algebra of diagonal matrices. Then the set $\mathfrak{A}$ of upper triangular matrices is a maximal subdiagonal

algebra. We note that $Z(D) = D \not\subset Z(B) = \mathbb{C}$. Let $\mathfrak{M} = L^2(B)$. Then $\mathfrak{M}$ is of $\mathfrak{A}$-type I. But there exist no partial isometry $v \in \pi(B)'$ such that $\mathfrak{M} = vH^2$, because $dim\ \mathfrak{M} = 4$ and $dim\ vH^2 \leq dim\ H^2 = 3$.

REFERENCES

1. W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math., 89 (1967), 578-642.
2. R. Exel, *Maximal subdiagonal algebras*, Amer. J. Math., 110 (1988), 775-782.
3. P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. 208 (1961), 102-112.
4. H. Helson and D. Lowdenslanger, *Invariant subspaces*, Proc. Internat. Symp. Linear Spaces, Jerusalem, 1960, 251-262, Macmillan (Pergamon), New York, 1961.
5. S. Kawamura and J. Tomiyama, *On subdiagonal algebras associated with flows in von Neumann algebras*, J. Math. Soc. Japan 29 (1977), 73-90.
6. R. I. Loebl and P. S. Muhly, *Analyticity and flows in von Neumann algeras*, J. Funct. Anal. 29 (1978), 214-252.
7. M. McAsey, P. S. Muhly and K. -S. Saito, *Non-selfadjoint crossed products (Invarant subspaces and maximality)*, Trans. Amer. Math. Soc. 248 (1978), 381-409.
8. M. McAsey, P. S. Muhly and K. -S. Saito, *Non-selfadjoint crossed products II*, J. Math. Soc. Japan 33 (1981), 485-495.
9. T. Nakazi, *Invariant subspaces of weak-*Dirichlet algebras*, Pacific J. Math., 69 (1977), 151-167.
10. T. Nakazi, *Extended weak-*Dirichlet algebras*, Pacific J. Math., 81 (1979), 493-513.
11. B. Solel, *The invariant subspace structure of nonselfadjoint crossed product*, Trans. Amer. Math. soc. 279 (1983), 825-840.
12. T. P. Srinivasan and Ju-Kwei Wang, *Weak-*Dirichlet algebras*, Proc. Internat. Sympos. on Functional Algebras (Tulane UNiv. 1965), Scott-Foresman, Chicago, *III.*, 1966, 216-249.

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO, 060, JAPAN

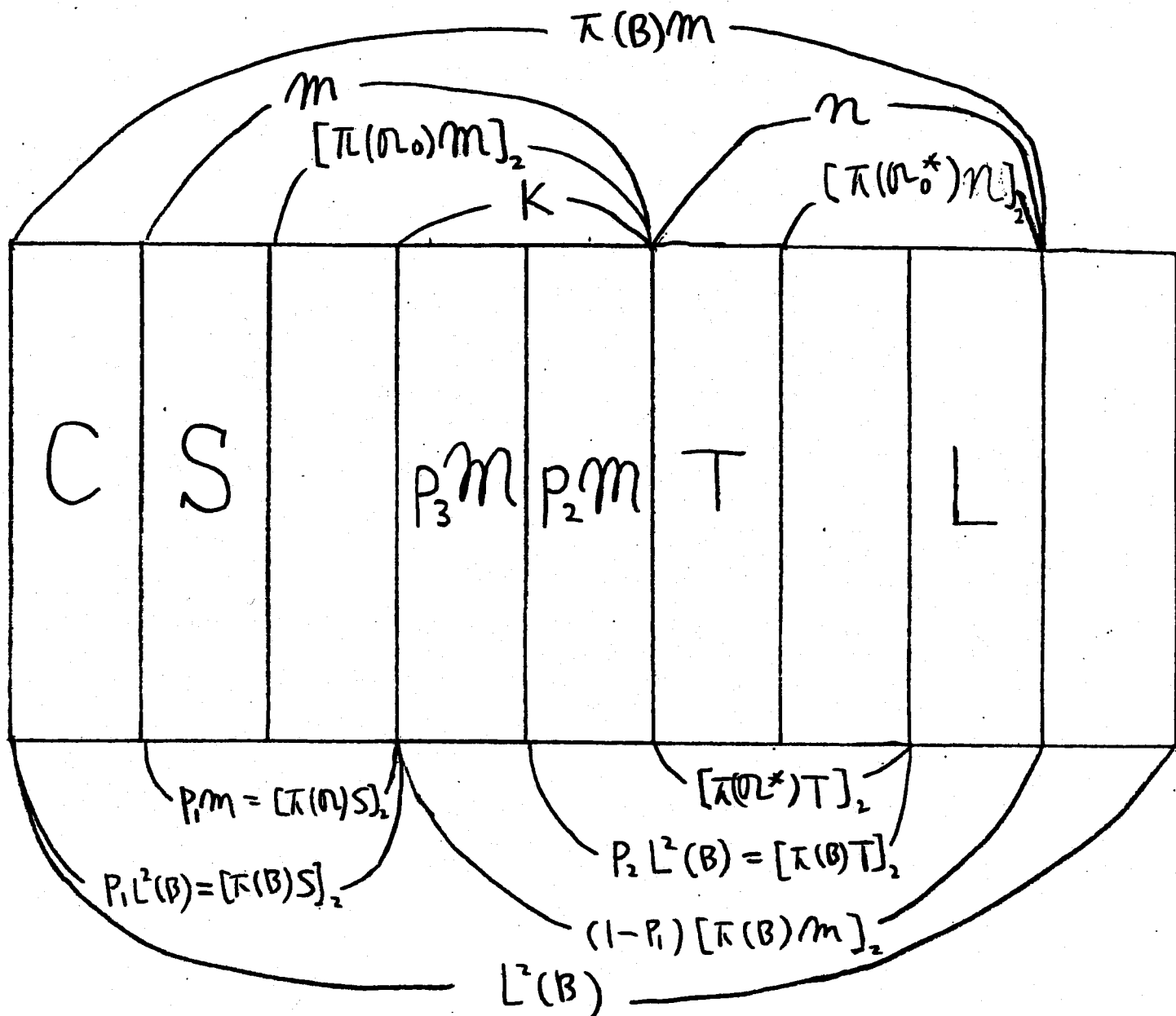GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, ROPPONMATSU, FUKUOKA, 810, JAPAN

Figure 1