THE DRASIN-SHEA-JORDAN THEOREM FOR HANKEL TRANSFORMS OF ARBITRARILY LARGE ORDER

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Series #356. November 1996

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THE DRASIN-SHEA-JORDAN THEOREM FOR HANKEL TRANSFORMS OF ARBITRARILY LARGE ORDER

N. H. BINGHAM and A. INOUE

§1. Introduction and result.

For $\rho \in \mathbb{R}$, we write R_ρ for the class of functions f *regularly varying* (at infinity) with *index* ρ: f is measurable, positive for large enough x, and

$$f(\lambda x)/f(x) \to \lambda^\rho \qquad (x \to \infty) \qquad \forall \lambda > 0;$$

see [2] for background. We are concerned here with comparisons between the asymptotic behaviour of f and that of integral transforms of f. We will write these in Mellin-convolution form as

$$(k * f)(x) := \int_0^\infty f(t)k(x/t)dt/t \qquad (x > 0),$$

for suitable kernels k. The simplest results of this type are *Abelian*, and state that under suitable conditions

$$f(x) \sim x^\rho \ell(x) \qquad (x \to \infty) \tag{1}$$

with ℓ slowly varying (i.e. $\ell \in R_0$) implies

$$(k * f)(x) \sim cx^\rho \ell(x) \qquad (x \to \infty), \tag{2}$$

where, if the Mellin transform of the kernel k is

$$\check{k}(s) := \int_0^\infty t^{-s}k(t)dt/t = \int_0^\infty u^s k(u)du/u,$$

$$c = \check{k}(\rho).$$

Tauberian results supply a partial converse, under suitable side-conditions (*Tauberian conditions*); see e.g. Ch. 4 of [2] for a detailed treatment.

In the circumstances above, one has

$$(k * f)(x)/f(x) \to c \qquad (x \to \infty). \tag{3}$$

The question arises of whether one can obtain (1) - and so (2) - from (3), with ρ such that $c = \check{k}(\rho)$. Such results are *Mercerian*, and are treated in [2], Ch. 5. The prototypes are due to Drasin and Shea ([5], Th. 6.2; [2], Th. 5.2.1), with k non-negative, and Jordan [7], where k can change sign. The proofs, which are rather long, involve reduction - by real analysis and Pólya's lemma - to an integral equation, uniqueness of solution of which is guaranteed by Fourier analysis (Titchmarsh [14], Th. 146); such uniqueness questions are related to the Wiener Tauberian theory.

The Drasin-Shea-Jordan theory as expounded in [2] uses absolute integrals and requires *absolute convergence* of the Mellin transform $\check{k}(s)$ (in an appropriate strip in the complex s-plane). However, the most interesting and important classical integral transforms - most particularly, Fourier sine and cosine transforms and Hankel transforms (Titchmarsh [14] and Watson [15]) - have Mellin transforms which are only *conditionally convergent.* A Drasin-Shea-Jordan theorem for Fourier and Hankel transforms was obtained recently by the authors in [3]. Two main extra complications arise:

(i) One is compelled to make use of special properties of the kernels, and so one's results lose the generality which is one of the characteristic advantages of the Drasin-Shea-Jordan theory (and of the Wiener Tauberian theory related to it);

(ii) One obtains a more complicated integral equation, whose solution requires a far-reaching extension of the Wiener Tauberian theory due to Nyman [11] and Korenblum [8], [9]; see Gurarii [6] for an accessible treatment in English rather than Russian.

This paper is devoted to the removal of a third difficulty which arises in [3]. There, we use the kernel k given by

$$k(x) := x^{-3/2} J_\nu(1/x)$$

with J_ν the Bessel function of order $\nu \geq -1/2$. We write F_ν, or simply F, for the Hankel transform of f, defined in Mellin-convolution form as

$$F(x) := \int_0^\infty f(t)(t/x)^{3/2} J_\nu(t/x)dt/t \qquad (x > 0).$$

Note that the Fourier cosine and sine transorms are the special cases $\nu = -1/2, 1/2$, as

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi}}\frac{\cos z}{\sqrt{z}}, \qquad J_{-1/2}(z) = \sqrt{\frac{2}{\pi}}\frac{\sin z}{\sqrt{z}},$$

also that the Mellin transform is given by *Weber's integral:*

$$a_\nu(s) := \check{k}(s) = \int_0^\infty t^{-s} k(t)dt/t = 2^{s+\frac{1}{2}} \frac{\Gamma(\frac{3}{4} + \frac{\nu}{2} + \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{\nu}{2} - \frac{s}{2})} \qquad (-\nu - \frac{3}{2} < \Re s < \nu + \frac{1}{2}) \quad (W)$$

(Watson [15], 13.24 (1), Titchmarsh [14], (7.4.1)). The results of [3] are restricted to 'small' ν: $\nu \le \nu_0$, where $\nu_0 \in (\frac{1}{2}, 1)$ is the unique root of the transcendental equation

$$\log 2 + \frac{\Gamma'(\frac{\nu_0}{2} + \frac{1}{2})}{\Gamma(\frac{\nu_0}{2} + \frac{1}{2})} = 0$$

($\nu_0 = 0.8660252365...$: thus the cosine and sine cases are included in $\nu \le \nu_0$). Our contribution here is two-fold: we remove the restriction $\nu \le \nu_0$, so covering all $\nu \ge -1/2$, and we simplify the proofs.

Recall that the *upper order* $\rho(f)$ of an ultimately positive function f is defined by

$$\rho(f) := \limsup_{x\to\infty} \log f(x)/\log x.$$

Our result is the following.

THEOREM. *Let $\nu > \nu_0$, $t^{\nu+\frac{1}{2}} f(t) \in L^1_{loc}[0,\infty)$, and f be ultimately decreasing to zero at infinity, with upper order $\rho := \rho(f)$ and Hankel transform $F = F_\nu$. If*

$$-\nu - \frac{3}{2} < \rho < 0, \qquad \rho \ne -\frac{1}{2}$$

and

$$F(x)/f(x) \to c \in (0,\infty) \qquad (x \to \infty),$$

then

$$c = a_\nu(\rho) = 2^{\rho+\frac{1}{2}} \frac{\Gamma(\frac{3}{4} + \frac{\nu}{2} + \frac{\rho}{2})}{\Gamma(\frac{1}{4} + \frac{\nu}{2} - \frac{\rho}{2})}$$

and $f \in R_\rho$.

We comment briefly on the conditions here. The condition that f be ultimately decreasing ensures that the integral giving the Hankel transform converges (conditionally). Some such condition is needed. For less stringent conditions of this type (such as quasi-monotonicity), see e.g. [2], §4.3; we use monotonicity for simplicity. This implies $\rho \le 0$. The condition $\rho \ne -1/2$ is essential; see [3] §7 for a counter-example showing this. The condition $\rho \in (-\nu - \frac{3}{2}, 0)$ ties the order of f to the convergence-strip of the Mellin transform of the kernel.

The condition $\nu \le \nu_0$ in [3] concerns the main point of the present paper, so we discuss it in detail. For a generic kernel k, let $\{z \in \mathbb{C} : \Re z \in (a,b)\}$ be the (absolute or conditional) convergence strip of $\check{k}$ (thus $(a,b) = (-\nu - \frac{3}{2}, \nu + \frac{1}{2})$ in our case $k(x) = x^{-3/2} J_\nu(1/x)$), and ρ be the upper order of f. The major difference between this paper and its predecessors, by Drasin and Shea [5], Jordan [7] and the authors [3], is that we dispense with the following condition:

$$\check{k} \quad \text{is monotone on either} \quad (a, \rho] \quad \text{or} \quad [\rho, b), \tag{A}$$

which is needed for the method of proof used there. In fact, for Hankel transforms with $\nu > \nu_0$, there appears a subinterval $(b_1, b_2) \subset (a, b)$ such that for $\rho \in (b_1, b_2)$ the above condition (A) is not satisfied (see §2). For those $\nu \in (-\frac{1}{2}, \nu_0)$ considered in [3], $\check{k}$ is strictly decreasing on (a, b), whence (A) holds for any $\rho \in (a, b)$. In [7], Theorems 1 and 1a, Jordan explicitly assumes (A), while in [5], where $k \geq 0$, (A) is automatically satisfied since $\check{k}'' > 0$ and $\check{k}$ is convex. Here we adopt a new method of proof, a type of localization, which enables us simultaneously to dispense with both condition (A) and use of a key tool in [5], [7] and [3], the Pólya Peak Theorem of Drasin and Shea, and thereby greatly to simplify the proofs.

§2. The Mellin transform a_ν.

Write

$$\psi(x) := \Gamma'(x)/\Gamma(x)$$

for the logarithmic derivative of the gamma function (the digamma function). Recall the series expansions

$$\psi(z) = -\gamma - \frac{1}{z} + z\sum_{n=1}^{\infty} \frac{1}{n(z+n)}, \qquad \psi'(z) = \sum_{n=0}^{\infty} 1/(z+n)^2$$

([16], §12.16, or [10], p. 14). Thus ψ is increasing, and ψ' decreasing, on $(0, \infty)$. As above, define $\nu_0 \in (\frac{1}{2}, 1)$ by

$$\log 2 + \psi(\frac{\nu_0}{2} + \frac{1}{2}) = 0.$$

PROPOSITION. *For $\nu > \nu_0$, there exist b_1 and b_2 with*

$$-\nu - \frac{3}{2} < b_1 < -\frac{1}{2} < b_2 < \nu + \frac{1}{2}$$

such that a_ν decreases on $(-\nu - \frac{3}{2}, b_1)$, increases on (b_1, b_2) and decreases on $(b_2, \nu + \frac{1}{2})$, with $a_\nu''(b_1) > 0$ and $a_\nu''(b_2) < 0$.

Proof. For $x \in (-\nu - \frac{3}{2}, \nu + \frac{1}{2})$, write

$$f_\nu(x) := \log a_\nu(x) = (x + \frac{1}{2})\log 2 + \log\Gamma(\frac{x}{2} + \frac{\nu}{2} + \frac{3}{4}) - \log\Gamma(-\frac{x}{2} + \frac{\nu}{2} + \frac{1}{4}).$$

Then

$$f_\nu'(x) = \log 2 + \frac{1}{2}\psi(\frac{x}{2} + \frac{\nu}{2} + \frac{3}{4}) + \frac{1}{2}\psi(-\frac{x}{2} + \frac{\nu}{2} + \frac{1}{4}),$$
$$f_\nu''(x) = \frac{1}{4}\psi'(\frac{x}{2} + \frac{\nu}{2} + \frac{3}{4}) - \frac{1}{4}\psi'(-\frac{x}{2} + \frac{\nu}{2} + \frac{1}{4}).$$

Since $\psi(x) \to -\infty$ as $x \downarrow 0$ (from the series expansion),

$$f_\nu'(x) \to -\infty \qquad \text{as} \qquad x \downarrow -\nu - \frac{3}{2} \qquad \text{or} \qquad x \uparrow \nu + \frac{1}{2}.$$

Also

$$f_\nu'(-\frac{1}{2}) = \log 2 + \psi(\frac{\nu}{2} + \frac{1}{2}) > 0$$

as $\nu > \nu_0$, and since ψ' decreases, f_ν'' is positive on $(-\infty, -\frac{1}{2})$, zero at $-\frac{1}{2}$ and negative on $(-\frac{1}{2}, \infty)$. Consequently, there exist b_1, b_2 in the ranges specified in the Proposition with

$$f_\nu'(b_1) = f_\nu'(b_2) = 0.$$

By considering the second derivative, we can easily check that the signs of the second derivatives at these points are as stated. •

COROLLARY. *For $\nu \geq -\frac{1}{2}$, $\lambda \in (-\nu - \frac{3}{2}, \nu + \frac{1}{2}) \setminus \{-\frac{1}{2}\}$, the root $z = \lambda$ of $a_\nu(z) - a_\nu(\lambda)$ is at most double.*

§3. Korenblum's theorem.

It is convenient in this section to work with additive convolutions and Fourier transforms, and to use k for a generic kernel (not the Bessel kernel of §1); when we apply our results later, we will revert to using multiplicative convolutions and Mellin transforms. We write the Fourier transform as

$$\hat{k}(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x) e^{-ixz} dx.$$

Recall first the key result of the Wiener Tauberian theory for $L_1(\mathbb{R})$, giving the equivalence of three statements:

(i) $\hat{k}(t)$ has no zeros $t \in \mathbb{R}$;

(ii) linear combinations of translates of k are dense in L_1;

(iii) the only solutions $g \in L_\infty(\mathbb{R})$ of the integral equation

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x - y) g(y) dy \equiv 0$$

are the trivial ones $g = 0$ a.e.
For background to this Wiener approximation theorem, see e.g. Reiter [13] I, §4.1, Bingham [1]. In particular, the resulting theory extends to finitely many zeros z_j, $j = 1, \cdots, r$ (or finitely many common zeros for classes of kernels). The non-trivial solutions g are now linear combinations of the $\exp\{ixz_j\}$, or of $x^{n-1}\exp\{ixz_j\}$ $(n = 1, \cdots, n_j)$ if z_j has multiplicity n_j.

The extension of this theory from (L_1, L_∞) to other pairs of function spaces in duality is due to Nyman and Korenblum; see e.g. Gurarii [6], Borichev [4], [3] §5. For $\alpha > 0$, one replaces $L_1(\mathbb{R})$ by the Banach algebra $L(\alpha)$ of functions k with norm

$$\|k\| := \int_{-\infty}^{\infty} |k(x)| e^{\alpha |x|} dx$$

and convolution as multiplication. The dual space $L(\alpha)^*$ of $L(\alpha)$ is the space of measurable functions g with norm

$$\|g\| := \text{ess-sup}\{|g(x)| e^{-\alpha|x|} : x \in \mathbb{R}\} < \infty;$$

the duality is given by

$$\langle k, g \rangle := \int_{-\infty}^{\infty} k(x) g(x) dx \qquad (k \in L(\alpha), g \in L(\alpha)^*).$$

For $k \in L(\alpha)$, set

$$\gamma^{\pm}(k) := \limsup_{x \to \pm\infty} \Big(\exp\Big\{ \frac{-\pi|x|}{2\alpha} \Big\} \log |\hat{k}(x)| \Big)$$

($\hat{k}(x) \to 0$ as $x \to \pm\infty$ by the Riemann-Lebesgue lemma, so $\gamma^{\pm}(k) \leq 0$). The Fourier transform $\hat{k}(z)$ is now defined in the strip $|\Im z| \leq \alpha$ in the complex z-plane. The analogue for $L(\alpha)$ of the Wiener condition that $\hat{k}$ be non-vanishing on $\mathbb{R}$ (except at finitely many zeros z_j of multiplicity n_j) is the non-vanishing of $\hat{k}$ in the strip (except at finitely many zeros), and in addition *Korenblum's condition*

$$\gamma^+(k) = \gamma^-(k) = 0,$$

which says that $\hat{k}$ should not decay to zero at infinity in the strip *too fast*. The Wiener results on non-trivial solutions $g \in L_\infty(\mathbb{R})$ to the integral equation (iii) now extend to $g \in L(\alpha)^*$. We shall need the case $r = 1, n_1 = 2$ of one double zero of $\hat{k}$ below.

§4. Proof of the Theorem.

We first note a crude but useful bound: for $k \geq 0$ with $\check{k}(\gamma) := \int_0^\infty t^{-\gamma} k(t)dt/t < \infty$, and f measurable with $f(x)/x^\gamma \leq d(\gamma)$ for $x > 0$, then $k * f$ exists and

$$(k * f)(x) = \int_0^\infty f(x/t)k(t)dt/t \leq d(\gamma)x^\gamma \int_0^\infty t^{-\gamma}k(t)dt/t = d(\gamma)\check{k}(\gamma)x^\gamma.$$

Next, as in [3], we introduce the positive kernels C, D defined by

$$C(x) := x^{\nu+\frac{1}{2}}e^{-x}, \qquad D(x) := \pi^{-\frac{1}{2}}2^{\nu+1}\Gamma(\nu + \frac{3}{2})\frac{x^{\nu+\frac{3}{2}}}{(1+x^2)^{\nu+\frac{3}{2}}}$$

(use of these kernels enables us to avoid the difficulties caused by sign-changes and conditional convergence; for a discussion of their roles, see [3], §2). The convergence-strips I_C, I_D of their Mellin transforms are

$$I_C = \{z \in \mathbb{C} : \Re z < \nu + \frac{1}{2}\}, \qquad I_D = \{z \in \mathbb{C} : -\nu - \frac{3}{2} < \Re z < \nu + \frac{3}{2}\}.$$

Define $\tilde{f}$, $\tilde{g}$, $\tilde{G}$ as in [3] §6:

$$\tilde{f}(x) := I_{[X,\infty)}(x)f(x) \qquad \text{for } X \text{ large enough,}$$

$$\tilde{g}(x) := (C * \tilde{f})(x), \qquad \tilde{G}(x) := (D * \tilde{f})(x).$$

By Propositions 1, 2 of [3] (or directly from the infinite-product representation and behaviour at infinity of the gamma function), we can choose $\epsilon > 0$ so small that $[\rho-2\epsilon, \rho+2\epsilon] \subset (-\nu - \frac{3}{2}, 0) \setminus \{-\frac{1}{2}\}$ and that $a_\nu(z)$ takes the value $a_\nu(\rho)$ only at $z = \rho$ in the strip $\rho - \epsilon \leq \Re z \leq \rho + \epsilon$.

Write p_1, p_2 for $\rho - \epsilon$, $\rho + \epsilon$, and consider

$$E_1(x) := I_{[1,\infty)}(x)x^{p_1}, \qquad E_2(x) := I_{(0,1]}(x)x^{p_2}, \qquad x > 0.$$

The convergence strips of $\check{E}_1$, $\check{E}_2$ are $\{\Re z > p_1\}$, $\{\Re z < p_2\}$. Write

$$h(x) := (E_2 * E_1 * \tilde{g})(x), \qquad H(x) := (E_2 * E_1 * \tilde{G})(x)$$

(for X large enough, $\tilde{f}$ is non-negative, and so are all the other functions involved; the integrals converge absolutely, and Fubini's theorem gives associativity of the convolutions: see below). Take $\gamma \in (\rho, p_2)$. By Lemma 3 of [3] (or by direct estimation),

$$\tilde{g}(x) \leq d(\gamma)x^\gamma \qquad (x > 0)$$

for some $d(\gamma) \in (0, \infty)$. The crude bound above then gives successively

$$(E_1 * \tilde{g})(x) \leq d(\gamma)\check{E}_1(\gamma)x^\gamma,$$

$$h(x) = E_2 * (E_1 * \tilde{g})(x) \leq d(\gamma)\check{E}_1(\gamma)\check{E}_2(\gamma)x^\gamma,$$

and similarly for H. Since $\gamma \in I_C \cap I_D$, $(D * h)(x)$ and $(C * H)(x)$ both exist for all $x > 0$, by the crude bound above. Fubini's theorem yields

$$\begin{aligned}(C * H)(x) &= E_2 * E_1 * (C * \tilde{G})(x) \\ &= E_2 * E_1 * (D * \tilde{g})(x) \\ &= (D * h)(x) \qquad \forall x \in (0, \infty)\end{aligned}$$

(for $C * \tilde{G} = D * \tilde{g}$, see [3], §2).

We pause to comment on the need to introduce the kernels E_1, E_2. For a generic kernel k, let $\{z \in \mathbb{C} : \Re z \in (a, b)\}$ be the convergence strip of $\check{k}$. The transform $f \mapsto E_1 * f$ has the effect of 'localizing' the problem from (a, b) to $[\rho - \epsilon, b)$, while $f \mapsto E_2 * f$ localizes to $(a, \rho + \epsilon]$. These transforms were used in [5] and [7], but only separately, while only the first was used in [3]. Under the assumption (A) discussed in §1, even such an incomplete localization suffices, for ϵ small enough, because of monotonicity of $\check{k}$. When (A), or monotonicity of $\check{k}$, is lacking, we need both transforms to effect a complete localization of the problem to $[\rho - \epsilon, \rho + \epsilon]$. This programme, which greatly simplifies the arguments of [5], [7] and [3], is carried out below.

Our key assumption is $F(x)/f(x) \to c(f)$ as $x \to \infty$. By [3] Theorem 1, this gives $c(f) = a_\nu(\rho)$. As in [3] §3, $F(x)/f(x) \to a_\nu(\rho)$ translates into $\tilde{G}(x)/\tilde{g}(x) \to a_\nu(\rho)$. Since

$$(E_1 * \tilde{G})(x) = x^{p_1} \int_0^x \tilde{G}(t)dt/t^{1+p_1},$$

and similarly for $E_1 * \tilde{g}$, this and (6.3) of [3] give

$$(E_1 * \tilde{G})(x)/(E_1 * \tilde{g})(x) \to a_\nu(\rho) \qquad (x \to \infty).$$

Since

$$H(x) = x^{p_2} \int_x^\infty (E_1 * \tilde{G})(t)dt/t^{1+p_2},$$

and similarly for h, $\tilde{g}$, this gives

$$\frac{H(x)}{h(x)} = \frac{\int_x^\infty (E_1 * \tilde{G})(t)dt/t^{1+p_2}}{\int_x^\infty (E_1 * \tilde{g})(t)dt/t^{1+p_2}} \to a_\nu(\rho) \qquad (x \to \infty).$$

Since $x^{-p_1}E_1(x)$ is increasing and

$$x^{-p_1}h(x) = \int_0^\infty (x/t)^{-p_1}E_1(x/t)(E_2 * \tilde{g})(t)dt/t^{1+p_1},$$

$x^{-p_1}h(x)$ is also increasing. Similarly, $x^{-p_2}h(x)$ is decreasing, and both conclusions hold with H in place of h. Thus

$$\frac{(xu)^{-p_1}h(ux)}{x^{-p_1}h(x)} \le 1 \qquad (0 < u \le 1, x > 0),$$

$$\frac{(xu)^{-p_2}h(ux)}{x^{-p_2}h(x)} \le 1 \qquad (1 \le u < \infty, x > 0).$$

Combining,

$$h(ux)/h(x) \le \max(u^{p_1}, u^{p_2}) \qquad (x, u > 0),$$

and similarly for H.

Now choose any sequence $x_n \uparrow \infty$. Consider

$$j_n(u) := h(x_n u)/h(x_n).$$

The functions $u^{-p_1}j_n(u) : (0,\infty) \to (0,\infty)$ are increasing, and - by the bounds above - uniformly bounded on compact u-sets in $(0,\infty)$. So by the Helly selection principle, we can find a sequence of integers $n' \to \infty$ such that $(u^{-p_1}j_{n'}(u))$ converges pointwise on $(0,\infty)$. So $(j_{n'}(u))$ converges also, to $j(u)$ say. Then $u^{-p_1}j(u)$ is increasing, $j(1) = 1$, and

$$j(u) \le \max(u^{p_1}, u^{p_2}) \qquad (u > 0).$$

Similarly for H: thus

$$h(x_{n'}u)/h(x_{n'}) \to j(u), \qquad H(x_{n'}u)/H(x_{n'}) \to j(u) \qquad (n' \to \infty) \qquad \forall u > 0.$$

From $C * H = D * h$,

$$\frac{H(x_{n'})}{h(x_{n'})}\int_0^\infty \frac{H(x.x_{n'}/t)}{H(x_{n'})}C(t)dt/t = \int_0^\infty \frac{h(x.x_{n'}/t)}{h(x_{n'})}D(t)dt/t.$$

We now have suitably dominated convergence of both integrands (note that $p_1, p_2 \in I_C \cap I_D$), and $H(x_{n'})/h(x_{n'}) \to a_\nu(\rho)$ by above. So

$$a_\nu(\rho)(C * j)(x) = (D * j)(x) \qquad (x > 0).$$

We write

$$m(x) := j(x)x^{-\rho}, \qquad K(x) := (D(x) - a_\nu(\rho)C(x))x^{-\rho} \qquad (x > 0).$$

Then the above becomes the integral equation

$$(K * m)(x) = 0 \qquad \forall x > 0.$$

The Mellin transform $\check{K}$ of K has the strip of absolute convergence $-\nu - \frac{3}{2} - \rho < \Re z < \nu + \frac{1}{2} - \rho$, and is given there by

$$\check{K}(z) = \Gamma(-z - \rho + \nu + \frac{1}{2})(a_\nu(z+\rho) - a_\nu(\rho))$$

(this is the kernel used in [3] §6, Step 5, after the shift $z \mapsto z - \rho$). Recall that

$$[-\epsilon, \epsilon] = [p_1 - \rho, p_2 - \rho] \subset (-\nu - \frac{3}{2} - \rho, \nu + \frac{1}{2} - \rho).$$

To apply Korenblum's theory, let

$$\phi(x) := m(e^x), \qquad k_0(x) := K(e^x) \qquad (x \in \mathbb{R}).$$

Then

$$\phi(x) \leq \max(e^{-\epsilon x}, e^{\epsilon x}) = e^{\epsilon|x|},$$

so $\phi(x) \in L(\epsilon)^*$, while $k_0 \in L(\epsilon)$. Our integral equation is

$$\int_{-\infty}^{\infty} k_0(x-y)\phi(y)dy = 0 \qquad \forall x \in \mathbb{R}.$$

The Fourier transform $\hat{k}_0$ of k_0 is given by

$$\hat{k}_0(z) = \frac{1}{\sqrt{2\pi}}\check{K}(iz) \qquad (|\Im z| \leq \epsilon).$$

So $\hat{k}_0(z)$ has a unique root $z = 0$ in the strip $|\Im z| \leq \epsilon$, which by the Corollary to the Proposition is at most double. By Propositions 2 and 4 of [3] - or by direct calculation from the complex form of Stirling's formula (see e.g. Rademacher [12], p. 38) -

$$\gamma^+(k_0) = \gamma^-(k_0) = 0.$$

Since the origin is at most a double root, Korenblum's theorem yields

$$\phi(x) = a_1 + a_2 x \qquad a.e. \qquad \forall x \in \mathbb{R}$$

for some $a_1, a_2 \in \mathbb{C}$. That is,

$$m(x) = a_1 + a_2 \log x \qquad a.e. \qquad \forall x > 0.$$

Since m is real, so are a_1, a_2. Since $m(x)x^{\rho - p_1}$ is increasing, $m(1-) = a_1 = m(1+)$, whence $a_1 = 1$. Since $m(.) \geq 0$, $a_2 = 0$. So $m(x-) = 1 = m(x+)$ for all x:

$$m(x) \equiv 1 \qquad \forall x \in (0, \infty).$$

So

$$j(x) \equiv x^{\rho} \qquad \forall x \in (0, \infty).$$

This says that the partial limit u^{ρ} of $h(ux_{n'})/h(x_{n'})$ does not depend on the particular sequence (x_n) chosen. We thus have

$$h(xu)/h(x) \to u^{\rho} \qquad (x \to \infty) \qquad \forall u > 0,$$

and so $h \in R_{\rho}$.

We conclude as in [3], §6, Step 6. Since $E_1 * \tilde{g}(x) = x^{p_1} \int_0^x \tilde{g}(t)dt/t^{1+p_1}$, the function $x^{-p_1} E_1 * \tilde{g}(x)$ is increasing, whence $\log(x^{-(1+p_2)} E_1 * \tilde{g}(x))$ is slowly decreasing. So we can use a monotone density argument on

$$h(x) = E_1 * E_2 * \tilde{g}(x) = x^{p_2} \int_x^{\infty} t^{-(1+p_2)} E_1 * \tilde{g}(t)dt$$

to pass from $h \in R_{\rho}$ to $E_1 * \tilde{g} \in R_{\rho}$. Similarly, since

$$\tilde{g}(x) = x^{\nu + \frac{1}{2}} \int_0^{\infty} e^{-x/t} t^{-(\nu + \frac{3}{2})} \tilde{f}(t)dt,$$

the function $x^{-(\nu + \frac{1}{2})} \tilde{g}(x)$ is decreasing. So $\log(x^{-(1+p_1)} \tilde{g}(x))$ is slowly increasing. So we can apply a monotone density argument to

$$E_1 * \tilde{g}(x) = x^{p_1} \int_0^x t^{-(1+p_1)} \tilde{g}(t)$$

to obtain $\tilde{g} \in R_{\rho}$. Now $\tilde{g} = C * \tilde{f}$, i.e.

$$\tilde{g}(x) = x^{\nu + \frac{1}{2}} \int_0^{\infty} u^{\nu + \frac{1}{2}} e^{-xu} \tilde{f}(1/u)du/u,$$

and the convolution on the right is essentially a Laplace transform. The Hardy-Littlewood-Karamata theorem (or Karamata's Tauberian theorem for Laplace transforms - [2] §1.7) and a further monotone density argument give $\tilde{f} \in R_{\rho}$. This says that $f \in R_{\rho}$, as required. •

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