LOW ENERGY SCATTERING FOR NONLINEAR SCHRÖDINGER EQUATIONS IN FRACTIONAL ORDER SOBOLEV SPACES

M. Nakamura and T. Ozawa

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LOW ENERGY SCATTERING FOR NONLINEAR SCHRÖDINGER EQUATIONS IN FRACTIONAL ORDER SOBOLEV SPACES

Dedicated to Professor Rentaro AGEMI on his sixtieth birthday

M. NAKAMURA(*) and T. OZAWA
Department of Mathematics
Hokkaido University
Sapporo 060, Japan

Abstract. We consider the scattering problem for the nonlinear Schrödinger equations with interactions behaving as a power p at zero. In the critical and subcritical cases ($s \geq n/2 - 2/(p-1) \geq 0$), we prove the existence and asymptotic completeness of wave operators in the sense of Sobolev norm of order s on a set of asymptotic states with small homogeneous norm of order $n/2 - 2/(p-1)$ in space dimension $n \geq 1$.

(*)JSPS Fellow

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1. Introduction

In this paper we consider the scattering problem for the nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u = f(u), \tag{1.1}$$

where u is a complex-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, $\partial_t = \partial/\partial t, \Delta$ is the Laplacian in $\mathbf{R}^n$, and f is a complex-valued function, a typical form of which is the single power interaction

$$f(u) = \lambda |u|^{p-1} u \tag{1.2}$$

with $\lambda \in \mathbf{R}$ and $1 < p < \infty$.

There is a large literature on the Cauchy problem for the equation (1.1) and on the asymptotic behavior in time of the global solutions [2, 4, 5-9, 12-17, 22, 25, and references therein]. The Cauchy problem for (1.1) has been studied mainly in the Sobolev spaces H^m of integral order m, especially $m = 0, 1, 2$, while there arises a new interest in the treatment of the Cauchy problem in the Sobolev spaces $H^s = (1-\Delta)^{-s/2} L^2(\mathbf{R}^n)$ of fractional order s with $0 \leq s < n/2$. In [5], Cazenave and Weissler proved that the Cauchy problem for (1.1) with (1.2) has global solutions in H^s for the data $\phi \in H^s$ with $||(-\Delta)^{s/2}\phi; L^2||$ sufficiently small, provided that $p = 1 + 4/(n-2s)$ and $[s] < p-1$, where $[s]$ is the greatest integer that is less than or equal to s. In [14], Kato generalized the results in [5] in some directions. In [7], Ginibre, Ozawa, and Velo proved the existence and asymptotic completeness of the wave operators for (1.1) with a class of interactions including (1.2) on small asymptotic states in H^s, provided that $1 + 4/n \leq p \leq 1 + 4/(n-2s)$ and $s < \min(2, p)$. In [22], Pecher proved that the Cauchy problem for (1.1) with (1.2) has global solutions in H^s for small data in H^s, provided that $1 + 4/n \leq p < 1 + 4/(n-2s)$ and $1 < s < \min(4, p+1)$ or $4 \leq s < p + 2$. In connection with the H^s theory for (1.1) with (1.2), a homogeneity argument indicates that the power p in (1.2) is critical [resp. subcritical] at the level of H^s if and only if $p = 1 + 4/(n-2s)$ [resp. $p < 1 + 4/(n-2s)$]. To sum up with this definition, the critical case is studied in [5, 7, 14] and the subcritical case is studied [7, 14, 22].

The purpose of this paper is to study the H^s theory for (1.1) with a class of interactions including (1.2) in more detail both in the critical and subcritical cases in the framework of low energy scattering. We prove the existence and asymptotic completeness of the wave operators for (1.1) on small asymptotic states in H^s in the critical case with $s < \min(n/2, p)$ as well as in the subcritical case with $\max(s, 3) < p$. Moreover, smallness assumption is shown to be necessary only for the L^2 norm of the fractional derivative $(-\Delta)^{s_0/2}\phi$ of the data $\phi \in H^s$, where $s_0 \equiv n/2 - 2/(p-1)$ is the critical order associated with p in the sense that p is critical at level H^{s_0}. Note that $s_0 = s$ in the critical case and $s_0 < s$ in the subcritical case.

To state the results precisely, we use the following notation. For any r with $1 \leq r \leq \infty$, $L^r = L^r(\mathbf{R}^n)$ denotes the Lebesgue space on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $H^s_r = (1-\Delta)^{-s/2}L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbf{R}$ and any r, m with $1 \leq r, m \leq \infty$, $B^s_{r,m}$ denotes the Besov space defined as the space of distributions u such that $\{2^{sj}||\phi_j * u; L^r||\}_{j=0}^{\infty} \in \ell^m$, where $\{\phi_j\}$ is a dyadic decomposition on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $\dot{H}^s_r$ denotes the homogeneous Sobolev space defined as the space of classes of distributions u modulo polynomials such that $(-\Delta)^{s/2}u \in L^r$. For any $s \in \mathbf{R}$ and any r, m with $1 \leq r, m \leq \infty$, $\dot{B}^s_{r,m}$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj}||\psi_j * u; L^r||\}_{j=-\infty}^{\infty} \in \ell^m$, where $\{\psi_j\}$ is a dyadic decomposition on $\mathbf{R}^n \backslash \{0\}$. We refer to [1, 10, 24] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous versions. For simplicity, we put $H^s = H^s_2, \dot{H}^s = \dot{H}^s_2, B^s_r = B^s_{r,2}, \dot{B}^s_r = \dot{B}^s_{r,2}$. For any interval $I \subset \mathbf{R}$ and any Banach space X we denote by $C(I; X)$ the space of strongly continuous functions from I to X and by $L^q(I; X)$ the space of strongly measurable functions u from I to X such that $||u(\cdot); X|| \in L^q(I)$. Let $U(t) = \exp(it\Delta)$ be the free propagator, namely the one parameter group which solves the free Schrödinger equation. For any r with $2 \leq r \leq \infty$, we define $\delta(r) = n/2 - n/r$. Concerning the space-time integrability properties with respect to $U(\cdot)$, it is convenient to call a pair of exponents (q, r) admissible if $0 \leq 2/q = \delta(r) < 1$, which is understood to be $0 \leq 2/q = \delta(r) \leq 1/2$ when $n = 1$. The Cauchy problem for the equation (1.1) with data $u(t_0) = U(t_0)\phi$ at time t_0 will be treated in the form of the integral equation

$$\begin{aligned} u(t) &= U(t-t_0)u(t_0) - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau \\ &= U(t)\phi - i(G_{t_0}f(u))(t), \end{aligned} \tag{1.3}$$

where the second line is understood to define the integral operator G_{t_0}. The first line of (1.3) is formally equivalent to (1.1) with Cauchy data $u(t_0)$ given at finite time t_0, while the second line will be used to describe the Cauchy problem for (1.1) with data ϕ at time $t_0 = 0$ as well as at $t_0 = \pm\infty$. The integral equation (1.3) will be studied in the spaces X^s

and Y^s with $s \geq 0$ defined as

$$X^s = C(\mathbf{R}; H^s) \cap \bigcap_{0 \leq 2/q = \delta(r) < 1} L^q(\mathbf{R}; B_r^s),$$

$$Y^s = C(\mathbf{R}; H^s) \cap \bigcap_{0 \leq 2/q = \delta(r) < 1} L^q(\mathbf{R}; H_r^s).$$

Note that $X^s \subset Y^s$. For the nonlinear interaction f behaving as a power p at zero, we introduce the following assumptions (A)$_k$ and (B)$_k$ with integer k with $0 \leq k \leq p$.

(A)$_k$ $f \in C^k(\mathbf{C}; \mathbf{C})$ and $f^{(j)}(0) = 0$ for all j with $0 \leq j \leq k$. There exists a constant C such that for all $z_1, z_2 \in \mathbf{C}$

$$|f^{(k)}(z_1) - f^{(k)}(z_2)| \leq \begin{cases} C(|z_1|^{p-k-1} + |z_2|^{p-k-1})|z_1 - z_2| & \text{if} \quad p \geq k+1, \\ C|z_1 - z_2|^{p-k} & \text{if} \quad p < k+1. \end{cases}$$

(B)$_k$ $f \in C^k(\mathbf{C}; \mathbf{C})$ and $f^{(j)}(0) = 0$ for all j with $0 \leq j \leq \max(k-1, 0)$. There exists a constant C such that for all $z \in \mathbf{C}$

$$|f^{(k)}(z)| \leq C|z|^{p-k}.$$

Here $f^{(j)}$ denotes any of the j-th order derivatives of f with respect to z and $\overline{z}$ and $|f^{(j)}|$ denotes the maximum of the moduli of those derivatives. Note that $(A)_k$ implies $(B)_k$ and that $(A)_k$ [resp. $(B)_k$] implies $(A)_j$ [resp. $(B)_j$] for all j with $0 \leq j \leq k$. Single power interaction (1.2) satisfies $(A)_k$ with $0 \leq k < p$ (see [11]).

With the notation above we now state the main results in this paper. Theorem 1 is devoted to the critical case and Theorems 2 and 3 are devoted to the subcritical case. For any s, p, ϵ with $s \geq s_0 \equiv n/2 - 2/(p-1) \geq 0, \epsilon > 0$, we define

$$B_\epsilon = \{\psi \in H^s; \|\psi; \dot{H}^{s_0}\| < \epsilon\}.$$

Theorem 1. *(I) Let s and p satisfy*

$$0 < s < n/2,$$
$$s < p = 1 + 4/(n - 2s).$$

Let f satisfy $(A)_{[s]}$. Then there exists $\epsilon > 0$ with the following property.
(1) For any data $\phi \in B_\epsilon$ at time $t_0 = 0$ the equation (1.3) has a unique solution $u \in X^s$.

(2) For any data $\phi_+ \in B_\epsilon$ at time $t_0 = +\infty$ the equation (1.3) has a unique solution $u \in X^s$ such that

$$\|u(t) - U(t)\phi_+; H^s\| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty \tag{1.4$_+$}$$

(3) For any data $\phi_- \in B_\epsilon$ at time $t_0 = -\infty$ the equation (1.3) has a unique solution $u \in X^s$ such that

$$\|u(t) - U(t)\phi_-; H^s\| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow -\infty. \tag{1.4$_-$}$$

(4) For any $\phi \in B_\epsilon$ at time $t_0 = 0$ there exists a unique pair of asymptotic states $\phi_\pm \in H^s$ satisfying (1.4)$_\pm$, where u is the unique solution given by Part (1).

(II) Let an integer s and p satisfy

$$0 \le s < n/2,$$
$$s \le p = 1 + 4/(n - 2s).$$

Let f satisfy $(B)_s$. Then all the conclusions of Part (I) hold if X^s is replaced by Y^s throughout the statement of Part (I).

Remark 1[2, 6, 15]. The power $p = 1+4/(n-2s)$ comes out as a critical one in H^s in the sense that $\|u; \dot{H}^s\|$ is invariant under the dilation $u \mapsto u_\lambda$ if and only if $s = n/2 - 2/(p-1)$, where $u_\lambda(t, x) \equiv \lambda^{-2/(p-1)} u(\lambda^{-2}t, \lambda^{-1}x)$ with $\lambda > 0$ and the dilation above leaves (1.1) with (1.2) invariant. Another characterization is given as the power which makes the estimates of the form

$$\|G_0 f(u); L^q(\mathbf{R}; \dot{H}^s_r)\| \le C\|u; L^q(\mathbf{R}; \dot{H}^s_r)\|^p$$

with any admissible pair (q, r) invariant under the dilation $u \mapsto u_\lambda$, where $u_\lambda(t, x) \equiv u(\lambda^{-2}t, \lambda^{-1}x)$ with $\lambda > 0$.

Remark 2. In part (I) of Theorem 1, the assumption $s < n/2$ is required to keep the critical power finite, while the assumption $s < p$ is required to keep the smoothness of the nonlinearity f compatible with a power behavior such as (1.2) at zero when p is not an even integer. The condition

$$0 < s < \min(n/2, 1 + 4/(n - 2s))$$

is equivalent to:

(a) $s \in (0, n/2)$ if $n \le 7$,

(b) $s \in (0, s_-(n)) \cup (s_+(n), n/2)$ if $n \geq 8$, where

$$s_\pm(n) = (n + 2 \pm (n^2 - 4n - 28)^{1/2})/4.$$

Compare those two conditions (a) and (b) with those given in [14]. The restriction $s < p$ may be partially removed by taking into account the regularity in time direction in more detail(see [22]).

Remark 3. Theorem 1 shows the existence and asymptotic completeness of the wave operators $W_\pm$ defined on B_ϵ as the maps $\phi_\pm \mapsto u(0) = \phi$. The scattering operator operator S is then defined on B_ϵ as $S = W_+^{-1} \circ W_-$. Note that smallness assumption is imposed on the data only through the fractional derivative of critical order $n/2 - 2/(p-1)$, which is equal to s in the critical case.

Remark 4. The existence and asymptotic completeness of the wave operators has been proved in [25] for (1.1) with (1.2) with $p = 1 + 4/(n-2), n \geq 3$, on small asymptotic states in H^1. A part of the result in [25] was then reproduced in [17]. Part (1) of Theorem 1 (I) is proved for (1.2) with $[s] + 1 < p = 1 + 4/(n - 2s)$ and $0 < s < n/2$. Related results were proved by Pecher [19, 20] for the nonlinear Klein-Gordon equation in H^1 with $p = 1 + 4/(n-2)$ and $n \geq 3$.

Theorem 2. *(I) Let $s > 0$ and $p > 1 + 4/n$ satisfy*

$$s < p < \begin{cases} \infty & \text{if } s \geq n/2, \\ 1 + 4/(n-2s) & \text{if } s < n/2. \end{cases}$$

Let f satisfy $(A)_{[s]}$. Then there exists $\epsilon > 0$ with the following property.
(1) For any data $\phi \in B_\epsilon$ at time $t_0 = 0$ the equation (1.3) has a unique solution $u \in X^s$. Moreover, there exists a unique pair of asymptotic states $\phi_\pm \in H^s$ satisfying $(1.4)_\pm$.
(2) For any data $\phi_+ \in B_\epsilon$ at time $t_0 = +\infty$ [resp. $\phi_- \in B_\epsilon$ at time $t_0 = -\infty$] the equation (1.3) has a unique solution $u \in X^s$ satisfying $(1.4)_+$ [resp. $(1.4)_-$].
(II) Let $s > 0$ be an integer and let $p > 1 + 4/n$ satisfy

$$s \leq p < \begin{cases} \infty & \text{if } s \geq n/2, \\ 1 + 4/(n-2s) & \text{if } s < n/2. \end{cases}$$

Let f satisfy $(B)_s$. Then all the conclusions of Part (I) hold if X^s is replaced by Y^s throughout the statement of Part (I).

Theorem 3. *Let $n = 1$ and $p = 5$. Then Part (I) holds for $0 < s < 2$ and Part (II) for all $s > 0$.*

Remark 5. Theorem 2 and 3 show the existence and asymptotic completeness of the wave operators on B_ϵ for (1.1) in the subcritical case. Note that smallness assumption is imposed on the data only through the fractional derivative of critical order $n/2 - 2/(p-1)$, which is less than s in the subcritical case. For the Cauchy Problem in both critical and subcritical cases the observation of this kind is made by Kato[14].

Remark 6. The assumptions of Theorem 2 cover for instance the case where $n = 3, p = 1 + 4/(n-2) = 5, s = 2, s_0 = 1$ [25], and therefore the result of Theorem 2 gives a partial answer to Question 4 of Kenig, Ponce, and Vega[15] under the smallness assumption on $||\phi; \dot{H}^1||$. The assumption of Theorem 3 cover for instance the case where $n = 1, p = 1 + 4/n = 5, s = 1, s_0 = 0$. Related results were proved by Rauch[23] for the nonlinear Klein-Gordon equation with $n = 3, p = 5, s = 2, s_0 = 1$.

We prove Theorem 1 in Section 3 and Theorems 2 and 3 in Section 4. The method of proof of Theorem 1 follows closely that of [7]. A novelty here consists in Lemma 2.2 below, where the estimate of composite functions is stated in the form stronger than that of [5] for instance. By virtue of Lemma 2.2, we were able to make the lower bound on p down to s. The method of proof of Lemma 2.2 follows closely that of [7] in the sense that we make good use of an equivalent norm on Besov spaces in terms of modulus of continuity with the second differences, though the actual proof below is rather involved because of higher derivatives. Similar and simpler estimates based on the usual modulus of continuity are given in earlier papers [3, 10, 11, 21] for instance. The method of proof of Theorem 2 follows closely that of Theorem 1.

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2. Preliminary Estimates

In this section we collect preliminary estimates for the proofs of the main results.

Lemma 2.1. [4, 5, 7, 9, 13, 26] *The free propagator U satisfies the following estimates.*

(1)For any $s \in \mathbf{R}$ and any admissible pair (q, r)

$$\| U(\cdot)\phi; L^q(\mathbf{R}; \dot{B}^s_r)\| \le C\|\phi; \dot{H}^s\|. \tag{2.1}$$

(2)For any $s \in \mathbf{R}$, any admissible pairs (q_1, r_1) and (q_2, r_2), for interval I possibly unbounded, and for any $t_0 \in \overline{I}$, the operator G_{t_0} defined by (1.3) satisfies the estimate

$$\|G_{t_0}v; L^{q_1}(I; \dot{B}^s_{r_1})\| \le C\|v; L^{q_2'}(I; \dot{B}^s_{r_2'})\|, \tag{2.2}$$

where the constant C is independent of I and t_0 and p' is the exponent dual to p defined by $1/p + 1/p' = 1$.

Lemma 2.2. *Let p and s satisfy $1 \le p < \infty$ and $0 \le s \le p$. Let ℓ, r, m satisfy $1 < \ell \le r < \infty, 1 < m \le \infty$, $1/\ell = 1/r + (p-1)/m$. Let $f \in C^{[s]}(\mathbf{C}; \mathbf{C})$.*
(1)When s is not an integer, assume in addition that $r, m \ge 2$ and $s < p$ and that f satisfies $(A)_{[s]}$. Then

$$\begin{aligned} \|f(u); \dot{B}^s_\ell\| &\le C\|u; \dot{B}^0_m\|^{p-1}\|u; \dot{B}^s_r\| && \text{if } m < \infty, && (2.3)\\ \|f(u); \dot{B}^s_\ell\| &\le C(\|u; \dot{B}^0_\infty\| + \|u; L^\infty\|)^{p-1}\|u; \dot{B}^s_\ell\| && \text{if } m = \infty. && (2.4) \end{aligned}$$

(2)When s is an integer, assume that f satisfies $(B)_{[s]}$. Then

$$\|f(u); \dot{H}^s_\ell\| \le C\|u; L^m\|^{p-1}\|u; \dot{H}^s_r\|. \tag{2.5}$$

Proof. We prove the lemma for real-valued functions for simplicity since the proof for complex-valued functions is analogous if we regard f as a function of two variables u and $\overline{u}$. The argument below follows that of [7], where Part (1) of the lemma is proved for $0 \le s < 2$. We prove Part (1) of the lemma only in the case where $s = N + \sigma$ with $1 \le \sigma < 2$ and an integar $N \ge 1$. We use the following equivalent norm on the homogeneous Besov space $\dot{B}^s_{\ell,q}$

$$\|v; \dot{B}^s_{\ell,q}\| \simeq \sum_{|\alpha|=N} (\int_0^\infty t^{-1-\sigma q} \sup_{|y|\le t} \|\tau_y\partial^\alpha v + \tau_{-y}\partial^\alpha v - 2\partial^\alpha v; L^\ell\|^q dt)^{1/q}, \tag{2.6}$$

which holds for $s = N + \sigma$ with $0 < \sigma < 2$, where τ_y is the translation by $y \in \mathbf{R}^n$, as well as the norm

$$\|v; \dot{B}^s_{\ell,q}\| \simeq \sum_{|\alpha|=N} (\int_0^\infty t^{-1-\sigma q} \sup_{|y|\le t} \|\tau_y\partial^\alpha v - \partial^\alpha v; L^\ell\|^q dt)^{1/q}, \tag{2.7}$$

which holds for $s = N + \sigma$ with $0 < \sigma < 1$. For any multi-index α with $|\alpha| = N$, we have

$$\partial^\alpha(f(u)) = \sum_{k=1}^{N} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_j|\geq 1}} \frac{\alpha!}{k!\,\prod_{j=1}^{k}\beta_j!} f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u. \tag{2.8}$$

We substitute (2.8) into (2.6) and consider the second differences of terms of the form $f^{(k)}(u)\partial^{\beta_1}u\cdots\partial^{\beta_k}u$ in L^ℓ. For simplicity we put

$$F(u) = f^{(k)}(u),\ v = \tau_y u,\ w = \tau_{-y}u,$$
$$u_j = \partial^{\beta_j}u,\ v_j = \partial^{\beta_j}v,\ w_j = \partial^{\beta_j}w.$$

Then the second difference of the term in question is written as

$$\begin{aligned}
&\tau_y(f^{(k)}(u)\prod_j \partial^{\beta_j}u) + \tau_{-y}(f^{(k)}(u)\prod_j \partial^{\beta_j}u) - 2f^{(k)}(u)\prod_j \partial^{\beta_j}u \\
=&F(v)\prod_j v_j + F(w)\prod_j w_j - 2F(u)\prod_j u_j \\
=&F(u)(\prod_j u_j + \prod_j w_j - 2\prod_j u_j) + (F(v)-F(u))(\prod_j v_j - \prod_j u_j) \\
&+ (F(u)-F(w))(\prod_j u_j - \prod_j w_j) + (F(v)+F(w)-2F(u))\prod_j u_j \\
=&F(u)\sum_j (v_j + w_j - 2u_j)\prod_{a<j} v_a \prod_{b>j} u_b \\
&+ F(u)\sum_j\sum_{i<j}(u_j - w_j)(v_i - u_i)\prod_{a<j} v_a \prod_{\substack{b>i \\ b\neq j}} u_b \\
&+ F(u)\sum_j\sum_{i>j}(u_j - w_j)(u_i - w_i)\prod_{\substack{a<i \\ a\neq j}} u_a \prod_{b>i} w_b \\
&+ (F(v)-F(u))\sum_j (v_j - u_j)\prod_{a<j} v_a \prod_{b>j} u_b \\
&+ (F(u)-F(w))\sum_j (u_j - w_j)\prod_{a<j} u_a \prod_{b>j} w_b \\
&+ F'(u)(v + w - 2u)\prod_j u_j \\
&+ \int_0^1 (F'(\theta v + (1-\theta)u) - F'(u))d\theta(v-u)\prod_j u_j \\
&+ \int_0^1 (F'(\theta w + (1-\theta)u) - F'(u))d\theta(w-u)\prod_j u_j.
\end{aligned} \tag{2.9}$$

According to the decomposition above, we denote by I, II, $\cdots$, VIII the first, second, $\cdots$, eighth term on the RHS of the last inequality of (2.9), respectively. In order to estimate I in L^ℓ by the Hölder inequality we define the exponents m_a with $1 \le a \le k$ by

$$1/m_a = \mu_a/r + (1-\mu_a)/m$$

where

$$\mu_a = \begin{cases} |\beta_a|/s & \text{if } a \neq j, \\ (|\beta_j| + \sigma)/s & \text{if } a = j. \end{cases}$$

Therefore

$$\begin{aligned} &\|\mathrm{I}; L^\ell\| \\ \le& \sum_j \|F(u); L^{m/(p-k)}\| \, \|v_j + w_j - 2u_j; L^{m_j}\| \prod_{a<j} \|v_a; L^{m_a}\| \prod_{b>j} \|u_b; L^{m_b}\| \\ \le& C \sum_j \|u; L^m\|^{p-k} \|v_j + w_j - 2u_j; L^{m_j}\| \prod_{a \neq j} \|u; \dot{H}^{|\beta_a|}_{m_a}\| \end{aligned} \tag{2.10}$$

since $\sum_a 1/m_a + (p-k)/m = 1/\ell$ and $|f^{(k)}(u)| \le C|u|^{p-k}$. On the RHS of the last inequality of (2.10), the dependence on $y \in \mathbf{R}^n$ is given only by the second factor and hence by (2.6)

$$\begin{aligned} &\Big(\int_0^\infty t^{-1-2\sigma} \sup_{|y|\le t} \|\mathrm{I}; L^\ell\|^2 dt\Big)^{1/2} \\ \le& C\|u; L^m\|^{p-k} \sum_j \|u; \dot{B}^{|\beta_j|+\sigma}_{m_j}\| \prod_{a\neq j} \|u; \dot{H}^{|\beta_a|}_{m_a}\|. \end{aligned} \tag{2.11}$$

We estimate the last norms in (2.11) by interpolation inequalities and embeddings between the homogeneous Besov and Sobolev spaces. By the definition of m_j

$$\|u; \dot{B}^{|\beta_j|+\sigma}_{m_j}\| \le \|u; \dot{B}^s_r\|^{\mu_j} \|u; \dot{B}^0_m\|^{1-\mu_j}. \tag{2.12}$$

By a similar computation and the embedding $\dot{B}^s_p \hookrightarrow \dot{H}^s_p$ with $2 \le p < \infty$,

$$\|u; \dot{H}^{|\beta_a|}_{m_a}\| \le C\|u; \dot{B}^{|\beta_a|}_{m_a}\| \le C\|u; \dot{B}^s_r\|^{\mu_a}\|u; \dot{B}^0_m\|^{1-\mu_a}, \tag{2.13}$$

$$\|u; L^m\| \le C\|u; \dot{B}^0_m\| \quad \text{if} \quad m < \infty. \tag{2.14}$$

By (2.12), (2.13), and (2.14), the RHS of (2.11) is estimated by

$$\begin{cases} C\|u; \dot{B}^0_m\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad m < \infty, \\ C(\|u; \dot{B}^0_\infty\| + \|u; L^\infty\|)^{p-1}\|u; \dot{B}^s_\ell\| & \text{if} \quad m = \infty. \end{cases} \tag{2.15}$$

A similar argument works on VI and the LHS of (2.11) with I replaced by VI is also estimated by (2.15).

We next consider the contribution of II. We define the exponents m_a with $1 \le a \le k$ by

$$1/m_a = \mu_a/r + (1-\mu_a)/m$$

where

$$\mu_a = \begin{cases} |\beta_a|/s & \text{if } \ a \ne i,j, \\ (|\beta_a| + \sigma/2)/s & \text{if } \ a = i,j. \end{cases}$$

By the Hölder inequality and (2.13) for $a \ne i,j$, we estimate II in L^ℓ in the same way as in the preceding argument and we consider the integral on the LHS of (2.11) with I replaced by II on the basis of the resulting inequality for II in L^ℓ, where the dependence on y is given only by the factors $\|u_j - w_j; L^{m_j}\|$ and $\|v_i - u_i; L^{m_i}\|$. Then we use the Schwarz inequality for the integral with respect to t to reconstruct Besov norms in the form of (2.7). Collecting everything, we obtain

$$\begin{aligned} &\left(\int_0^\infty t^{-1-2\sigma} \sup_{|y|\le t} \|\mathrm{II}; L^\ell\|^2 dt\right)^{1/2} \\ &\le C \sum_j \sum_{i<j} \|u; L^m\|^{p-k} \|\partial^{\beta_j} u; \dot{B}^{\sigma/2}_{m_j,4}\| \; \|\partial^{\beta_i} u; \dot{B}^{\sigma/2}_{m_i,4}\| \prod_{a \ne i,j} \|u; \dot{H}^{|\beta_a|}_{m_a}\| \\ &\le C \sum_j \sum_{i<j} \|u; L^m\|^{p-k} \|u; \dot{B}^{|\beta_j|+\sigma/2}_{m_j,4}\| \; \|u; \dot{B}^{|\beta_i|+\sigma/2}_{m_i,4}\| \prod_{a \ne i,j} \|u; \dot{B}^s_r\|^{\mu_a} \|u; \dot{B}^0_m\|^{1-\mu_a}. \end{aligned} \tag{2.16}$$

We now use the interpolation inequality

$$\|u; \dot{B}^{|\beta_j|+\sigma/2}_{m_a}\| \le \|u; \dot{B}^s_r\|^{\mu_a} \|u; \dot{B}^0_m\|^{1-\mu_a}$$

for $a = i,j$ and the embedding $\dot{B}^s_m \hookrightarrow \dot{B}^s_{m,4}$ on the RHS of the last inequality of (2.16) to conclude that the integral for II of (2.16) is estimated by (2.15). A similar argument works on III, IV, V, and also for VII and VIII except when $k = N$ and $p < N+2$.

We are therefore left with the task of estimating VII and VIII when $k = N$ and $p < N+2$. It follows from (2.8) that $|\beta_j| = 1$ for all j with $1 \le j \le N$. By the assumption of f at the level $[s] = N+1$, we have

$$|\mathrm{VII}| \le C|v-u|^{p-N} |\nabla u|^N. \tag{2.17}$$

We define the exponents m_a with $a = 0,1$, by

$$1/m_a = \mu_a/r + (1-\mu_a)/m$$

where

$$\mu_0 = \sigma/(s(p-N)), \ \mu_1 = 1/s.$$

In the same way as above, we substitute (2.17) into the integral as in the LHS of (2.11), we apply the Hölder inequality to estimate the L^ℓ norm, and we reconstruct Besov norms in the form of (2.7) to obtain

$$\begin{aligned}(\int_0^\infty & t^{-1-2\sigma} \sup_{|y|\le t} \|\mathrm{VII}; L^\ell\|^2 dt)^{1/2} \\ \le & C\|u; \dot{B}^{\sigma/(p-N)}_{m_0,2(p-N)}\|^{p-N} \|\nabla u; L^{m_1}\|^N, \end{aligned} \tag{2.18}$$

where $\sigma/(p-N) < 1$ and $2(p-N) \ge 2\sigma \ge 2$. We now use the interpolation inequality

$$\|u; \dot{B}^{\sigma/(p-N)}_{m_0}\| \le \|u; \dot{B}^s_r\|^{\mu_0} \|u; \dot{B}^0_m\|^{1-\mu_0}$$

with the embedding $\dot{B}^s_m \hookrightarrow \dot{B}^s_{m,q}$ with $q \ge 2$ and

$$\|\nabla u; L^{m_1}\| \le C\|u; \dot{B}^1_{m_1}\| \le C\|u; \dot{B}^s_r\|^{\mu_1}\|u; \dot{B}^0_m\|^{1-\mu_1}$$

to conclude that the RHS of (2.18) is estimated by (2.15). The same proof works on VIII. This proves Part (1) of the lemma.

We finally prove Part (2). It suffices to estimate $\partial^\alpha(f(u))$ in L^ℓ for all α with $|\alpha| = s$ with resulting inequalities dominated by the RHS of (2.5). In view of (2.8) with $|\alpha| = s = N$, we consider the terms of the form $f^{(k)}(u)\partial^{\beta_1}u \cdots \partial^{\beta_k}u$ in L^ℓ. We define the exponents m_j with $1 \le j \le k$ by

$$1/m_j = \mu_j/r + (1-\mu_j)/m$$

where $\mu_j = |\beta_j|/s$. By the Hölder and Gagliardo-Nirenberg inequalities, we obtain

$$\begin{aligned}\|f^{(k)}(u)\prod_j \partial^{\beta_j}u; L^\ell\| &\le \|f^{(k)}(u); L^{m/(p-k)}\| \prod_j \|\partial^{\beta_j}u; L^{m_j}\| \\ &\le C\|u; L^m\|^{p-k} \prod_j \|u; \dot{H}^s_r\|^{\mu_j}\|u; L^m\|^{1-\mu_j} \\ &\le C\|u; \dot{H}^s_r\| \ \|u; L^m\|^{p-1},\end{aligned}$$

from which we obtain Part (2). QED

3. Proof of Theorem 1

In this section we prove Theorem 1. For $\delta \in [0,1) \cap [0,n/2]$, we introduce the Banach space X_δ^s by

$$X_\delta^s = C(\mathbf{R}; H^s) \cap \bigcap_{0 \le 2/q = \delta(r) \le \delta} L^q(\mathbf{R}; B_r^s)$$

with norm

$$\|u; X_\delta^s\| = \sup_{0 \le 2/q = \delta(r) \le \delta} \|u; L^q(\mathbf{R}; B_r^s)\|.$$

An equivalent norm on X_δ^s is given by

$$\|u; X_\delta^s\| \simeq \sup_{0 \le 2/q = \delta(r) \le \delta} (\|u; L^q(\mathbf{R}; L^r)\| + \|u; L^q(\mathbf{R}; \dot{B}_r^s)\|).$$

Note that for $n = 1$

$$X^s = X_{1/2}^s$$

and X^s is a Banach space with the norm on $X_{1/2}^s$ and that for $n \ge 2$

$$X^s = \bigcap_{0 \le \delta < 1} X_\delta^s$$

and X^s is a Fréchet space as a projective limit of Banach spaces $\{X_\delta^s; 0 \le \delta < 1\}$. For $\rho > 0$ let $B(\rho)$ be the closed ball in X_δ^s with radius ρ and center at the origin. We solve the integral equation (1.3) by a contraction argument on $B(\rho)$ with metric induced from the norm on $L^q(\mathbf{R}; L^r)$ where ρ is to be fixed later and (q, r) satisfy

$$0 \le 2/q = \delta(r) \le \delta, \tag{3.1}$$

where we may assume $\delta \ge 2/(p+1)$ without loss of generality. For that purpose we prove that for $\phi \in B_\epsilon$ the RHS of (1.3) leaves $B(\rho)$ invariant and the norms corresponding to $u \in L^q(\mathbf{R}; L^r)$ are contracted. Let (q_0, r_0) satisfy

$$0 \le 2/q_0 = \delta(r_0) \le \delta. \tag{3.2}$$

Let m satisfy

$$n/m = n/r_0 - s. \tag{3.3}$$

Let (q_1, r_1) satisfy

$$1/r_1' = (p-1)/m + 1/r, \tag{3.4}$$

$$0 \le 2/q_1 = \delta(r_1) \le \delta. \tag{3.5}$$

By Lemma 2.2, the embedding $\dot{B}^s_{r_0} \hookrightarrow \dot{B}^0_m$, and the Hölder inequality in space and time, for $u, v \in X^s_\delta$ we estimate

$$\|f(u); L^{q_1'}(\mathbf{R}; \dot{B}^s_{r_1'})\| \leq C\|u; L^{q_0}(\mathbf{R}; \dot{B}^s_{r_0})\|^{p-1}\|u; L^q(\mathbf{R}; \dot{B}^s_r)\|, \tag{3.6}$$

$$\begin{aligned} &\|f(u) - f(v); L^{q_1'}(\mathbf{R}; L^{r_1'})\| \\ &\leq C(\|u; L^{q_0}(\mathbf{R}; \dot{B}^s_{r_0})\|^{p-1} + \|v; L^{q_0}(\mathbf{R}; \dot{B}^s_{r_0})\|^{p-1})\|u - v; L^q(\mathbf{R}; L^r)\|. \end{aligned} \tag{3.7}$$

If we can prove that there exist $(q, r), (q_0, r_0), (q_1, r_1)$ satisfying (3.1)-(3.5), then it follows from Lemma 2.1, (3.6), (3.7) that the RHS of (1.3) leaves $B(\rho)$ invariant and is a contraction with respect to $L^q(\mathbf{R}; L^r)$ for $\epsilon > 0$ small enough. By (3.3) and (3.4),

$$\delta(r_1) + \delta(r) = 2 - (p-1)\delta(r_0). \tag{3.8}$$

When $n = 1$ and $\delta(r_1) = \delta(r) = 1/2$, (3.8) implies $\delta(r_0) = 1/(p-1)$, and therefore (3.2) is realized for $p \geq 3$. When $n \geq 2$ and $\delta(r_1) = \delta(r) = \delta$, (3.8) implies $\delta(r_0) = 2(1-\delta)/(p-1)$, and therefore (3.2) is realized for $\delta \geq 2/(p+1)$. The choice above makes the contraction argument go through, as required. We remark here that $\rho > 0$ must satisfy $\rho \leq C\epsilon$ with constant C independent of ϵ if ϵ is small enough. This proves Part (1). In the same way (1.3) is proved to have a unique solution $u \in X^s$ for data $\phi_+ \in B_\epsilon$ at $t_0 = +\infty$ if $\epsilon > 0$ is small enough. By Lemma 2.1, and (3.6), the solution u satisfy

$$\begin{aligned} &\|u - U(\cdot)\phi_+; L^\infty([t, +\infty); H^s)\| \\ &\leq C\|u; L^{q_0}([t, +\infty); \dot{B}^s_{r_0})\|^{p-1}\|u; L^q([t, +\infty); \dot{B}^s_r)\| \\ &\longrightarrow 0 \end{aligned}$$

as $t \to +\infty$. This proves Part (2). Similarly Part (3) follows. For Part (4) it is sufficient to prove that $\{U(-t)u(t)\}$ is Cauchy in H^s as both $t \to \pm\infty$, which follows from (1.3), Lemma 2.1, and (3.6). Part (II) follows in the same way as above. QED

4. Proofs of Theorems 2 and 3

We consider only Part (I) since the proof of Part (II) is similar and simpler.

Proof of Theorem 2. Let X^s_δ and X^s be as in the proof of Theorem 1. By Theorem 1, (1.3) has a unique solution $u \in X^{s_0}$ for $\|u_0; H^{s_0}\|$ sufficiently small, since s_0 is the critical index associated with p on the basis of the relation $p = 1 + 4/(n - 2s_0)$. Here, in addition to this fact we prove that a contraction argument to (1.3) works in $B(\rho, R) \equiv$

$\{u \in X^s_\delta; ||u; X^{s_0}_\delta|| \le \rho,\ ||u; X^s_\delta|| \le R\}$ with metric induced from the norm on $L^q(\mathbb{R}; L^r)$, where ρ and R are to be fixed later and (q, r) satisfy

$$0 \le 2/q = \delta(r) \le \delta.$$

Let $(q_0, r_0), (q_1, r_1), m$ satisfy

$$\begin{aligned}
&0 \le 2/q_0 = \delta(r_0) \le \delta,\\
&n/m = n/r_0 - s_0,\\
&1/r_1' = (p-1)/m + 1/r,\\
&0 \le 2/q_1 = \delta(r_1) \le \delta.
\end{aligned}$$

Then in the same way as in (3.6) and (3.7), for $u, v \in X^s_\delta$ we estimate

$$\begin{aligned}
&||f(u); L^{q_1'}(\mathbb{R}; \dot{B}^s_{r_1'})|| \le C||u; L^{q_0}(\mathbb{R}; \dot{B}^{s_0}_{r_0})||^{p-1}\ ||u; L^q(\mathbb{R}; \dot{B}^s_r)||,\\
&||f(u) - f(v); L^{q_1'}(\mathbb{R}; L^{r_1'})||\\
&\le C(||u; L^{q_0}(\mathbb{R}; \dot{B}^{s_0}_{r_0})||^{p-1} + ||v; L^{q_0}(\mathbb{R}; \dot{B}^{s_0}_{r_0})||^{p-1})||u - v; L^q(\mathbb{R}; L^r)||,
\end{aligned}$$

where we have used the embedding $\dot{B}^{s_0}_{r_0} \hookrightarrow \dot{B}^0_m$ and the relation $1/q_1' = (p-1)/q_0 + 1/q$. Then a contraction argument yields a unique solution $u \in B(\rho, R)$ provided $\rho > 0$ is sufficiently small, which is guaranteed by the smallness of $||u_0; H^{s_0}||$. The rest of the proof proceeds in the same way as in the proof of Theorem 1. QED

Proof of Theorem 3. When $p = 5$ and $0 < s < 2$, we need not estimate the Besov norm to estimate the nonlinear coefficient to the linear inequalities (see [7]) and use directly the integrability in time of $||u; L^\infty||^4$, which is guaranteed by the range of admissible powers when $n = 1$. QED

Remark. Under the additional assumption that $p > 3$, the argument of [25] using Gronwall's inequality works with the help of Miyakawa's inequality [18, Proposition 3.1].

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