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NONLINEAR SCHRÖDINGER EQUATIONS IN THE SOBOLEV SPACE OF CRITICAL ORDER

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Abstract. The Cauchy problem for the nonlinear Schrödinger equations is considered in the Sobolev space $H^{n/2}(\mathbf{R}^n)$ of critical order $n/2$, where the embedding into $L^\infty(\mathbf{R}^n)$ breaks down and any power behavior of interaction works as a subcritical nonlinearity. Under the interaction of exponential type the existence and uniqueness is proved for global $H^{n/2}$-solutions with small Cauchy data.

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1 Introduction

In this paper we study the nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u = f(u), \tag{1.1}$$

where u is a complex-valued function of $(t,\ x) \in \mathbf{R}\times\mathbf{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbf{R}^n$, and $f(u)$ is the nonlinear interaction given by a complex-valued function f on $\mathbf{C}$. There arises a new interest in the Cauchy problem for (1.1) in the fractional Sobolev spaces $H^s = (1-\Delta)^{-s/2}L^2(\mathbf{R}^n)$ of order $s \geq 0$ [3, 4, 7, 8, 11-15, 18, 23-25]. To illustrate some of the latest developments of the H^s-theory for (1.1), take for example the single power interaction

$$f(u) = \lambda |u|^{p-1}u, \tag{1.2}$$

where $\lambda \in \mathbf{R}$ and $1 < p < \infty$. A homogeneity argument on (1.1) with (1.2) indicates that the power p is critical [resp. subcritical] at the level of H^s if and only if $p = 1+4/(n-2s)$ [resp. $p < 1+4/(n-2s)$], where we follow the convention $0 \leq s < n/2$ to keep the critical exponent $1+4/(n-2s)$ finite with the exception $1 < p < \infty$ to denote the subcritical case at the level of H^s with $s \geq n/2$. Both in the critical and subcritical cases the available results show the existence and uniqueness of local H^s-solutions and, under the smallness assumption on the fractional derivative $(-\Delta)^{s_0/2}\phi$ in L^2 for the Cauchy data $\phi \in H^s$ with $s_0 \equiv n/2 - 2/(p-1)$, the existence and uniqueness of global H^s-solutions. Here, when p is not an odd integer, an additional assumption such as $s < p$ is required to keep the smoothness of f compatible with the behavior at zero. Concerning the number s_0, we notice the following simple facts: (1) $s = s_0$ in the critical case. (2) $s_0 < s$ in the subcritical case. (3) p is critical at the level of H^{s_0}. (4) $0 \leq s_0 < n/2$.

As we see above, as regards the H^s-theory with $0 \leq s < n/2$, the power behavior of the nonlinearity determines the order of the Sobolev space where the smallness of the data is imposed to ensure the existence and uniqueness of global H^s-solutions. This is the right phenomenon, as is usual with other nonlinear evolution equations with dilation structure, such as the heat equation with single power interaction and the Navier-Stokes equations, see [4, 5] and references there for instance.

In contrast, when $s > n/2$, no specific behavior for the nonlinearity is required for the H^s-theory for (1.1) at least locally in time. In fact, when

$s > n/2$, for the existence and uniqueness of local H^s-solutions one has only to assume that $f \in C^k(\mathbf{C}; \mathbf{C})$ with $f(0) = 0$, where differentiability refers to the real sense and k is the smallest integer greater than or equal to s. The proof depends on the usual Sobolev embedding $H^s \subset L^\infty$ for $s > n/2$ in an essential way.

The case $s = n/2$ may therefore be regarded as the borderline in two aspects: (1) No power behavior of interaction amounts to the critical nonlinearity at the level of $H^{n/2}$. (2) Pointwise control of solutions falls beyond the scope of the $H^{n/2}$-theory, so that any argument similar to that of the H^s-theory with $s > n/2$ breaks down even for local theory without specific behavior of interaction.

In addition to the critical phenomena described above, $H^{n/2}$-solutions deserve attention as finite energy solutions for $n = 2$ and as strong solutions for $n = 4$.

The purpose of this paper is to determine the critical behavior of nonlinearity that ensures the existence and uniqueness of global $H^{n/2}$-solutions. We prove the existence and uniqueness of global $H^{n/2}$-solutions to (1.1) with small Cauchy data under the nonlinearity of exponential type. This is reminiscent of Trudinger's inequality [16, 17, 19, 22], which replaces the Sobolev embedding in the limiting case on the basis of the exponential estimates in terms of functions in the critical order Sobolev space $H^{n/2}$. To state the main result more precisely we introduce the following notation.

For any r with $1 \le r \le \infty$, $L^r = L^r(\mathbf{R}^n)$ denotes the Lebesgue space on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $H^s_r = (1 - \Delta)^{-s/2} L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any r with $1 < r < \infty$, $\dot{H}^s_r$ denotes the homogeneous Sobolev space defined as the space of classes of distributions u modulo polynomials such that $(-\Delta)^{s/2} u \in L^r$. For any $s \in \mathbf{R}$ and any r, m with $1 \le r, m \le \infty$, $\dot{B}^s_{r,m}$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj} \| \phi_j * u; L^r \|\}_{-\infty}^{\infty} \in l^m$, where $*$ denotes the convolution in $\mathbf{R}^n$ and $\{\phi_j\}_{-\infty}^{\infty}$ is a sequence of functions such that the Fourier transformed sequence $\{\hat{\phi}_j\}$ forms the Paley-Littlewood dyadic decomposition on the space $\mathbf{R}^n \setminus \{0\}$ of nonvanishing momenta. Analogously the usual Besov space $B^s_{r,m}$ is defined on the basis of the Paley-Littlewood dyadic decomposition on the whole momentum space $\mathbf{R}^n$. We refer to [1, 10, 21] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity we set $H^s = H_2^s$, $\dot{H}^s = \dot{H}_2^s$, $B_r^s = B_{r,\,2}^s$, $\dot{B}_r^s = \dot{B}_{r,\,2}^s$. For any interval $I \subset \mathbf{R}$ and any Banach space X we denote by $C(I;X)$ the space of strongly continuous functions from I to X and by $L^q(I;X)$ the space of measurable functions u from I to X such that $\|u(\cdot);X\| \in L^q(I)$. Let $U(t) = \exp(it\Delta)$ be the free propagator, namely the one parameter group which solves the free Schrödinger equation. For any r with $2 \le r \le \infty$, we define $\delta(r) = n/2 - n/r$, namely the optimal decay rate in time for the L^r -estimate of the free propagator. Regarding the space-time integrability properties for the free propagator, it is covenient to call a pair of exponents $(q,\ r)$ admissible if $0 \le 2/q = \delta(r) < 1$. The Cauchy problem for the equation (1.1) with data $u(t_0) = U(t_0)\phi$ at time t_0 will be treated in the form of the integral equation

$$\begin{aligned} u(t) &= U(t-t_0)u(t_0) - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau \\ &= U(t)\phi - i(G_{t_0}f(u))(t), \end{aligned} \tag{1.3}$$

where the second line is understood to define the integral operator G_{t_0} . The first line of (1.3) is formally equivalent to (1.1) with Cauchy data $u(t_0)$ given at finite time t_0 , while the second line will be used to denote the Cauchy problem for (1.1) with Cauhy data ϕ at time $t_0 = 0$ as well as at $t_0 = \pm\infty$. To describe the nonlinear interaction f with an exponential growth at infinity as well as with a vanishing behavior as a power at zero, for $\lambda > 0$ we introduce the following assumptions $(A)_m$ with $m \ge 1$ and $(B)_m$ with $m \ge 0$.

$(A)_1 : f \in C^1(\mathbf{C};\mathbf{C})$ and $f(0) = 0$.

There exists a constant C such that for all $z \in \mathbf{C}$

$$|f'(z)| \le Ce^{\lambda|z|^2}|z|^2.$$

$(A)_m$ for $m \ge 2$: $f \in C^m(\mathbf{C};\mathbf{C})$ and $f(0) = 0$. There exists a constant C such that for all $z \in \mathbf{C}$ and $2 \le k \le m$

$$|f'(z)| \le Ce^{\lambda|z|^2}|z|,$$

$$|f^{(k)}(z)| \le Ce^{\lambda|z|^2}.$$

$(B)_0 : f \in C(\mathbf{C};\mathbf{C})$ and $f(0)=0$. There exists a constant C such that for all $z_1,\ z_2 \in \mathbf{C}$

$$|f(z_1)-f(z_2)| \le C(e^{\lambda|z_1|^2}|z_1|^4 + e^{\lambda|z_2|^2}|z_2|^4)|z_1-z_2|.$$

$(B)_m$ for $m \ge 1$: In addition to $(A)_m$, $f^{(m)}$ satisfies the estimate for all $z_1,\ z_2 \in \mathbf{C}$

$$|f^{(m)}(z_1)-f^{(m)}(z_2)| \le C(e^{\lambda|z_1|^2} + e^{\lambda|z_2|^2})|z_1-z_2|.$$

Here $f^{(k)}$ denotes any of the k -th order derivatives of f with respect to z and $\bar{z}$ and $|f^{(k)}|$ denotes the maximum of the moduli of those derivatives. We solve the equation (1.3) in the Banach space X defined by

$$\begin{aligned} X &= C(\mathbf{R};H^{n/2}) \cap \bigcap_{0\le 2/q=\delta(r)<1} L^q(\mathbf{R};H_r^{n/2}) \quad \text{if } n \text{ is even,} \\ X &= C(\mathbf{R};H^{n/2}) \cap \bigcap_{0\le 2/q=\delta(r)<1} L^q(\mathbf{R};\dot{B}_r^0 \cap B_r^{n/2}) \quad \text{if } n \text{ is odd,} \end{aligned}$$

with norm

$$\|u;X\| = \sup_{0\le 2/q=\delta(r)<1} \|u;L^q(\mathbf{R};H_r^{n/2})\| \quad \text{if } n \text{ is even,}$$

$$\|u;X\| = \sup_{0\le 2/q=\delta(r)<1} (\|u;L^q(\mathbf{R};\dot{B}_r^0)\| + \|u;L^q(\mathbf{R};B_r^{n/2})\|) \quad \text{if } n \text{ is odd.}$$

With the notation above we now state the main result.

Theorem 1 *Let $n \ge 1$. If n is even let f satisfy $(A)_{n/2}$ for some $\lambda > 0$. If n is odd let f satisfy $(B)_{(n-1)/2}$ for some $\lambda > 0$. Then there exists $\epsilon > 0$ with the following property.*

(1) For any data $\phi \in B_\epsilon$ at time $t_0 = 0$ the equation (1.3) has a unique solution $u \in X$, where B_ϵ is the ball in $H^{n/2}$ with center 0 and radius ϵ. Moreover there exists a unique pair $(\phi_+,\ \phi_-) \in H^{n/2} \oplus H^{n/2}$ such that

$$\|u(t)-U(t)\phi_+;H^{n/2}\| \to 0 \quad as \quad t \to +\infty, \tag{1.4}$$

$$\|u(t)-U(t)\phi_-;H^{n/2}\| \to 0 \quad as \quad t \to -\infty. \tag{1.5}$$

(2) For any data $\phi_+ \in B_\epsilon$ *at time* $t_0 = +\infty$ *the equation* (1.3) *has a unique solution* $u \in X$*. Moreover* u *satisfies* (1.4).

(3) For any data $\phi_- \in B_\epsilon$ *at time* $t_0 = -\infty$ *the equation* (1.3) *has a unique solution* $u \in X$*. Moreover* u *satisfies* (1.5).

Remark 1. The assumptions of the theorem above cover for instance the nonlinearities of the form

$$\begin{aligned} f(u) &= \pm(e^{\lambda|u|^2} - 1 - \lambda|u|^2)u \quad &&\text{for } n = 1, \\ f(u) &= \pm(e^{\lambda|u|^2} - 1)u &&\text{for } n = 2,\ 3, \\ f(u) &= \pm(e^{\lambda|u|^2} - 1) &&\text{for } n \geq 4, \end{aligned}$$

with $\lambda > 0$, which need not be the same as that of $(A)_m$ or of $(B)_m$. The nonlinearity of exponential type was studied in [2, 6] only in the special case where $n = 2,\ 3$ and $\lambda < 0$.

Remark 2. In the framework of pure H^s -theory the nonlinearity is required to behave as a power u^p at least $p \geq 1 + 4/n$ at the origin. On the other hand, the nonlinearity is required to have the differentiability of order greater than or equal to $n/2$ at the origin. To take those requirements into account, it is sufficient to suppose that the nonlinearity should behave as a power u^5 for $n = 1$, u^3 for $n = 2,\ 3,$ and u^2 for $n \geq 4$ to keep everything smooth. This is the reason why we have imposed additional power behavior at the origin of the nonlinearity. Although there is a room to reduce the order of power behavior at the origin to the minimal value $1 + 4/n$, that is outside the purpose of this paper since we intend to keep the exposition not too technical.

Remark 3. To our knowledge, both in the local and global cases there is no other work to treat the Schrödinger equation with nonlinearity of exponential growth in the H^s -theory with $s \leq n/2$. In view of Trudinger's inequality the growth rate as $e^{\lambda|z|^2}$ at infinity seems to be optimal at the level of $H^{n/2}$. Note that the L^∞ -norm is out of control of the $H^{n/2}$ -norm even when the latter is infinitesimally small.

Remark 4. The theorem above proves the existence and asymptotic completeness of the wave operators $W_\pm : \phi_\pm \mapsto u(0) = \phi$ on the small asymptotic states $\phi_\pm$ in $H^{n/2}$.

We prove the theorem in the next section. The method of the proof depends on a contraction argument on (1.3) in X with metric on $L^q(\mathbf{R}; L^r)$ for an admissible pair of exponents (q, r) in which the Strichartz type estimates for the free propagator fit naturally. For that purpose we prove that all the norms appearing in the definition of X are reproduced by the right hand side of (1.3) and that the metric on $L^q(\mathbf{R}; L^r)$ is contracted. As is usual with the estimation of the nonlinearity of exponential growth, we take the corresponding power series expansion and reduce the problem to majorizing each term of the expansion in terms of the Sobolev and Besov norms so that the resulting series of norms should converge. This requires detailed information on the individual terms of the expansion regarding the growth rate with respect to the exponent r of the L^r -norm in the space variables, since the argument here should be based exclusively on the $H^{n/2}$-norm but be independent of the L^∞ -norm. For that purpose we derive a sharp form of the Gagliardo-Nirenberg inequality which states precise dependence of the exponent r of the L^r -norm on the associated bound, see Lemma 2.2 below. As well as the problem described above on the norms with respect to the space variables, there arises a problem on the norms with respect to the time variable since no explicit dependence of the size T of the time interval $[0, T]$ on the nonlinear estimate is required to make the contraction argument work on the whole time interval $(-\infty, +\infty)$. To be more specific, the contribution of time on the estimates occuring in the contraction argument should be given in terms of the norm of X so that the argument could go through regardless of the size of the time interval. The problem is naturally solved by a simple power counting with respect to the time variable on the basis of admissible pairs arising from the Strichartz estimates, where we remark that no contribution of the time interval is given by the quantities associated with the $H^{n/2}$ -norm since X is continuously embedded in $L^\infty(\mathbf{R}; H^{n/2})$.

Before concluding the introduction we should mention that it was M. Tsutsumi [23] who pointed out the lack of the $H^{n/2}$ -theory in the Sobolev scale for the nonlinear Schrödinger equations. Actually that has been the major motivation for this work and we would appreciate his insight on that matter.

2 Proof of the theorem

In this section we prove the theorem. We first recall the basic estimates for the free propagator on the homogeneous Besov spaces. For any $s \in \mathbf{R}$ and any r, m with $1 \leq r, m \leq \infty$, the homogeneous Besov space $\dot{B}^s_{r,m}$ is defined as the space of classes of distributions u modulo polynomials such that

$$\|u; \dot{B}^s_{r,m}\| \equiv \Big(\sum_{j=-\infty}^{\infty} 2^{sjm} \|\phi_j * u; L^r\|^m \Big)^{1/m} < \infty$$

with obvious modification for $m = \infty$, where ϕ_j is defined through the Fourier transform by

$$\hat{\phi}_j(\xi) = \hat{\psi}(2^{-j}\xi) - \hat{\psi}(2^{-j+1}\xi)$$

with $\hat{\psi} \in C_0^\infty(\mathbf{R}^n)$ satisfying $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$, and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$. We now state the Strichartz type estimates for the free propagator U on the homogeneous Besov spaces $\dot{B}^s_r \equiv \dot{B}^s_{r,2}$.

Lemma 2. 1 [4,8]. *The free propagator U satisfies the following estimates.*
(1) For any $s \in \mathbf{R}$ and any admissible pair (q, r)

$$\|U(\cdot)\phi; L^q(\mathbf{R}; \dot{B}^s_r)\| \leq C\|\phi; \dot{H}^s\|.$$

(2) For any $s \in \mathbf{R}$ and any admissible pair (q_1, r_1) and (q_2, r_2), any interval $I \subset \mathbf{R}$ possibly unbounded, and any t_0 in the closure $\bar{I}$ of I in the extended real line $\mathbf{R} \cup \{\pm\infty\}$, the operator G_{t_0} defined by (1.3) satisfies the estimate

$$\|G_{t_0} f; L^{q_1}(\mathbf{R}; \dot{B}^s_{r_1})\| \leq C\|f; L^{q'_2}(\mathbf{R}; \dot{B}^s_{r'_2})\|,$$

where the constant C is independent of I and t_0 and p' is the exponent dual to p defined by $1/p + 1/p' = 1$.

Remark. Except for the case where $(q, r) = (4, \infty)$ for $n = 1$, the usual Strichartz estimates [3, 7, 9, 12, 20, 26] are derived immediately from Lemma 2.1 on the basis of the embeddings

$$\dot{B}^s_r \subset \dot{H}^s_r \quad \text{if } 2 \leq r < \infty,$$

$$\dot{H}^s_r \subset \dot{B}^s_r \quad \text{if } 1 < r \leq 2.$$

Conversely, Lemma 2.1 is derived from the usual Strichartz estimates by using the definition of the norm on $\dot{B}^s_r$, the Minkowski inequality, and the commutation relation between U and $\varphi *$.

The following lemma is crucial in the proof of convergence of the series of exponential type.

Lemma 2. 2 *Let* $1 < r < \infty$. *Then there exists a constant* $C_0 > 0$ *such that for any* q *with* $r \leq q < \infty$ *the following estimates hold.*

$$\|u; L^q\| \leq C_0 q^{1/2+(r-2)/(2q)} \|u; \dot{H}^{n/2}\|^{1-r/q} \|u; L^r\|^{r/q},$$

$$\|u; \dot{B}^0_q\| \leq C_0 q^{1/2+(r-2)/(2q)} \|u; \dot{H}^{n/2}\|^{1-r/q} \|u; \dot{B}^0_r\|^{r/q}.$$

Proof. We first recall the following estimate from [17, Inequality(2.6)]. For any p with $1 < p < \infty$ there exists a constant $C > 0$ such that for any q with $p \leq q < \infty$

$$\|u; L^q\| \leq C q^{1/p'} \|u; \dot{H}^{n/p-n/q}_p\|.$$

For any r with $1 < r < \infty$ and any q with $r < q < \infty$ we define a and p by $a = 1 - r/q$ and $1/p = a/2 + (1-a)/r$. Note that $0 < a < 1$, $p < q < \infty$, $n/p - n/q = na/2$. By an interpolation inequality (see [10, Lemma A.1] for instance), we have

$$\|u; \dot{H}^{n/p-n/q}_p\| \leq C \|u; \dot{H}^{n/2}\|^a \|u; L^r\|^{1-a},$$

where C is independent of q since two numbers $n/p - n/q$ and p range over compact intervals. Combining two inequalities above, we obtain the first inequality of the lemma since $1/p' = 1/2 + (r-2)/(2q)$ by definition. The second inequality of the lemma follows from the first by substituting u by $\varphi_j * u$ and taking the norm on $l^2(L^q)$. We note here that the $l^2(\dot{H}^{n/2})$ norm of $\{\varphi_j * u\}$ is equivalent to the $\dot{H}^{n/2}$ norm of u.

QED

Lemma 2. 3 *Let* (q_0, r_0) *satisfy*

$$\begin{aligned} n/2 < r_0 \le \infty, &\quad 1 < r_0 \le \infty, \\ n/2 < q_0 < \infty, &\quad 1 \le q_0 < \infty, \\ 2/q_0 + n/r_0 = 2. & \end{aligned}$$

Let f *be as in Theorem 1. Then there exists a continuous, nonnegative function* F *defined on the interval* $[0, \rho_0]$ *with* $\rho_0 > 0$ *such that the following estimates hold: For any admissible pairs* (q_1, r_1) *and* (q_2, r_2) *with* $\delta(r_1) + \delta(r_2) = n/r_0$*, any* ρ *with* $0 < \rho \le \rho_0$*, and any* $u, v \in X$ *with* $\|u; X\| \le \rho$*,* $\|v; X\| \le \rho$*,*

$$\|f(u) - f(v); L^{q_2'}(\mathbf{R}; L^{r_2'})\| \le F(\rho)\|u - v; L^{q_1}(\mathbf{R}; L^{r_1})\|, \tag{2.1}$$

$$\|f(u) - f(v); L^{q_2'}(\mathbf{R}; \dot{B}^0_{r_2'})\| \le F(\rho)\|u - v; L^{q_1}(\mathbf{R}; \dot{B}^0_{r_1})\|, \tag{2.2}$$

$$\|f(u); L^{q_2'}(\mathbf{R}; H^{n/2}_{r_2'})\| \le F(\rho)\|u; L^{q_1}(\mathbf{R}; H^{n/2}_{r_1})\| \quad \textit{for } n \textit{ even}, \tag{2.3}$$

$$\|f(u); L^{q_2'}(\mathbf{R}; B^{n/2}_{r_2'})\| \le F(\rho)\|u; L^{q_1}(\mathbf{R}; B^{n/2}_{r_1})\| \quad \textit{for } n \textit{ odd}. \tag{2.4}$$

Moreover, F *satisfies*

$$\begin{aligned} F(\rho) &= O(\rho^4) \quad \textit{for } n = 1, \\ F(\rho) &= O(\rho^2) \quad \textit{for } n = 2, 3, \\ F(\rho) &= O(\rho) \quad \textit{for } n \ge 4, \end{aligned}$$

as $\rho \to 0$.

Proof. We prove the lemma for real-valued functions for simplicity since the proof for complex-valued functions is analogous if we regard f as a function of two variables z and $\bar{z}$. By the assumptions on exponents, (q_j, r_j), $j = 0, 1, 2$, satisfy

$$\begin{aligned} 1/r_2' &= 1/r_0 + 1/r_1, \\ 1/q_2' &= 1/q_0 + 1/q_1. \end{aligned}$$

Moreover, $(4q_0/n, 4r_0/n)$ forms an admissible pair.

We first consider the case where $n \ge 4$. By the assumption $(A)_{[n/2]}$, we decompose f as $f = f_1 + f_2$ with $f_1(0) = f_2(0) = 0$,

$$\begin{aligned} |f_1'(u)| &\le C|u|, \\ |f_2'(u)| &\le C(e^{\lambda|u|^2} - 1), \end{aligned}$$

where $\lambda > 0$ may be taken large enough if needed. We treat separately the contributions of f_1 and f_2. By the Hölder inequality in space and time, we estimate

$$\begin{aligned} &\|f_1(u) - f_1(v); L^{q_2'}(\mathbf{R}; L^{r_2'})\| \\ \leq\ & C(\|u; L^{q_0}(\mathbf{R}; L^{r_0})\| + \|v; L^{q_0}(\mathbf{R}; L^{r_0})\|)\|u - v; L^{q_1}(\mathbf{R}; L^{r_1})\|, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\|f_2(u) - f_2(v); L^{q_2'}(\mathbf{R}; L^{r_2'})\| \\ \leq\ & C\sum_{j=1}^{\infty} \frac{\lambda^j}{j!}(\|u^{2j}; L^{q_0}(\mathbf{R}; L^{r_0})\| + \|v^{2j}; L^{q_0}(\mathbf{R}; L^{r_0})\|)\|u - v; L^{q_1}(\mathbf{R}; L^{r_1})\|. \end{aligned} \tag{2.6}$$

We estimate the first two norms on the RHS of (2.5) and (2.6) by using Lemma 2.2 for the L^{r_0} norm in space and taking the L^{q_0} norm in time of the resulting inequality to obtain for instance

$$\begin{aligned} \|u; L^{q_0}(\mathbf{R}; L^{r_0})\| &\leq C\|u; L^\infty(\mathbf{R}; \dot{H}^{n/2})\|^{1-4/n}\|u; L^{4q_0/n}(\mathbf{R}; L^{4r_0/n})\|^{4/n} \\ &\leq C\|u; X\|, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\|u^{2j}; L^{q_0}(\mathbf{R}; L^{r_0})\| = \|\|u; L^{2jr_0}\|^{2j}; L^{q_0}\| \\ \leq\ & C_0^{2j}(2jr_0)^{j+2/(nq_0)}\|u; L^\infty(\mathbf{R}; \dot{H}^{n/2})\|^{2j-4/n}\|u; L^{4q_0/n}(\mathbf{R}; L^{4r_0/n})\|^{4/n} \\ \leq\ & C_0^{2j}(2jr_0)^{j+2/(nq_0)}\|u; X\|^{2j}, \end{aligned} \tag{2.8}$$

where C_0 is a constant independent of j and we have used Lemma 2.2 with $q = 2jr_0$ and $r = 4r_0/n$. Combining (2.5) and (2.7), for $u, v \in X$ with norm bounded by ρ we have

$$\|f_1(u) - f_1(v); L^{q_2'}(\mathbf{R}; L^{r_2'})\| \leq C\rho\|u - v; L^{q_1}(\mathbf{R}; L^{r_1})\|. \tag{2.9}$$

Combining (2.6) and (2.8), for $u, v \in X$ with norm bounded by ρ we have

$$\begin{aligned} &\|f_2(u) - f_2(v); L^{q_2'}(\mathbf{R}; L^{r_2'})\| \\ \leq\ & C\sum_{j=1}^{\infty} \frac{j^{j+2/(nq_0)}}{j!}(2\lambda r_0 C_0^2\rho^2)^j\|u - v; L^{q_1}(\mathbf{R}; L^{r_1})\|, \end{aligned} \tag{2.10}$$

where the last series converges for all ρ with $0 \le \rho < \rho_0 \equiv 1/((2\lambda r_0 e)^{1/2} C_0)$. The inequality (2.1) therefore follows from (2.9) and (2.10), while (2.2) follows by the embeddings $\dot{B}^0_r \subset L^r$ and $L^{r'} \subset \dot{B}^0_{r'}$ for $r \ge 2$. We next prove (2.3) for even $n \ge 4$. In view of (2.1) with $f(0) = 0$, it suffices to prove the inequality of the form

$$\|f(u); L^{q'_2}(\mathbf{R}; \dot{H}^{n/2}_{r'_2})\| \le F(\rho)\|u; L^{q_1}(\mathbf{R}; \dot{H}^{n/2}_{r_1})\|. \tag{2.11}$$

We estimate the LHS of (2.11) as

$$\begin{aligned}
&\|f(u); L^{q'_2}(\mathbf{R}; \dot{H}^{n/2}_{r'_2})\| \\
\le\ & C \sum_{|\alpha|=n/2} \|f'_1(u)\partial^\alpha u; L^{q'_2}(\mathbf{R}; L^{r'_2})\| + C \sum_{|\alpha|=n/2} \|f'_2(u)\partial^\alpha u; L^{q'_2}(\mathbf{R}; L^{r'_2})\| \\
&+ C \sum_{\substack{|\alpha|=n/2 \\ 2\le k\le n/2}} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_\ell|\ge 1}} \|f^{(k)}(u) \prod_{\ell=1}^{k} \partial^{\beta_\ell} u; L^{q'_2}(\mathbf{R}; L^{r'_2})\|,
\end{aligned} \tag{2.12}$$

where α and $\{\beta_\ell\}$ stand for the usual multi-indices for derivatives and f_1 and f_2 are as in the preceding argument. We consider separately the contributions of the derivatives $f^{(k)}$ with $1 \le k \le n/2$ on the RHS of (2.12). The terms with $f' = f'_1 + f'_2$ are estimated in the same way as in the preceding argument and the first and second sums on the RHS of (2.12) are bounded respectively by the RHS of (2.9) and (2.10) with $\|u - v; L^{q_1}(\mathbf{R}; L^{r_1})\|$ replaced by $\|u; L^{q_1}(\mathbf{R}; \dot{H}^{n/2}_{r_1})\|$. For the terms with $f^{(k)}$ with $2 \le k \le n/2$, we define p_ℓ with $1 \le \ell \le k$ by

$$1/p_\ell = \mu_\ell/r_1 + (1 - \mu_\ell)/s,$$

where $\mu_\ell = |\beta_\ell|/|\alpha|$ and $s = (2j + k - 1)r_0$ with $j \ge 0$. By the Hölder inequality in space, the convexity inequalities between the homogeneous Sobolev spaces [10;Lemma A.1], and Lemma 2.2 with $q = s = (2j + k - 1)r_0$ and $r = 4r_0/n$, we estimate the last norm in space on the RHS of (2.12) by

$$\begin{aligned}
&\|f^{(k)}(u) \prod_{\ell=1}^{k} \partial^{\beta_\ell} u; L^{r'_2}\| \\
\le\ & C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \|u^{2j} \prod_{\ell=1}^{k} \partial^{\beta_\ell} u; L^{r'_2}\|
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\|u;L^s\|^{2j}\prod_{\ell=1}^{k}\|u;\dot H^{|\beta_\ell|}_{p_\ell}\| \\
&\le C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\|u;L^s\|^{2j+k-1}\|u;\dot H^{n/2}_{r_1}\| \\
&\le C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}C_0^{2j+k-1}((2j+k-1)r_0)^{j+(k-1)/2+2/(nq_0)}\|u;\dot H^{n/2}\|^{2j+k-1-4/n} \\
&\qquad \cdot\|u;L^{4r_0/n}\|^{4/n}\|u;\dot H^{n/2}_{r_1}\|, \qquad (2.13)
\end{aligned}
$$

where C_0 is a constant independent of j. We take the $L^{q_2'}$ norm in time of (2.13) for $u\in X$ with norm bounded by ρ and use the Hölder inequality in time to obtain

$$
\begin{aligned}
&\|f^{(k)}(u)\prod_{\ell=1}^{k}\partial^{\beta_\ell}u;L^{q_2'}(\mathbf{R};L^{r_2'})\| \\
\le\ & C\rho^{k-1}\sum_{j=0}^{\infty}\frac{(2j+k-1)^{j+(k-1)/2+2/(nq_0)}}{j!}(C_0\lambda^{1/2}r_0^{1/2}\rho)^{2j}\|u;L^{q_1}(\mathbf{R};\dot H^{n/2}_{r_1})\|, \qquad (2.14)
\end{aligned}
$$

where the last series converges for all ρ with $0\le\rho<\rho_0$, so that the LHS of (2.14) is bounded by the RHS of (2.11). Collecting those estimates above yields (2.11), as required. We now prove (2.4) for odd $n\ge 5$. In view of (2.1) with $f(0)=0$, it suffices to prove the inequality of the form

$$
\|f(u);L^{q_2'}(\mathbf{R};\dot B^{n/2}_{r_2'})\|\le F(\rho)\|u;L^{q_1}(\mathbf{R};\dot B^{n/2}_{r_1})\|. \qquad (2.15)
$$

We write n as $n=2m+1$ with $m\ge 2$ and we estimate the $\dot B^{n/2}_{r_2'}$ norm on the LHS of (2.15) as

$$
\begin{aligned}
&\|f(u);\dot B^{m+1/2}_{r_2'}\| \\
\le\ & C\sum_{|\alpha|=m}\|\partial^\alpha(f(u));\dot B^{1/2}_{r_2'}\| \\
\le\ & C\sum_{|\alpha|=m}\|f_1'(u)\partial^\alpha u;\dot B^{1/2}_{r_2'}\|+C\sum_{|\alpha|=m}\|f_2'(u)\partial^\alpha u;\dot B^{1/2}_{r_2'}\| \\
&+C\sum_{\substack{|\alpha|=m\\ 2\le k\le m}}\ \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\ |\beta_\ell|\ge 1}}\|f^{(k)}(u)\prod_{\ell=1}^{k}\partial^{\beta_\ell}u;\dot B^{1/2}_{r_2'}\|, \qquad (2.16)
\end{aligned}
$$

where we have decomposed f' as $f' = f_1' + f_2'$ with $f_1'(0) = f_2'(0) = 0$,

$$\begin{aligned}
&|f_1'(u) - f_1'(v)| \le C|u - v|, \\
&|f_2'(u) - f_2'(v)| \le C\Big((e^{\lambda|u|^2} - 1) + (e^{\lambda|v|^2} - 1)\Big)|u - v|.
\end{aligned}$$

We consider separately the contributions of f_1', f_2', and $f^{(k)}$ with $2 \le k \le m$ on the RHS of (2.16). As for the latter, it follows by assumption that for all k with $2 \le k \le m$, $f^{(k)}(0) = 0$ and

$$|f^{(k)}(u) - f^{(k)}(v)| \le C(e^{\lambda|u|^2} + e^{\lambda|v|^2})|u - v|.$$

To estimate the Besov norms on the RHS of (2.16) we use the following equivalent norm on $\dot{B}^{\sigma}_{r,m}$ with $0 < \sigma < 1$

$$\|u; \dot{B}^{\sigma}_{r,m}\| \simeq \Big(\int_0^\infty t^{-1-\sigma m} \sup_{|y|\le t} \|\tau_y u - u; L^r\|^m dt \Big)^{1/m},$$

where τ_y is the space translation by $y \in \mathbf{R}^n$. For the terms with f_1' on the RHS of (2.16) we define p_0 and p_1 by

$$1/p_0 = \mu_0/r_0 + (1 - \mu_0)/r_1,$$

$$1/p_1 = \mu_1/r_0 + (1 - \mu_1)/r_1,$$

where $\mu_0 = 1 - 1/n$ and $\mu_1 = 1/n$. The terms with f_1' are estimated as

$$\begin{aligned}
&\|f_1'(u)\partial^\alpha u; \dot{B}^{1/2}_{r_2'}\| \\
\le\ & C\Big(\int_0^\infty t^{-2} \sup_{|y|\le t} \|f_1'(\tau_y u) - f_1'(u); L^{p_0}\|^2 dt \Big)^{1/2} \|\partial^\alpha u; L^{p_1}\| \\
&\quad + C\|f_1'(u); L^{r_0}\|\ \|\partial^\alpha u; \dot{B}^{1/2}_{r_1}\| \\
\le\ & C\|u; \dot{B}^{1/2}_{p_0}\|\ \|u; \dot{H}^{(n-1)/2}_{p_1}\| + C\|u; L^{r_0}\|\ \|u; \dot{B}^{n/2}_{r_1}\| \\
\le\ & C\|u; \dot{B}^{0}_{r_0}\|\ \|u; \dot{B}^{n/2}_{r_1}\|,
\end{aligned} \tag{2.17}$$

where we have used the Hölder inequality, the embeddings $\dot{B}^0_{r_0} \subset L^{r_0}$ and $\dot{B}^{n/2}_{r_1} \subset \dot{H}^{n/2}_{r_1}$, and the following convexity inequalities

$$\|u; \dot{B}^{1/2}_{p_0}\| \le C\|u; \dot{B}^0_{r_0}\|^{\mu_0}\|u; \dot{B}^{n/2}_{r_1}\|^{\mu_1},$$

$$\|u; \dot{H}^{(n-1)/2}_{p_1}\| \le C\|u; L^{r_0}\|^{\mu_1}\|u; \dot{H}^{n/2}_{r_1}\|^{\mu_0}.$$

Taking the $L^{q_2'}$ norm in time of (2.17), we conclude that the contribution of f_1' on the RHS of (2.15) is bounded by the LHS, as required.
For the terms with f_2' on the RHS of (2.16) we define p_0 and p_1 by

$$1/p_0 = \mu_0/s + (1-\mu_0)/r_1,$$

$$1/p_1 = \mu_1/s + (1-\mu_1)/r_1,$$

where $\mu_0 = 1 - 1/n$, $\mu_1 = 1/n$, and $s = (2j+1)r_0$ with $j \ge 1$. The terms with f_2' are estimated as

$$\begin{aligned}
&\|f_2'(u)\partial^\alpha u; \dot{B}^{1/2}_{r_2'}\| \\
\le\ & C\Big(\int_0^\infty t^{-2}\sup_{|y|\le t}\|(f_2'(\tau_y u) - f_2'(u))\partial^\alpha u; L^{r_2'}\|^2 dt\Big)^{1/2} \\
&+C\|f_2'(u); L^{r_0}\|\ \|\partial^\alpha u; \dot{B}^{1/2}_{r_1}\| \\
\le\ & C\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}\|u; L^s\|^{2j}\|u; \dot{B}^{1/2}_{p_0}\|\|\partial^\alpha u; L^{p_1}\| + C\|f_2'(u); L^{r_0}\|\|u; \dot{B}^{n/2}_{r_1}\| \\
\le\ & C\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}\|u; \dot{B}^0_s\|^{2j+1}\|u; \dot{B}^{n/2}_{r_1}\| \\
\le\ & C\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}C_0^{2j+1}((2j+1)r_0)^{j+1/2+2/(nq_0)}\|u; \dot{H}^{n/2}\|^{2j+1-4/n} \\
&\cdot\|u; \dot{B}^0_{4r_0/n}\|^{4/n}\|u; \dot{B}^{n/2}_{r_1}\|,
\end{aligned} \tag{2.18}$$

where we have used the Hölder inequality with $1/r_2' = 1/r_0 + 1/r_1 = 2j/s + 1/p_0 + 1/p_1$, the embeddings and convexity inequalities as in the preceding argument with r_0 replaced by s, and Lemma 2.2. Taking the $L^{q_2'}$ norm in time of (2.18), we conclude that the contribution of f_2' on the RHS of (2.15) is bounded by the LHS for any $u \in X$ with norm bounded by ρ in the interval $[0,\ \rho_0)$, which ensures convergence of the resulting series as before.

For the terms with $f^{(k)}$ on the RHS of (2.16), namely $f^{(k)}(u)\partial^{\beta_1}u\cdots\partial^{\beta_k}u$ with $\beta_1+\cdots+\beta_k=\alpha$ and $\beta_1,\cdots,\beta_k\neq 0$, we write the corresponding first difference in the equivalent Besov norm as

$$\begin{aligned}
&\tau_y(f^{(k)}(u)\prod_\ell \partial^{\beta_\ell}u)-f^{(k)}(u)\prod_\ell \partial^{\beta_\ell}u\\
=\ &(f^{(k)}(\tau_y u)-f^{(k)}(u))\prod_\ell \partial^{\beta_\ell}u\\
&+\sum_\ell f^{(k)}(u)(\tau_y\partial^{\beta_\ell}u-\partial^{\beta_\ell}u)\prod_{a<\ell}\tau_y\partial^{\beta_a}u\prod_{b>\ell}\partial^{\beta_b}u \qquad (2.19)
\end{aligned}$$

and we denote by I and II the first and second line on the RHS of (2.18), respectively. For the term I, we define p_a with $0\le a\le k$ by

$$1/p_a=\mu_a/r_1+(1-\mu_a)/s,$$

where $\mu_0=1/n$, $\mu_a=2|\beta_a|/n$ for $a\neq 0$, and $s=(2j+k)r_0$ for $j\ge 0$. We estimate

$$\begin{aligned}
&\Big(\int_0^\infty t^{-2}\sup_{|y|\le t}\|\mathrm{I};L^{r_2'}\|^2dt\Big)^{1/2}\\
\le\ &C\sum_{j=0}^\infty\frac{\lambda^j}{j!}\|u;L^s\|^{2j}\|u;\dot B^{1/2}_{p_0}\|\prod_\ell\|\partial^{\beta_\ell}u;L^{p_\ell}\|\\
\le\ &C\sum_{j=0}^\infty\frac{\lambda^j}{j!}\|u;\dot B^0_s\|^{2j+k}\|u;\dot B^{n/2}_{r_1}\|\\
\le\ &C\sum_{j=0}^\infty\frac{\lambda^j}{j!}C_0^{2j+k}((2j+k)r_0)^{j+k/2+2/(nq_0)}\|u;\dot H^{n/2}\|^{2j+k-4/n}\\
&\cdot\|u;\dot B^0_{4r_0/n}\|^{4/n}\|u;\dot B^{n/2}_{r_1}\|, \qquad (2.20)
\end{aligned}$$

where we have used the Hölder inequality with $1/r_2'=1/r_0+1/r_1=2j/s+\sum_a 1/p_a$, the embeddings and convexity inequalities between the homogeneous Sobolev and Besov spaces, and Lemma 2.2. For the term II, we difine p_a with $1\le a\le k$ by

$$1/p_a=\mu_a/r_1+(1-\mu_a)/s,$$

where $\mu_a=2|\beta_a|/n$ for $a\neq\ell$, $\mu_\ell=(2|\beta_\ell|+1)/n$, and $s=(2j+k-1)r_0$ for $j\ge 0$. In the same way as above, we see that the integral on the LHS of

(2.20) with I replaced by II is estimated by the RHS of (2.20) with k replaced by $k-1$. Taking the $L^{q_2'}$ norm of (2.20) and the corresponding inequality for II, we have for $u \in X$ with norm bounded by ρ

$$\begin{aligned}
&\|f^{(k)}(u) \prod_{\ell=1}^{k} \partial^{\beta_\ell} u; L^{q_2'}(\mathbf{R}; \dot{B}^{1/2}_{r_2'})\| \\
\leq\ & C(\rho^{k-1} + \rho^k) \sum_{j=0}^{\infty} \frac{(2j+k)^{j+k/2+2/(nq_0)}}{j!} (C_0 \lambda^{1/2} r_0^{1/2} \rho)^{2j} \\
&\cdot \|u; L^{q_1}(\mathbf{R}; \dot{B}^{n/2}_{r_1})\|, \qquad (2.21)
\end{aligned}$$

where the last series converges for all ρ with $0 \leq \rho < \rho_0$, so that the LHS of (2.21) is bounded by the RHS of (2.15). Collecting those estimates yields (2.15), as required. This proves the lemma for $n \geq 4$.

We now comment briefly on the lower dimensional case $n \leq 3$. The method of proof above works as well with minor modifications and the only essential difference arises in connection with the power behavior of nonlinearity as $u \to 0$. To be specific, the power p at the origin should satisfy $p \geq 1 + 4/n$. For instance, to ensure (2.8) for all dimensions, one has to impose the condition that $2j \geq 4/n$. This determines the behavior of $f'(u)$ as $u \to 0$, so that the assumptions of the theorem fit into the argument above.

QED

Proof of the theorem. We solve the equation (1.3) by a contraction argument. For that purpose it suffices to show that all the norms appearing in the definition of X are reproduced by the RHS of (1.3) and that the metric on $L^q(\mathbf{R}; L^r)$ or on $L^q(\mathbf{R}; \dot{B}^0_r)$ is contracted for an admissible pair (q, r). The smallness assumption on the data is required to ensure the invariance of a closed ball in X under the mapping (1.3) with restect to $u \in X$. The size of radius of the ball in X is accordingly chosen to be small enough. The contraction argument on (1.3) is carried out on the basis of Lemmas 2.1 and 2.3 and yields a unique fixed point $u \in X$ for any data ϕ at $t_0 = 0$ or at $t_0 = \pm\infty$ with the $H^{n/2}$ norm sufficiently small. The asymptotic conditions (1.4) and (1.5) are proved in the same way if one replaces the whole time interval by the interval $I = [t, +\infty)$ or $(-\infty, t]$ in the relevant estimates, so that the required convergence follows from the integrability in time of solutions in X. The existence of asymptotic states $\phi_\pm$ follows by the standard Cook-Kuroda method.

QED

References

[1] J.Bergh, J.Löfström, "Interpolation Spaces," Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] T.Cazenave, *Equation de Schrödinger non linéaires en dimension deux*, Proceedings of the Royal Society of Edinburgh, **84A**(1979), 327-346.

[3] T.Cazenave, "An Introduction to Nonlinear Schrödinger Equations," Textos de Métodos Matemáticos, **22**, Instituto de Matemática, Rio de Janeiro, 1989.

[4] T.Cazenave, F.B.Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. TMA, **14**(1990), 807-836.

[5] T.Cazenave, F.B.Weissler, *Asymptotically self-similar solutions of the nonlinear Schrödinger and heat equations*, (preprint).

[6] Chen Yunmei, $L^\infty(\mathbf{R}^n)$ *decay for solutions to a class of Schrödinger equations*, Applicable Analysis, **39**(1990), 209-226.

[7] J.Ginibre, *An Introduction to Nonlinear Schrödinger Equations*, in "Nonlinear Waves," R.Agemi, Y.Giga, T.Ozawa(Eds), GAKUTO International Series, Mathematical Sciences and Applications, Gakkōtosho, Tokyo(in press).

[8] J.Ginibre, T.Ozawa, G.Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Physique théorique, **60**(1994), 211-239.

[9] J.Ginibre, G.Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, J.Math.pures et appl., **64**(1985), 363-401.

[10] J.Ginibre, G.Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math.Z., **189**(1985), 487-505.

[11] T.Kato, *On nonlinear Schrödinger equations*, Ann.Inst.Henri Poincaré, Physique théorique, **46**(1987), 113-129.

[12] T.Kato, *Nonlinear Schrödinger Equations*, in "Schrödinger Operators," H.Holden, A.Jensen(Eds), Lecture Notes in Physics, **345**, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[13] T.Kato, *On nonlinear Schrödinger equations II. H^s-solutions and unconditional well-posedness*, J.d'Anal.Math., **67**(1995), 281-306.

[14] C.E.Kenig, G.Ponce, L.Vega, *On the IVP for the nonlinear Schrödinger equations*, Contemporary Mathematics, **189**(1995), 353-367.

[15] M.Nakamura, T.Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spacs*, Rev. Math. Phys. (in press).

[16] T.Ogawa, T.Ozawa, *Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem*, J.Math.Anal.Appl., **155**(1991), 531-540.

[17] T.Ozawa, *On critical cases of Sobolev's inequalities*, J.Funct.Anal., **127**(1995), 259-269.

[18] H.Pecher, *Solutions of semilinear Schrödinger equations in H^s*, (preprint).

[19] R.S.Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ.Math.J., **21**(1972), 841-842.

[20] R.S.Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math.J., **44**(1977), 705-714.

[21] H.Triebel, "Theory of Function Spaces," Birkhäuser, Basel, 1983.

[22] N.S.Trudinger, *On imbeddings into Orlicz spaces and some applications*, J.Math.Mech., **17**(1967), 473-483.

[23] M.Tsutsumi, *Nonlinear Schrödinger equations*, SUGAKU, **47**(1995), 18-37, (in Japanese).

[24] Y.Tsutsumi, L^2 *-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funk.Ekva., **30**(1987), 115-125.

[25] Y.Tsutsumi, *Global strong solutions for nonlinear Schrödinger equations*, Nonlinear Anal.TMA, **11**(1987), 1143-1154.

[26] K.Yajima, *Existence of solutions for Schrödinger evolution equations*, Commun.Math. Phys., **110**(1987), 415-426.