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The Commutator Ideal In Toeplitz Algebras For Uniform Algebras And The Analytic Structure

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Abstract. Let A be a uniform algebra and H^2 the abstract Hardy space defined by A. Under some conditions on A, we show that its maximal ideal space has some analytic structure when the commutator ideal of a Toeplitz algebra on H^2 is just a set of all compact operators on H^2.

§1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Fix a nonzero complex homomorphism τ on A and a representing measure m for τ whose support is X. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined to the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak star closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$. $M(A)$ denotes the maximal ideal space of A and X is regarded as a closed subset of $M(A)$. The Shilov boundary ∂A of A then becomes a closed subset of X. The τ fixed above belongs to $M(A)$. We set $A_\tau = \{f \in A \, ; \, \tau(f) = 0\}$ and $H^p_\tau = \{f \in H^p \, ; \, \int f dm = 0\}$.

For ϕ in L^∞, the Toeplitz operator T_ϕ is defined by

$$T_\phi(f) = P(\phi f) \qquad (f \in H^2)$$

where P is the orthogonal projection onto H^2, and the Hankel operator H_ϕ is defined by

$$H_\phi(f) = (I - P)(\phi f) \qquad (f \in H^2).$$

Let $\mathcal{T}(C(X))$ be the C^*-algebra generated by the operators $\{T_\phi \, ; \, \phi \in C(X)\}$ and $\mathcal{C}(C(X))$ be the commutator ideal of $\mathcal{T}(C(X))$. $\mathcal{LC}(H^2)$ denotes the set of all compact operators on H^2. In this paper, we study the analytic structure on $M(A)$ when $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. Let $\mathcal{A}$ be the set of all continuous functions on the closed unit disc $\partial\triangle \cup \triangle$ which are analytic on the open unit disc $\triangle$. If $A = \mathcal{A}|\partial\triangle, X = \partial\triangle$ and m is the normalized Lebesgue measure on $\partial\triangle$, then H^2 is the classical Hardy space. If $A = \mathcal{A}, X = \partial\triangle \cup \triangle$ and m is the normalized area measure on $\partial\triangle \cup \triangle$, then H^2 is called the Bergman space. In both examples, it is known (*cf.*[5, Theorem 7.23] and [1, Theorem 4.12]) that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. Let z be a coordinate function in $\mathbf{C}$. If $A = \mathcal{A}|\partial\triangle$, then T_z is an isometry and if $A = \mathcal{A}$, then T_z is an essential isometry. In this paper, these two uniform algebras are typical examples such that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$.

We call A a uniform algebra with generalized approximation in modulus property when the set of all finite sums $\sum_i \mid f_i \mid^2$, $f_i \in A$, is dense in the cone of positive and continuous functions on X(*cf.* [7]). A lot of uniform algebras have such modulus property. If a function q in A is unimodular on X, we call q an inner function. T is called isometry when $T^*T = 1$ and T is called essential isometry when $1 - T^*T$ is compact. $G(\tau)$ denotes the Gleason part of τ, that is, $G(\tau) = \{\sigma \in M(A) \, ; \, \|\tau - \sigma\| < 2\}$. $[E]_2$ denotes the closure of the subset of E in L^2.

In this paper, we assume that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. In §2, if there exists a nonconstant function q in A such that T_q is an isometry and not a unitary, we show that $G(\tau)$ is nontrivial and under some conditions on A and q, it is contained in a subset of $M(A)$ given an analytic structure which is determined by q. In §3, we consider the same problem when T_q is an essential isometry. In §4, we study it without the existence of T_q.

§2. Isometric Toeplitz operator

The typical example in this section is $A = \mathcal{A} \mid \partial\Delta$ and then T_z is an isometry.

Theorem 1. If $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A such that T_q is an isometry and not a unitary, then $G(\tau) \neq \{\tau\}$.

Proof. Since T_q is an isometry, $T_q^* T_q$ is the identity operator and $1 - T_q T_q^*$ is an orthogonal projection from H^2 onto $H^2 \ominus qH^2$. Because $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and the range of a compact operator contains no closed infinite-dimensional subspace, $1 - T_q T_q^*$ is a compact operator and $\dim(H^2 \ominus qH^2) < \infty$. Let $B = H^2 \cap L^\infty$ and $D = (qH^2) \cap L^\infty$. If $\dim(H^2 \ominus qH^2) = n$, then $\dim(B/D) \leq n$. In fact, if $\dim(B/D) > n$, there exists $\{f_j\}_{j=1}^{n+1}$ in B such that $\{f_j + D\}_{j=1}^{n+1}$ is linearly independent in B/D. By hypothesis, $\{f_j + qH^2\}_{j=1}^{n+1}$ is not linearly independent in H^2/qH^2 and so there exists $\{\alpha_j\}_{j=1}^{n+1}$ in $\mathbf{C}$ with $\langle \alpha_1, \cdots, \alpha_{n+1} \rangle \neq \langle 0, \cdots, 0 \rangle$ such that $\alpha_1 f_1 + \cdots + \alpha_{n+1} f_{n+1}$ belongs to $(qH^2) \cap L^\infty = D$. This contradiction implies that $\dim(B/D) \leq n$. If $D = B$, then $1 \in D$ and so $qH^2 = H^2$. This contradicts that T_q is not a unitary. This contradiction shows that D is a weak star closed proper ideal of B. Since $\dim(B/D) < \infty$, there exists ϕ in $M(B)$ such that $B_\phi = \{f \in B\ ;\ \phi(f) = 0\} \supseteq D$ and B_ϕ is weak star closed. Since $1 \notin B_\phi$, there exists a function h in L^1 such that

$$a = \int h dm \neq 0 \text{ and } \int hg dm = 0 \qquad (g \in B_\phi).$$

Hence $\phi(f) = \int f h dm / a$ for all f in B. Now there exists a (positive) representing measure for ϕ that is absolutely continuous with respect to hdm/a (*cf.* [2, Theorem 2.1.1]). Thus $\phi \mid A$ belongs to $G(\tau)$. If $\tau(q) \neq 0$, then $\phi \mid A \neq \tau$. and so $G(\tau) \neq \{\tau\}$. If $\tau(q) = 0$, put $Q = (q + a)/(1 + \bar{a}q)$ for some nonzero constant a with $\mid a \mid < \|q\|_\infty$, then Q belongs to A and T_Q is an isometry. Hence the argument above is valid for the new nonconstant function Q with $\tau(Q) \neq 0$. This completes the proof.

Theorem 2. Suppose $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A such that T_q is an isometry and not a unitary.

(1) Suppose $H^2 \cap C(X) = A$ and $M(A)$ has no isolated point. If $\mid q \mid = 1$ on X, then

$$\hat{q}(\partial A) = \partial\Delta, \hat{q}(M(A)) = \bar{\Delta} \text{ and } G(\tau) \subseteq \hat{q}^{-1}(\Delta).$$

Moreover $\hat{q}^{-1}(\Delta) \subseteq M(A)$ can be given the structure of a (possibly disconnected) open Riemann surface with discrete point identifications, and hence the Gelfand transforms of functions in A become analytic on $\hat{q}^{-1}(\Delta)$. $\hat{q}$ is an n-sheeted analytic cover of $\hat{q}^{-1}(\Delta)$ over Δ where $1 \leq n \leq \dim\ker T_q^* < \infty$.

(2) If A is an algebra with generalized approximation in modulus property, then (1) is valid without assuming that $\mid q \mid = 1$ on X.

(3) If m is a unique representing measure for τ, then (1) is valid with $G(\tau) = \hat{q}^{-1}(\triangle)$ without any assumptions in (1).

Proof. (1) Since $H^2 \cap C(X) = A$, we can show $\dim(A/qA) < \infty$ as in the proof of Theorem 1. By hypothesis on q, $\hat{q}(\partial A) \subseteq \partial\triangle$. We will show that $\hat{q}(\partial A) = \partial\triangle$. If $\hat{q}(\partial A) \neq \partial\triangle$, then $W = \mathbf{C}\backslash\hat{q}(\partial A)$ is connected and unbounded. Since q is a nonconstant function because T_q is not a unitary, qA is a proper closed ideal in A and so there exists ϕ in $M(A)$ such that $\phi(q) = 0$. This implies $0 \in W$ and hence $(q-\lambda)A \subsetneq A$ for any $\lambda \in W$ by the index theory for Fredholm operators (*cf.* [6, p157]). Since W is unbounded, there exists $\lambda \in W$ such that $(q-\lambda)^{-1} \in A$ and so $(q-\lambda)A = A$. This contradiction shows that $\hat{q}(\partial A) = \partial\triangle$ and $\mathbf{C}\backslash\hat{q}(\partial A) = \triangle \cup \bar{\triangle}^c$. Now $\hat{q}(M(A)) = \bar{\triangle}$. For if $\hat{q}(M(A)) \neq \bar{\triangle}$, then there exists $\lambda \in \triangle$ such that $(q-\lambda)A = A$ because $\hat{q}(M(A)) \supseteq \partial\triangle$. On the other hand, $qA \subsetneq A$ and so $(q-\lambda)A \subsetneq A$ for any $\lambda \in \triangle$ by the index theory for Fredholm operators. This contradiction shows that $\hat{q}(M(A)) = \bar{\triangle}$. By [6, Chapter VI, Theorem 6.3], $\hat{q}^{-1}(\triangle) \subseteq M(A)$ can be given the structure of a (possibly disconnected) open Riemann surface with discrete point identifications, and hence the Gelfand transform of functions in A become analytic on $\hat{q}^{-1}(\triangle)$. Since $\dim(A/qA) = \dim\ker T_q^*$, by [6, Chapter VI, Theorem 6.3] $\hat{q}$ is an n-sheeted analytic cover of $\hat{q}^{-1}(\triangle)$ over $\triangle$ where $1 \leq n \leq \dim\ker T_q^* < \infty$. Now we will show that $G(\tau) \subseteq \hat{q}^{-1}(\triangle)$ and then the proof will be compleded. If $|\tau(q)| = 1$, then $\int |q| \, dm = |\int q dm|$ and so q is constant. This contradiction implies that $|\tau(q)| < 1$. Put $\tau(q) = a$ and $Q = (q-a)/(1-\bar{a}q)$, then $Q \in A$ and $|Q| = 1$. and $\tau(Q) = 0$. If $\phi \in G(\tau)$ and $\phi \notin \hat{q}^{-1}(\triangle)$, then $|\phi(q)| = 1$ and so $|\phi(Q)| = 1$. This and [6, Chapter VI, Theorem 2.1] imply that $\phi \notin G(\tau)$ and hence $G(\tau) \subseteq \hat{q}^{-1}(\triangle)$.

(2) Since T_q is an isometry, $\int |q|^2 |f|^2 \, dm = \int |f|^2 \, dm$ for all $f \in A$. By the hypothesis on A, $\int |q|^2 \, udm = \int udm$ for any positive function u in $C(X)$ and so $|q| = 1$ on X.

(3) Since T_q is an isometry, $\int |q|^2 |f|^2 \, dm = \int |f|^2 \, dm$ for all $f \in H^2$. If u is a nonnegative function in L^∞ and $\varepsilon > 0$, then there exists a function f_ε in H^∞ such that $u + \varepsilon = |f_\varepsilon|^2$ (*cf.* [6, Chapter V, Theorem 5.4]). This implies that $|q| = 1$ on X. By Theorem 1, $G(\tau) \neq \{\tau\}$ and so there exists a unimodular function Z such that $H^\infty_\tau = ZH^\infty$ (*cf.* [6, Chapter VI, Theorem 7.1]). By the proof of Theorem 1, $\dim H^\infty/qH^\infty < \infty$ and so $qH^\infty \subset H^\infty_\phi$ for some $\phi \in G(\tau)$ because $H^2 \cap L^\infty = H^\infty$(*cf.* [6, Chapter V, Theorem 6.1]). Put $a_1 = \phi(Z)$, then $H^\infty_\phi = \dfrac{Z - a_1}{1 - \bar{a}_1 Z} H^\infty$ and so

$$q = \frac{Z - a_1}{1 - \bar{a}_1 Z} q_1, \; q_1 \in H^\infty \text{ and } |q_1| = 1 \quad a.e..$$

If $q_1 H^\infty \neq H^\infty$, then by the same argument

$$q_1 = \frac{Z - a_2}{1 - \bar{a}_2 Z} q_2, \; q_2 \in H^\infty \text{ and } |q_2| = 1 \quad a.e.$$

where $|a_2| < 1$. Repeating the above process, we can show that for some finite integer n,

$$q = \prod_{j=1}^{n} \frac{Z - a_j}{1 - \bar{a}_j Z} \text{ and } | a_j | < 1 \quad (j = 1, \cdots, n).$$

By the same argument in (1), $\hat{q}(\partial A) = \partial\Delta$, $\hat{q}(M(A)) = \bar{\Delta}$ and $G(\tau) \subseteq \hat{q}^{-1}(\Delta)$. it is well known (*cf.* [6, Chapter VI, Theorem 7.2]) that the Gefand transforms of functions in A become analytic on $G(\tau)$. Hence $G(\tau) = \hat{q}^{-1}(\Delta)$. It is not difficult to see that $\hat{q}$ is an n-sheeted analytic cover of $G(\tau)$ over Δ where $1 \leq n = \dim \ker T_q^* < \infty$.

Even if A is a Dirichlet algebra and $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$, A need not be isometrically isomorphic to the disc algebra $\mathcal{A} \mid \partial\Delta$. In fact, suppose $A = H^\infty(\partial\Delta) \cap QC$, $C(X) = QC$ and m is the normalized Lebesgue measure on $\partial\Delta$, where $H^\infty(\partial\Delta)$ is the weak star closure of the disc algebra $\mathcal{A} \mid \partial\Delta$ in $L^\infty(m)$ and $QC = (H^\infty(\partial\Delta) + C(\partial\Delta)) \cap (\overline{H^\infty(\partial\Delta) + C(\partial\Delta)})$. Then $A \underset{\neq}{\supset} \mathcal{A} \mid \partial\Delta$, A is a Dirichlet algebra, A is not generated by inner functions in $H^\infty(\partial\Delta)$ and $\mathcal{C}(C(X)) = \mathcal{C}(C(\partial\Delta)) = \mathcal{LC}(H^2)$.

Let Γ be a discrete abelian group, let G be its dual group, and let Γ_+ be a subsemigroup of Γ such that $\Gamma_+ - \Gamma_+ = \Gamma$. Put

$$A = \{f \in C(G) \ ; \ \int \phi\bar{\chi} dm = 0 \quad (\chi \notin \Gamma_+)\}$$

and let m be the Haar measure on G. Then A is a uniform algebra on $X = G$. *G.J.* Murphy [8] showed that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ if and only if $\Gamma = Z$ where Z is a set of all integers. (1) of the following theorem shows this.

Theorem 3. Suppose A is a nontrivial uniform algebra generated by inner functions $\{q_\alpha\}_{\alpha\in\Lambda}$.

(1) $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$ if and only if $\dim(H^2 \ominus q_\alpha H^2) < \infty$ for all $\alpha \in \Lambda$.

(2) When $A = H^2 \cap C(X)$, $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$ if and only if $\dim(A/q_\alpha A) < \infty$ for all $\alpha \in \Lambda$.

Proof. (1) By the proof of Theorem 1, if $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$ then $\dim(H^2 \ominus q_\alpha H^2) < \infty$ for all $\alpha \in \Lambda$. Conversely if $\dim(H^2 \ominus q_\alpha H^2) < \infty$ for all $\alpha \in \Lambda$, then $T_{q_\alpha}^* T_{q_\alpha} - T_{q_\alpha} T_{q_\alpha}^* = 1 - T_{q_\alpha} T_{q_\alpha}^*$ is in $\mathcal{LC}(H^2)$. Our situation is a little different from that in [9] but the proof of [9, Corollary 2.3] shows that $\mathcal{T}(C(X))$ is irreducible. By [5, Theorem 5.39], $\mathcal{LC}(H^2) \subseteq \mathcal{C}(C(X))$. Let ϕ be arbitrary function in $C(X)$ and $\alpha, \beta \in \Lambda$, then

$$T_\phi T_{\bar{q}_\beta q_\alpha} - T_{\bar{q}_\beta q_\alpha} T_\phi = (T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi) T_{q_\alpha} + (T_{\bar{q}_\beta \phi} T_{q_\alpha} - T_{q_\alpha} T_{\bar{q}_\beta \phi}) + (T_{q_\alpha} T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_{q_\alpha}) T_\phi$$

because $T_{\bar{q}_\beta} T_\phi T_{q_\alpha} = T_{\bar{q}_\beta \phi} T_{q_\alpha}$ and $T_{q_\alpha} T_{\bar{q}_\beta \phi} = T_{q_\alpha} T_{\bar{q}_\beta} T_\phi$. Since $(T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi) T_{q_\beta} = 0$ and $\dim \ker T_{q_\beta}^* < \infty$ for arbitrary $\beta \in \Lambda$, both $T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi$, $T_{\bar{q}_\beta \phi} T_{q_\alpha} - T_{q_\alpha} T_{\bar{q}_\beta \phi}$ and $T_{q_\alpha} T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_{q_\alpha}$ are compact. Thus $T_\phi T_{\bar{q}_\beta q_\alpha} - T_{\bar{q}_\beta q_\alpha} T_\phi$ is compact. Since A is generated by $\{q_\alpha\}_{\alpha\in\Lambda}$, $C(X)$ is generated by $\{q_\alpha, \bar{q}_\alpha\}_{\alpha\in\Lambda}$. Therefore for any ϕ, ψ in $C(X)$ $T_\phi T_\psi - T_\psi T_\phi$ is compact and so $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$.

(2) As in the proof Theorem 1, using $H^2 \cap C(X) = A$, we can show that $\dim(H^2 \ominus q_\alpha H^2) = \dim(A/q_\alpha A)$. Now (2) follows from (1).

§3. Essential isometric Toeplitz operator

The typical example in this section is $A = \mathcal{A}$. Then T_z is an essential isometry and bounded below.

Theorem 4. If $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A such that T_q is an essential isometry and does not have a dense range, then $G(\tau) \neq \{\tau\}$.

Proof. By the hypothesis, $1 - T_q^* T_q \in \mathcal{LC}(H^2)$ and $T_q^* T_q - T_q T_q^* \in \mathcal{LC}(H^2)$. Hence $1 - T_q T_q^*$ belongs to $\mathcal{LC}(H^2)$ and range $(1 - T_q T_q^*) \supseteq \ker T_q^*$. This implies that $\dim \ker T_q^* < \infty$ and so $\dim(H^2 \ominus [qH^2]) < \infty$, because the range of a compact operator contains no closed infinite-dimensional subspace. Now the proof of Theorem 1 shows that $G(\tau) \neq \{\tau\}$.

Theorem 5. Suppose $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A_τ such that T_q is an essential isometry and bounded below. If $H^2 \cap C(X) = A$ and $(qH^2) \cap C(X) = qA$, then $1 \leq \dim A/qA < \infty$. Hence if $M(A)$ has no isorated point and W is the connected component of $\mathbf{C} \backslash \hat{q}(\partial A)$ which contains 0, then $\hat{q}^{-1}(W) \subseteq M(A)$ can be given the structure of a (possiblely disconnected) open Riemann surface, with discrete point identifications, so that the Gelfand transforms of functions in A become analytic on $\hat{q}^{-1}(W)$. $\hat{q}$ is an n-sheeted analytic cover of $\hat{q}^{-1}(W)$ over W where $1 \leq n \leq \dim \ker T_q^* < \infty$.

Proof. Since T_q is bounded below, qH^2 is closed in H^2. By Theorem 4 and the proof, $\dim(H^2 \ominus qH^2) < \infty$ and by the proof of Theorem 1, $\dim\{H^2 \cap C(X)/(qH^2) \cap C(X)\} < \infty$. By the hypothesis, $1 \leq \dim(A/qA) < \infty$. The second statement is just [5, Chapter VI, Theorem 6.3]

If $1 - T_q^* T_q$ is finite rank or $\ker(1 - T_q^* T_q)$ is nontrivial, under some conditions on q and A, T_q is just an isometry. If T_q is an essential isometry and m is a unique representing measure for τ, then T_q is just an isometry. In genertal, when each compact Toepliz operator is zero, if T_q is an essential isometry then T_q is just isometry. In Theorem 5, if $|q| = 1$ on ∂A, then (1) in Theorem 2 is true but T_q may not be an isometry when $\partial A \neq X$.

§4. Hankel operator

In the previous sections, under the exstence of some special function q in A, we could find some analytic structure on $G(\tau)$ or some subset of $M(A)$. In this section, we do not put on such a hypothesis. If A is a polydisc algebra, that is, the set of all continuous functions on the closure $\overline{\triangle^n}$ of the unit polydisk $\triangle^n$ whose restriction to $\triangle^n$ is holomorplic there, then by Theorem 3 $\mathcal{C}(C(X)) \neq \mathcal{LC}(H^2)$. (1) of Theorem 6 also shows it because each compact Hankel operator is zero. (see [4]).

Theorem 6. Suppose $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$.

(1) There exists at least one nonzero compact Hankel operator whose symbol is in $C(X)$.

(2) If m is a unique representing measure for τ, then $G(\tau) \neq \{\tau\}$ and hence there is a one-to-one continuous map σ of $\triangle$ onto $G(\tau)$, such that $\hat{f}\circ\sigma$ is analytic on $\triangle$ for all f in A.

Proof. (1) Suppose each compact Hankel operator is zero. For any $f \in A$,

$$H_{\bar{f}}^* H_{\bar{f}} = T_f^* T_f - T_f T_f^*$$

is compact because $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. Hence $H_{\bar{f}}$ is compact and so by the hypothesis, $H_{\bar{f}} = 0$. For any $g \in H^2$,

$$\bar{f}g = (M_{\bar{f}} - H_{\bar{f}})g = T_{\bar{f}}(g)$$

where $M_{\bar{f}}$ is multiplication operator by $\bar{f}$. Thus both f and $\bar{f}$ belong to H^2 and so

$$\int \mid f - \tau(f) \mid^2 dm = \int (f - \tau(f))dm \int (\bar{f} - \tau(\bar{f}))dm = 0$$

Hence $f = \tau(f)$ *a.e.* and we get a contradiction. (2) If m is a unique representing measure and $G(\tau) = \{\tau\}$, then it is known [3] that each compact Hankel operator is zero.

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