



|                  |   |
|------------------|---|
| Title            | The commutator ideal in Toeplitz algebras for uniform algebras and the analytic structure |
| Author(s)        | Nakazi, T.; Sawada, H.  |
| Citation         | Hokkaido University Preprint Series in Mathematics, 371, 1-9                              |
| Issue Date       | 1997-2-1  |
| DOI              | 10.14943/83517  |
| Doc URL          | <a href="http://hdl.handle.net/2115/69121">http://hdl.handle.net/2115/69121</a>           |
| Type             | bulletin (article)  |
| File Information | pre371.pdf  |



[Instructions for use](#)

**The Commutator Ideal In  
Toeplitz Algebras For Uniform Algebras  
And The Analytic Structure**

**T. Nakazi and H. Sawada**

Series #371. February 1997

HOKKAIDO UNIVERSITY  
PREPRINT SERIES IN MATHEMATICS

- #347 T. Nakazi and T. Yamamoto, Weighted Norm Inequalities For Some Singular Integral Operators, 17 pages. 1996.
- #348 Y. Ito and I. Nakamura, Hilbert schemes and simple singularities  $A_n$  and  $D_n$ , 22 pages. 1996
- #349 R. Agemi and K. Yokoyama, The null condition and global existence of solutions to systems of wave equations with different speeds, 42 pages. 1996
- #350 F. Hiroshima, Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field, 21 pages. 1996.
- #351 T. Nakazi and Y. Watatani, Invariant subspace theorems for subdiagonal algebras, 22 pages. 1996.
- #352 Y. Nishiura and H. Suzuki, Nonexistence of stable turing patterns with smooth limiting interfacial configurations in higher dimensional spaces, 21 pages. 1996.
- #353 Y.-G. Chen, Y. Giga and K. Sato, On instant extinction for very fast diffusion equations, 9 pages. 1996.
- #354 A. Gyoja and H. Yamashita, Associated variety, Kostant-Sekiguchi correspondence, and locally free  $U(n)$ -action on Harish-Chandra modules, 25 pages. 1996.
- #355 G. Ishikawa, Topology of plane trigonometric curves and the strangeness of plane curves derived from real pseudo-line arrangements, 18 pages. 1996.
- #356 N.H. Bingham and A. Inoue, The Drasin-Shea-Jordan theorem for Hankel transforms of arbitrarily large order, 13 pages. 1996.
- #357 S. Izumiya, Singularities of solutions for first order partial differential equations, 20 pages. 1996.
- #358 N. Hayashi, P.I. Naumkin and T. Ozawa, Scattering theory for the Hartree equation, 14 pages. 1996.
- #359 I. Tsuda and K. Tadaki, A logic-based dynamical theory for a genesis of biological threshold, 49 pages. 1996.
- #360 I. Tsuda and A. Yamaguchi, Singular-continuous nowhere-differentiable attractors in neural systems, 40 pages. 1996.
- #361 M. Nakamura and T. Ozawa, Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces, 17 pages. 1996.
- #362 I. Nakamura, Hilbert schemes and simple singularities  $E_6$ ,  $E_7$  and  $E_8$ , 21 pages. 1996.
- #363 T. Mikami, Equivalent conditions on the central limit theorem for a sequence of probability measures on  $R$ , 7 pages. 1996.
- #364 S. Izumiya and T. Sano, Generic affine differential geometry of space curves, 23 pages. 1996.
- #365 T. Tsukada, Stability of reticular optical caustics, 12 pages. 1996.
- #366 A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, 40 pages. 1996.
- #367 A. Arai, A class of representations of the  $*$ -algebra of the canonical commutation relations over a Hilbert space and instability of embedded eigenvalues in quantum field models, 12 pages. 1996.
- #368 K. Ito, BV-solutions of a hyperbolic-elliptic system for a radiating gas, 33 pages. 1997.
- #369 M. Nakamura and T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, 20 pages. 1997.
- #370 N.H. Bingham and A. Inoue, An Abel-Tauber theorem for Hankel transforms, 8 pages. 1997.

The Commutator Ideal In Toeplitz Algebras For Uniform Algebras And The Analytic Structure

by

Takahiko Nakazi \*

and

Hironari Sawada

Department of Mathematics

Hokkaido University

Sapporo 060, Japan

and

Higashiyama Gakuen

Sakyoku Eikando Cho

Kyoto 606, Japan

AMS Subject Classification(1991) ; 46 J 15, 47 B 35

Key Words And Phrases ; Toeplitz algebra, commutator ideal, compact operator, uniform algebra, Gleason part.

\* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

Abstract. Let  $A$  be a uniform algebra and  $H^2$  the abstract Hardy space defined by  $A$ . Under some conditions on  $A$ , we show that its maximal ideal space has some analytic structure when the commutator ideal of a Toeplitz algebra on  $H^2$  is just a set of all compact operators on  $H^2$ .

## §1. Introduction

Let  $X$  be a compact Hausdorff space, let  $C(X)$  be the algebra of complex-valued continuous functions on  $X$ , and let  $A$  be a uniform algebra on  $X$ . Fix a nonzero complex homomorphism  $\tau$  on  $A$  and a representing measure  $m$  for  $\tau$  whose support is  $X$ . The abstract Hardy space  $H^p = H^p(m)$ ,  $1 \leq p \leq \infty$ , determined by  $A$  is defined to be the closure of  $A$  in  $L^p = L^p(m)$  when  $p$  is finite and to be the weak star closure of  $A$  in  $L^\infty = L^\infty(m)$  when  $p = \infty$ .  $M(A)$  denotes the maximal ideal space of  $A$  and  $X$  is regarded as a closed subset of  $M(A)$ . The Shilov boundary  $\partial A$  of  $A$  then becomes a closed subset of  $X$ . The  $\tau$  fixed above belongs to  $M(A)$ . We set  $A_\tau = \{f \in A; \tau(f) = 0\}$  and  $H_\tau^p = \{f \in H^p; \int f dm = 0\}$ .

For  $\phi$  in  $L^\infty$ , the Toeplitz operator  $T_\phi$  is defined by

$$T_\phi(f) = P(\phi f) \quad (f \in H^2)$$

where  $P$  is the orthogonal projection onto  $H^2$ , and the Hankel operator  $H_\phi$  is defined by

$$H_\phi(f) = (I - P)(\phi f) \quad (f \in H^2).$$

Let  $\mathcal{T}(C(X))$  be the  $C^*$ -algebra generated by the operators  $\{T_\phi; \phi \in C(X)\}$  and  $\mathcal{C}(C(X))$  be the commutator ideal of  $\mathcal{T}(C(X))$ .  $\mathcal{LC}(H^2)$  denotes the set of all compact operators on  $H^2$ . In this paper, we study the analytic structure on  $M(A)$  when  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ . Let  $\mathcal{A}$  be the set of all continuous functions on the closed unit disc  $\partial\Delta \cup \Delta$  which are analytic on the open unit disc  $\Delta$ . If  $A = \mathcal{A}|_{\partial\Delta}$ ,  $X = \partial\Delta$  and  $m$  is the normalized Lebesgue measure on  $\partial\Delta$ , then  $H^2$  is the classical Hardy space. If  $A = \mathcal{A}$ ,  $X = \partial\Delta \cup \Delta$  and  $m$  is the normalized area measure on  $\partial\Delta \cup \Delta$ , then  $H^2$  is called the Bergman space. In both examples, it is known (cf. [5, Theorem 7.23] and [1, Theorem 4.12]) that  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ . Let  $z$  be a coordinate function in  $\mathbb{C}$ . If  $A = \mathcal{A}|_{\partial\Delta}$ , then  $T_z$  is an isometry and if  $A = \mathcal{A}$ , then  $T_z$  is an essential isometry. In this paper, these two uniform algebras are typical examples such that  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ .

We call  $A$  a uniform algebra with generalized approximation in modulus property when the set of all finite sums  $\sum_i |f_i|^2$ ,  $f_i \in A$ , is dense in the cone of positive and continuous functions on  $X$  (cf. [7]). A lot of uniform algebras have such modulus property. If a function  $q$  in  $A$  is unimodular on  $X$ , we call  $q$  an inner function.  $T$  is called isometry when  $T^*T = 1$  and  $T$  is called essential isometry when  $1 - T^*T$  is compact.  $G(\tau)$  denotes the Gleason part of  $\tau$ , that is,  $G(\tau) = \{\sigma \in M(A); \|\tau - \sigma\| < 2\}$ .  $[E]_2$  denotes the closure of the subset of  $E$  in  $L^2$ .

In this paper, we assume that  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ . In §2, if there exists a nonconstant function  $q$  in  $A$  such that  $T_q$  is an isometry and not a unitary, we show that  $G(\tau)$  is nontrivial and under some conditions on  $A$  and  $q$ , it is contained in a subset of  $M(A)$  given an analytic structure which is determined by  $q$ . In §3, we consider the same problem when  $T_q$  is an essential isometry. In §4, we study it without the existence of  $T_q$ .

## §2. Isometric Toeplitz operator

The typical example in this section is  $A = \mathcal{A} | \partial\Delta$  and then  $T_z$  is an isometry.

**Theorem 1.** If  $\mathcal{C}(C(X)) = \mathcal{L}\mathcal{C}(H^2)$  and there exists a function  $q$  in  $A$  such that  $T_q$  is an isometry and not a unitary, then  $G(\tau) \neq \{\tau\}$ .

*Proof.* Since  $T_q$  is an isometry,  $T_q^*T_q$  is the identity operator and  $1 - T_qT_q^*$  is an orthogonal projection from  $H^2$  onto  $H^2 \ominus qH^2$ . Because  $\mathcal{C}(C(X)) = \mathcal{L}\mathcal{C}(H^2)$  and the range of a compact operator contains no closed infinite-dimensional subspace,  $1 - T_qT_q^*$  is a compact operator and  $\dim(H^2 \ominus qH^2) < \infty$ . Let  $B = H^2 \cap L^\infty$  and  $D = (qH^2) \cap L^\infty$ . If  $\dim(H^2 \ominus qH^2) = n$ , then  $\dim(B/D) \leq n$ . In fact, if  $\dim(B/D) > n$ , there exists  $\{f_j\}_{j=1}^{n+1}$  in  $B$  such that  $\{f_j + D\}_{j=1}^{n+1}$  is linearly independent in  $B/D$ . By hypothesis,  $\{f_j + qH^2\}_{j=1}^{n+1}$  is not linearly independent in  $H^2/qH^2$  and so there exists  $\{\alpha_j\}_{j=1}^{n+1}$  in  $\mathbb{C}$  with  $(\alpha_1, \dots, \alpha_{n+1}) \neq (0, \dots, 0)$  such that  $\alpha_1 f_1 + \dots + \alpha_{n+1} f_{n+1}$  belongs to  $(qH^2) \cap L^\infty = D$ . This contradiction implies that  $\dim(B/D) \leq n$ . If  $D = B$ , then  $1 \in D$  and so  $qH^2 = H^2$ . This contradicts that  $T_q$  is not a unitary. This contradiction shows that  $D$  is a weak star closed proper ideal of  $B$ . Since  $\dim(B/D) < \infty$ , there exists  $\phi$  in  $M(B)$  such that  $B_\phi = \{f \in B ; \phi(f) = 0\} \supseteq D$  and  $B_\phi$  is weak star closed. Since  $1 \notin B_\phi$ , there exists a function  $h$  in  $L^1$  such that

$$a = \int h dm \neq 0 \text{ and } \int hg dm = 0 \quad (g \in B_\phi).$$

Hence  $\phi(f) = \int f h dm / a$  for all  $f$  in  $B$ . Now there exists a (positive) representing measure for  $\phi$  that is absolutely continuous with respect to  $h dm / a$  (cf. [2, Theorem 2.1.1]). Thus  $\phi | A$  belongs to  $G(\tau)$ . If  $\tau(q) \neq 0$ , then  $\phi | A \neq \tau$ . and so  $G(\tau) \neq \{\tau\}$ . If  $\tau(q) = 0$ , put  $Q = (q + a)/(1 + \bar{a}q)$  for some nonzero constant  $a$  with  $|a| < \|q\|_\infty$ , then  $Q$  belongs to  $A$  and  $T_Q$  is an isometry. Hence the argument above is valid for the new nonconstant function  $Q$  with  $\tau(Q) \neq 0$ . This completes the proof.

**Theorem 2.** Suppose  $\mathcal{C}(C(X)) = \mathcal{L}\mathcal{C}(H^2)$  and there exists a function  $q$  in  $A$  such that  $T_q$  is an isometry and not a unitary.

(1) Suppose  $H^2 \cap C(X) = A$  and  $M(A)$  has no isolated point. If  $|q| = 1$  on  $X$ , then

$$\hat{q}(\partial A) = \partial\Delta, \hat{q}(M(A)) = \bar{\Delta} \text{ and } G(\tau) \subseteq \hat{q}^{-1}(\Delta).$$

Moreover  $\hat{q}^{-1}(\Delta) \subseteq M(A)$  can be given the structure of a (possibly disconnected) open Riemann surface with discrete point identifications, and hence the Gelfand transforms of functions in  $A$  become analytic on  $\hat{q}^{-1}(\Delta)$ .  $\hat{q}$  is an  $n$ -sheeted analytic cover of  $\hat{q}^{-1}(\Delta)$  over  $\Delta$  where  $1 \leq n \leq \dim \ker T_q^* < \infty$ .

(2) If  $A$  is an algebra with generalized approximation in modulus property, then (1) is valid without assuming that  $|q| = 1$  on  $X$ .

(3) If  $m$  is a unique representing measure for  $\tau$ , then (1) is valid with  $G(\tau) = \hat{q}^{-1}(\Delta)$  without any assumptions in (1).

Proof. (1) Since  $H^2 \cap C(X) = A$ , we can show  $\dim(A/qA) < \infty$  as in the proof of Theorem 1. By hypothesis on  $q$ ,  $\hat{q}(\partial A) \subseteq \partial\Delta$ . We will show that  $\hat{q}(\partial A) = \partial\Delta$ . If  $\hat{q}(\partial A) \neq \partial\Delta$ , then  $W = \mathbf{C} \setminus \hat{q}(\partial A)$  is connected and unbounded. Since  $q$  is a nonconstant function because  $T_q$  is not a unitary,  $qA$  is a proper closed ideal in  $A$  and so there exists  $\phi$  in  $M(A)$  such that  $\phi(q) = 0$ . This implies  $0 \in W$  and hence  $(q - \lambda)A \subsetneq A$  for any  $\lambda \in W$  by the index theory for Fredholm operators (cf. [6, p157]). Since  $W$  is unbounded, there exists  $\lambda \in W$  such that  $(q - \lambda)^{-1} \in A$  and so  $(q - \lambda)A = A$ . This contradiction shows that  $\hat{q}(\partial A) = \partial\Delta$  and  $\mathbf{C} \setminus \hat{q}(\partial A) = \Delta \cup \bar{\Delta}^c$ . Now  $\hat{q}(M(A)) = \bar{\Delta}$ . For if  $\hat{q}(M(A)) \neq \bar{\Delta}$ , then there exists  $\lambda \in \Delta$  such that  $(q - \lambda)A = A$  because  $\hat{q}(M(A)) \supseteq \partial\Delta$ . On the other hand,  $qA \subsetneq A$  and so  $(q - \lambda)A \subsetneq A$  for any  $\lambda \in \Delta$  by the index theory for Fredholm operators.

This contradiction shows that  $\hat{q}(M(A)) = \bar{\Delta}$ . By [6, Chapter VI, Theorem 6.3],  $\hat{q}^{-1}(\Delta) \subseteq M(A)$  can be given the structure of a (possibly disconnected) open Riemann surface with discrete point identifications, and hence the Gelfand transform of functions in  $A$  become analytic on  $\hat{q}^{-1}(\Delta)$ . Since  $\dim(A/qA) = \dim \ker T_q^*$ , by [6, Chapter VI, Theorem 6.3]  $\hat{q}$  is an  $n$ -sheeted analytic cover of  $\hat{q}^{-1}(\Delta)$  over  $\Delta$  where  $1 \leq n \leq \dim \ker T_q^* < \infty$ . Now we will show that  $G(\tau) \subseteq \hat{q}^{-1}(\Delta)$  and then the proof will be completed. If  $|\tau(q)| = 1$ , then  $\int |q| dm = \int |q| dm$  and so  $q$  is constant. This contradiction implies that  $|\tau(q)| < 1$ . Put  $\tau(q) = a$  and  $Q = (q - a)/(1 - \bar{a}q)$ , then  $Q \in A$  and  $|Q| = 1$  and  $\tau(Q) = 0$ . If  $\phi \in G(\tau)$  and  $\phi \notin \hat{q}^{-1}(\Delta)$ , then  $|\phi(q)| = 1$  and so  $|\phi(Q)| = 1$ . This and [6, Chapter VI, Theorem 2.1] imply that  $\phi \notin G(\tau)$  and hence  $G(\tau) \subseteq \hat{q}^{-1}(\Delta)$ .

(2) Since  $T_q$  is an isometry,  $\int |q|^2 |f|^2 dm = \int |f|^2 dm$  for all  $f \in A$ . By the hypothesis on  $A$ ,  $\int |q|^2 u dm = \int u dm$  for any positive function  $u$  in  $C(X)$  and so  $|q| = 1$  on  $X$ .

(3) Since  $T_q$  is an isometry,  $\int |q|^2 |f|^2 dm = \int |f|^2 dm$  for all  $f \in H^2$ . If  $u$  is a nonnegative function in  $L^\infty$  and  $\varepsilon > 0$ , then there exists a function  $f_\varepsilon$  in  $H^\infty$  such that  $u + \varepsilon = |f_\varepsilon|^2$  (cf. [6, Chapter V, Theorem 5.4]). This implies that  $|q| = 1$  on  $X$ . By Theorem 1,  $G(\tau) \neq \{\tau\}$  and so there exists a unimodular function  $Z$  such that  $H_\tau^\infty = ZH^\infty$  (cf. [6, Chapter VI, Theorem 7.1]). By the proof of Theorem 1,  $\dim H^\infty/qH^\infty < \infty$  and so  $qH^\infty \subset H_\phi^\infty$  for some  $\phi \in G(\tau)$  because  $H^2 \cap L^\infty = H^\infty$  (cf. [6, Chapter V, Theorem 6.1]). Put  $a_1 = \phi(Z)$ , then  $H_\phi^\infty = \frac{Z - a_1}{1 - \bar{a}_1 Z} H^\infty$  and so

$$q = \frac{Z - a_1}{1 - \bar{a}_1 Z} q_1, \quad q_1 \in H^\infty \text{ and } |q_1| = 1 \quad a.e..$$

If  $q_1 H^\infty \neq H^\infty$ , then by the same argument

$$q_1 = \frac{Z - a_2}{1 - \bar{a}_2 Z} q_2, \quad q_2 \in H^\infty \text{ and } |q_2| = 1 \quad a.e.$$

where  $|a_2| < 1$ . Repeating the above process, we can show that for some finite integer  $n$ ,



$$q = \prod_{j=1}^n \frac{Z - a_j}{1 - \bar{a}_j Z} \text{ and } |a_j| < 1 \quad (j = 1, \dots, n).$$

By the same argument in (1),  $\hat{q}(\partial A) = \partial \Delta$ ,  $\hat{q}(M(A)) = \bar{\Delta}$  and  $G(\tau) \subseteq \hat{q}^{-1}(\Delta)$ . It is well known (cf. [6, Chapter VI, Theorem 7.2]) that the Gelfand transforms of functions in  $A$  become analytic on  $G(\tau)$ . Hence  $G(\tau) = \hat{q}^{-1}(\Delta)$ . It is not difficult to see that  $\hat{q}$  is an  $n$ -sheeted analytic cover of  $G(\tau)$  over  $\Delta$  where  $1 \leq n = \dim \ker T_q^* < \infty$ .

Even if  $A$  is a Dirichlet algebra and  $\mathcal{C}(C(X)) = \mathcal{L}\mathcal{C}(H^2)$ ,  $A$  need not be isometrically isomorphic to the disc algebra  $\mathcal{A} | \partial \Delta$ . In fact, suppose  $A = H^\infty(\partial \Delta) \cap QC$ ,  $C(X) = QC$  and  $m$  is the normalized Lebesgue measure on  $\partial \Delta$ , where  $H^\infty(\partial \Delta)$  is the weak star closure of the disc algebra  $\mathcal{A} | \partial \Delta$  in  $L^\infty(m)$  and  $QC = (H^\infty(\partial \Delta) + C(\partial \Delta)) \cap (\overline{H^\infty(\partial \Delta) + C(\partial \Delta)})$ . Then  $A \supsetneq \mathcal{A} | \partial \Delta$ ,  $A$  is a Dirichlet algebra,  $A$  is not generated by inner functions in  $H^\infty(\partial \Delta)$  and  $\mathcal{C}(C(X)) = \mathcal{C}(C(\partial \Delta)) = \mathcal{L}\mathcal{C}(H^2)$ .

Let  $\Gamma$  be a discrete abelian group, let  $G$  be its dual group, and let  $\Gamma_+$  be a subsemigroup of  $\Gamma$  such that  $\Gamma_+ - \Gamma_+ = \Gamma$ . Put

$$A = \{f \in C(G) ; \int \phi \bar{\chi} dm = 0 \quad (\chi \notin \Gamma_+)\}$$

and let  $m$  be the Haar measure on  $G$ . Then  $A$  is a uniform algebra on  $X = G$ . G.J. Murphy [8] showed that  $\mathcal{C}(C(X)) = \mathcal{L}\mathcal{C}(H^2)$  if and only if  $\Gamma = Z$  where  $Z$  is a set of all integers. (1) of the following theorem shows this.

**Theorem 3.** Suppose  $A$  is a nontrivial uniform algebra generated by inner functions  $\{q_\alpha\}_{\alpha \in \Lambda}$ .

(1)  $\mathcal{L}\mathcal{C}(H^2) = \mathcal{C}(C(X))$  if and only if  $\dim(H^2 \ominus q_\alpha H^2) < \infty$  for all  $\alpha \in \Lambda$ .

(2) When  $A = H^2 \cap C(X)$ ,  $\mathcal{L}\mathcal{C}(H^2) = \mathcal{C}(C(X))$  if and only if  $\dim(A/q_\alpha A) < \infty$  for all  $\alpha \in \Lambda$ .

**Proof.** (1) By the proof of Theorem 1, if  $\mathcal{L}\mathcal{C}(H^2) = \mathcal{C}(C(X))$  then  $\dim(H^2 \ominus q_\alpha H^2) < \infty$  for all  $\alpha \in \Lambda$ . Conversely if  $\dim(H^2 \ominus q_\alpha H^2) < \infty$  for all  $\alpha \in \Lambda$ , then  $T_{q_\alpha}^* T_{q_\alpha} - T_{q_\alpha} T_{q_\alpha}^* = 1 - T_{q_\alpha} T_{q_\alpha}^*$  is in  $\mathcal{L}\mathcal{C}(H^2)$ . Our situation is a little different from that in [9] but the proof of [9, Corollary 2.3] shows that  $\mathcal{T}(C(X))$  is irreducible. By [5, Theorem 5.39],  $\mathcal{L}\mathcal{C}(H^2) \subseteq \mathcal{C}(C(X))$ . Let  $\phi$  be arbitrary function in  $C(X)$  and  $\alpha, \beta \in \Lambda$ , then

$$T_\phi T_{\bar{q}_\beta q_\alpha} - T_{\bar{q}_\beta q_\alpha} T_\phi = (T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi) T_{q_\alpha} + (T_{\bar{q}_\beta \phi} T_{q_\alpha} - T_{q_\alpha} T_{\bar{q}_\beta \phi}) + (T_{q_\alpha} T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_{q_\alpha}) T_\phi$$

because  $T_{\bar{q}_\beta} T_\phi T_{q_\alpha} = T_{\bar{q}_\beta \phi} T_{q_\alpha}$  and  $T_{q_\alpha} T_{\bar{q}_\beta \phi} = T_{q_\alpha} T_{\bar{q}_\beta} T_\phi$ . Since  $(T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi) T_{q_\alpha} = 0$  and  $\dim \ker T_{q_\beta}^* < \infty$  for arbitrary  $\beta \in \Lambda$ , both  $T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi$ ,  $T_{\bar{q}_\beta \phi} T_{q_\alpha} - T_{q_\alpha} T_{\bar{q}_\beta \phi}$  and  $T_{q_\alpha} T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_{q_\alpha}$  are compact. Thus  $T_\phi T_{\bar{q}_\beta q_\alpha} - T_{\bar{q}_\beta q_\alpha} T_\phi$  is compact. Since  $A$  is generated by  $\{q_\alpha\}_{\alpha \in \Lambda}$ ,  $C(X)$  is generated by  $\{q_\alpha, \bar{q}_\alpha\}_{\alpha \in \Lambda}$ . Therefore for any  $\phi, \psi$  in  $C(X)$   $T_\phi T_\psi - T_\psi T_\phi$  is compact and so  $\mathcal{L}\mathcal{C}(H^2) = \mathcal{C}(C(X))$ .

(2) As in the proof Theorem 1, using  $H^2 \cap C(X) = A$ , we can show that  $\dim(H^2 \ominus q_\alpha H^2) = \dim(A/q_\alpha A)$ . Now (2) follows from (1).

### §3. Essential isometric Toeplitz operator

The typical example in this section is  $A = \mathcal{A}$ . Then  $T_z$  is an essential isometry and bounded below.

Theorem 4. If  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$  and there exists a function  $q$  in  $A$  such that  $T_q$  is an essential isometry and does not have a dense range, then  $G(\tau) \neq \{\tau\}$ .

Proof. By the hypothesis,  $1 - T_q^* T_q \in \mathcal{LC}(H^2)$  and  $T_q^* T_q - T_q T_q^* \in \mathcal{LC}(H^2)$ . Hence  $1 - T_q T_q^*$  belongs to  $\mathcal{LC}(H^2)$  and  $\text{range}(1 - T_q T_q^*) \supseteq \ker T_q^*$ . This implies that  $\dim \ker T_q^* < \infty$  and so  $\dim(H^2 \ominus [qH^2]) < \infty$ , because the range of a compact operator contains no closed infinite-dimensional subspace. Now the proof of Theorem 1 shows that  $G(\tau) \neq \{\tau\}$ .

Theorem 5. Suppose  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$  and there exists a function  $q$  in  $A_\tau$  such that  $T_q$  is an essential isometry and bounded below. If  $H^2 \cap C(X) = A$  and  $(qH^2) \cap C(X) = qA$ , then  $1 \leq \dim A/qA < \infty$ . Hence if  $M(A)$  has no isolated point and  $W$  is the connected component of  $\mathbb{C} \setminus \hat{q}(\partial A)$  which contains 0, then  $\hat{q}^{-1}(W) \subseteq M(A)$  can be given the structure of a (possibly disconnected) open Riemann surface, with discrete point identifications, so that the Gelfand transforms of functions in  $A$  become analytic on  $\hat{q}^{-1}(W)$ .  $\hat{q}$  is an  $n$ -sheeted analytic cover of  $\hat{q}^{-1}(W)$  over  $W$  where  $1 \leq n \leq \dim \ker T_q^* < \infty$ .

Proof. Since  $T_q$  is bounded below,  $qH^2$  is closed in  $H^2$ . By Theorem 4 and the proof,  $\dim(H^2 \ominus qH^2) < \infty$  and by the proof of Theorem 1,  $\dim\{H^2 \cap C(X)/(qH^2) \cap C(X)\} < \infty$ . By the hypothesis,  $1 \leq \dim(A/qA) < \infty$ . The second statement is just [5, Chapter VI, Theorem 6.3]

If  $1 - T_q^* T_q$  is finite rank or  $\ker(1 - T_q^* T_q)$  is nontrivial, under some conditions on  $q$  and  $A$ ,  $T_q$  is just an isometry. If  $T_q$  is an essential isometry and  $m$  is a unique representing measure for  $\tau$ , then  $T_q$  is just an isometry. In general, when each compact Toeplitz operator is zero, if  $T_q$  is an essential isometry then  $T_q$  is just isometry. In Theorem 5, if  $|q| = 1$  on  $\partial A$ , then (1) in Theorem 2 is true but  $T_q$  may not be an isometry when  $\partial A \neq X$ .

#### §4. Hankel operator

In the previous sections, under the existence of some special function  $q$  in  $A$ , we could find some analytic structure on  $G(\tau)$  or some subset of  $M(A)$ . In this section, we do not put on such a hypothesis. If  $A$  is a polydisc algebra, that is, the set of all continuous functions on the closure  $\overline{\Delta^n}$  of the unit polydisk  $\Delta^n$  whose restriction to  $\Delta^n$  is holomorphic there, then by Theorem 3  $\mathcal{C}(C(X)) \neq \mathcal{LC}(H^2)$ . (1) of Theorem 6 also shows it because each compact Hankel operator is zero. (see [4]).

Theorem 6. Suppose  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ .

(1) There exists at least one nonzero compact Hankel operator whose symbol is in  $C(X)$ .

(2) If  $m$  is a unique representing measure for  $\tau$ , then  $G(\tau) \neq \{\tau\}$  and hence there is a one-to-one continuous map  $\sigma$  of  $\Delta$  onto  $G(\tau)$ , such that  $\bar{f} \circ \sigma$  is analytic on  $\Delta$  for all  $f$  in  $A$ .

Proof. (1) Suppose each compact Hankel operator is zero. For any  $f \in A$ ,

$$H_{\bar{f}}^* H_{\bar{f}} = T_{\bar{f}}^* T_f - T_f T_{\bar{f}}^*$$

is compact because  $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ . Hence  $H_{\bar{f}}$  is compact and so by the hypothesis,  $H_{\bar{f}} = 0$ . For any  $g \in H^2$ ,

$$\bar{f}g = (M_{\bar{f}} - H_{\bar{f}})g = T_{\bar{f}}(g)$$

where  $M_{\bar{f}}$  is multiplication operator by  $\bar{f}$ . Thus both  $f$  and  $\bar{f}$  belong to  $H^2$  and so

$$\int |f - \tau(f)|^2 dm = \int (f - \tau(f)) dm \int (\bar{f} - \tau(\bar{f})) dm = 0$$

Hence  $f = \tau(f)$  a.e. and we get a contradiction. (2) If  $m$  is a unique representing measure and  $G(\tau) = \{\tau\}$ , then it is known [3] that each compact Hankel operator is zero.

#### References

1. S.Axler, Bergman spaces and their operators, Surveys of Some Recent Results in Operator Theory Vol.1 (J.B.Conway and B.B.Morrel, editors), Pitman Research Notes in Math. 171, 1988, 1-50.
2. A.Browder, Introduction to Function Algebras, Benjamin, New York, 1969.
3. R.E.Curto, P.S.Muhly, T.Nakazi and J.Xia, Hankel operators and uniform algebras, Arch. Math. 43(1984), 440-447.

4. R.E.Curto, P.S.Muhly and T.Nakazi, Uniform algebras, Hankel operators and invariant subspaces, *Operator Theory : Advances and Applications*, 17(1986), 109-119.
5. R.G.Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York-London, 1972.
6. T.Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs New Jersey, 1969.
7. J.Janas, Toeplitz spectral inclusion and generalized in modulus property, *Proc. Amer. Math. Soc.* Vol.104, 1988.
8. G.J.Murphy, Ordered groups and Toeplitz algebras, *J.Operator Theory* 18(1987), 303-326.
9. J.Tomiyama and K.Yabuta, Toeplitz operators for uniform algebras, *Tôhoku Math. J.* 30(1978), 117-129.