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Of Weighted Toeplitz Operators**

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# Brown-Halmos Type Theorems Of Weighted Toeplitz Operators

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Abstract. The spectra of the Toeplitz operators on the weighted Hardy space  $H^2(Wd\theta/2\pi)$  and the Hardy space  $H^p(d\theta/2\pi)$ , and the singular integral operators on the Lebesgue space  $L^2(d\theta/2\pi)$  are studied. For example, the theorems of Brown-Halmos type and Hartman-Wintner type are studied.

## §1. Introduction

Let  $m$  be the normalized Lebesgue measure on the unit circle  $T$  and let  $W$  be a non-negative integrable function on  $T$  which does not vanish identically. Suppose  $1 \leq p \leq \infty$ . Let  $L^p(W) = L^p(Wdm)$  and  $L^p(W) = L^p$  when  $W \equiv 1$ . Let  $H^p(W)$  denote the closure in  $L^p(W)$  of the set  $\mathcal{P}$  of all analytic polynomials when  $p \neq \infty$ . We will write  $H^p(W) = H^p$  when  $W \equiv 1$ , and then this is a usual Hardy space.  $H^\infty$  denotes the weak \* closure of  $\mathcal{P}$  in  $L^\infty$ .  $P$  denotes the projection from the set  $\mathcal{C}$  of all trigonometric polynomials to  $\mathcal{P}$ . For  $1 < p < \infty$ ,  $P$  can be extended to a bounded map of  $L^p(W)$  onto  $H^p(W)$  if and only if  $W$  satisfies the condition

$$(A_p) \quad \sup_I \left( \frac{1}{|I|} \int_I W dm \right) \left( \frac{1}{|I|} \int_I W^{-\frac{1}{p-1}} dm \right)^{p-1} < \infty$$

where the supremum is over all intervals  $I$  of  $T$ . This is the well known theorem of Hunt, Muckenhoupt and Wheeden [7], which is a generalization of the theorem of Helson and Szegő [6].

In this paper, we assume that the weight  $W$  satisfies the condition  $(A_p)$ . For  $\phi$  in  $L^\infty$ , the Toeplitz operator  $T_\phi^W$  is defined as a bounded map on  $H^p(W)$  by

$$T_\phi^W f = P(\phi f).$$

For  $\alpha$  and  $\beta$  in  $L^\infty$ , the singular integral operator  $S_{\alpha\beta}^W$  is defined as a bounded map on  $L^p(W)$  by

$$S_{\alpha\beta}^W f = \alpha P f + \beta(I - P)f$$

where  $I$  is an identity operator. If  $W \equiv 1$ , we will write  $T_\phi^W = T_\phi$  and  $S_{\alpha\beta}^W = S_{\alpha\beta}$ . Almost all results in this paper will be essentially shown using the following theorems. They are called the theorems of Widom, Devinatz and Rochberg (cf. [1], [10] and [9]).

**Theorem A.** Suppose  $1 < p < \infty$  and  $W = |h|^p$  satisfies the condition  $(A_p)$ , where  $h$  is an outer function in  $H^p$ . Then the following conditions on  $\phi$  and  $W$  are equivalent.

- (1)  $T_\phi^W$  is an invertible operator on  $H^p(W)$ .
- (2)  $\phi = k(\bar{h}_0/h_0)(h/\bar{h})$ , where  $k$  is an invertible function in  $H^\infty$  and  $h_0$  is an outer function in  $H^p$  with  $|h_0|^p$  satisfying the condition  $(A_p)$ .
- (3)  $\phi = \gamma \exp(U - iV)$ , where  $\gamma$  is constant with  $|\gamma| = 1$ ,  $U$  is a bounded real function,  $V$  is a real function in  $L^1$  and  $W \exp(\frac{p}{2}V)$  satisfies  $(A_p)$ .

**Theorem B.** Suppose  $1 < p < \infty$  and  $W = |h|^p$  satisfies the condition  $(A_p)$ , where  $h$  is an outer function in  $H^p$ .  $S_{\alpha\beta}^W$  is invertible on  $L^p(W)$  if and only if both  $\alpha$  and

$\beta$  are invertible in  $L^\infty$  and  $\alpha/\beta = \gamma \exp(U - i\tilde{V})$ , where  $\gamma$  is constant with  $|\gamma| = 1$ ,  $U$  is a bounded real function,  $V$  is a real function in  $L^1$  and  $W \exp(\frac{p}{2}V)$  satisfies  $(A_p)$ .

Theorem C. Suppose  $T_\phi$  and  $S_{\alpha\beta}$  are on  $L^2$ , where  $\phi, \alpha$  and  $\beta$  are invertible functions in  $L^\infty$ .

(1)  $T_\phi$  is invertible if and only if  $\phi$  has the form :  $\phi = |\phi| e^{it}$  where  $t$  is a real function in  $L^1$  such that

$$\|t\|' = \inf\{\|t - \tilde{s} - a\|_\infty ; s \in L^\infty_R \text{ and } a \in R\} < \pi/2$$

(2)  $S_{\alpha\beta}$  is invertible if and only if  $\alpha/\beta$  has the form :  $\alpha/\beta = |\alpha/\beta| e^{it}$  where  $t$  is the same to that of (1). Hence  $S_{\alpha\beta}$  is invertible if and only if  $T_{\alpha/\beta}$  is invertible.

In this paper, we are interested in  $\sigma(T_\phi^W)$  and  $\sigma(S_{\alpha\beta}^W)$ , that is , the spectra of  $T_\phi^W$  and  $S_{\alpha\beta}^W$ .

For  $\alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$  and  $\beta = \beta_1 + i\beta_2 \in \mathbb{C}$ , put  $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \alpha_2\beta_2$  and  $\theta(\alpha, \beta) = \arccos(\langle \alpha, \beta \rangle / |\alpha| |\beta|)$  for  $\alpha \neq 0$  and  $\beta \neq 0$ . Set

$$\ell_\alpha^+ = \{z \in \mathbb{C} ; \langle z, \alpha \rangle \geq 1\} \text{ and } \ell_\alpha^- = \{z \in \mathbb{C} ; \langle z, \alpha \rangle \leq 1\}$$

and  $\mathcal{E}_{\alpha\beta}^{ij}$  denotes  $\ell_\alpha^i \cap \ell_\beta^j$  where  $i = +$  or  $-$  and  $j = +$  or  $-$ . For each pair  $(\alpha, \beta)$ ,

$$\mathbb{C} = \mathcal{E}_{\alpha\beta}^{++} \cup \mathcal{E}_{\alpha\beta}^{+-} \cup \mathcal{E}_{\alpha\beta}^{-+} \cup \mathcal{E}_{\alpha\beta}^{--}$$

and if  $\ell = -i$  and  $m = -j$ , then

$$\overline{(\mathcal{E}^{\ell m})^c} = \overline{\mathbb{C} \setminus \mathcal{E}^{\ell m}} \supset \mathcal{E}_{\alpha\beta}^{ij}$$

For any bounded subset  $E$  in  $\mathbb{C}$ , there exists a pair  $(\alpha, \beta)$  such that  $\mathcal{E}_{\alpha\beta}^{ij} \supseteq E$  for some  $(i, j)$ . In fact, there are a lot of such pairs  $(\alpha, \beta)$ . Now we can define a set which contains  $E$  and is important in this paper. When  $|\theta(\alpha, \beta)| = \pi - 2t$  and  $0 \leq t < \pi/2$ , put

$$h^t(E) = \cap \{ \overline{(\mathcal{E}_{\alpha\beta}^{\ell m})^c} ; \mathcal{E}_{\alpha\beta}^{ij} \supseteq E \text{ and } \ell = -i, m = -j \}$$

for a subset  $E$  in  $\mathbb{C}$ . If  $t < s$ , then  $h^t(E) \subseteq h^s(E)$ . If  $t = 0$ , then  $h^0(E)$  is the closed convex hull of  $E$ . For example, if  $E = [a, b]$  then

$$h^t(E) = \Delta(c, r) \cap \Delta(\bar{c}, r)$$

$c = \frac{a+b}{2} - i \frac{a-b}{2} \cot 2t$  and  $r = -\frac{a-b}{2 \sin 2t}$  where  $\Delta(c, r)$  denotes the circle of center  $c$  and radius  $r$ . If  $E = \Delta(0, 1)$ , then  $h^t(E) = \Delta(0, 1/\cos t)$ . When  $T_\phi$  is a Toeplitz operator

on  $H^2$ , Brown and Halmos (*cf.* [2, Corollary 7.19]) showed that  $\sigma(T_\phi) \subseteq h^0(\mathcal{R}(\phi))$  where  $\mathcal{R}(\phi)$  is the essential range of  $\phi$ . In this paper we show this type results for Toeplitz operators on  $H^2(W)$  and  $H^p$  and for singular integral operators on  $L^2$ . When  $\phi$  is a real function and  $T_\phi$  is a Toeplitz operator on  $H^2$ , Hartman and Wintner (*cf.* [2, Theorem 7.20]) showed that  $\sigma(T_\phi) = h^0(\mathcal{R}(\phi))$ . In this paper, for real symbols we try to describe the spectra of Toeplitz operators on  $H^2(W)$  and  $H^p$ , and singular integral operators on  $L^2$ . When  $\phi$  is a continuous function,  $\sigma(T_\phi)$  is described using  $R(\phi)$  and the winding number of the curve determined by  $\phi$  (*cf.* [2, Corollary 7.28]). In this case it is known that  $\sigma(T_\phi^W) = \sigma(T_\phi^p) = \sigma(T_\phi)$  for arbitrary weight  $W$  satisfying the condition  $(A_2)$ , and for any  $p$  with  $1 < p < \infty$ ,  $T_\phi^p$  denotes the Toeplitz operator on  $H^p$ . In this paper, we study symbols  $\phi$  such that  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary weight  $W$ .

Now we collect the notations which will be used in this paper.  $R$  is the set of all real numbers and  $X_R$  denotes the set of the real parts of all elements in  $X$ .  $[X]^{cl}$  denotes the closure of  $X$ .  $D$  is the open unit disc.  $C$  is the set of all continuous functions on  $T$ . If  $v$  is a real function in  $L^1$ , then  $\tilde{v}$  denotes the harmonic conjugate function with  $v(0) = 0$ .

## §2. Toeplitz operators on $H^2(W)$ .

In this section, we fix arbitrary weight  $W$  satisfying the condition  $(A_2)$  or equivalently, a Helson-Szegő weight  $W$ . We call  $W$  a Helson-Szegő weight when  $W = e^{u+\tilde{v}}$ ,  $u$  and  $v$  are functions in  $L_R^\infty$  and  $\|v\|_\infty < \pi/2$ . For a Helson-Szegő weight  $W = e^{u+\tilde{v}}$ , put

$$t_W = \|v\|' = \inf\{\|v - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R\}.$$

When  $W \equiv 1$ , (1) of Theorem 1 is a theorem of Brown and Halmos (*cf.* [2, Corollary 7.19]) and (2) and (3) of Theorem 1 is a theorem of Hartman and Wintner (*cf.* [2, Theorem 7.20]). When  $\phi$  is a piecewise continuous function,  $\sigma(T_\phi^W)$  is described when  $W$  is arbitrary weight [10]. The symbol  $\phi$  in Corollary 2 and (3) of Corollary 3 is not necessarily piecewise continuous. It is known that  $\sigma(T_\phi^W) \neq \sigma(T_\phi)$  for some weight  $W$  and some piecewise continuous symbol  $\phi$  (*cf.* [4]). In Theorem 2, we determine weight  $W$  such that  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary symbol  $\phi$  in  $L^\infty$  and study symbols  $\phi$  such that  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary weight  $W$ . (1) of Corollary 3 is related with a particular (corresponding to  $p = 2$ ) case of [3, Theorem 6.1 and Corollary 6.2]. For if  $\log W \in VMO$  then  $\log W = u + \tilde{v}$  for some real functions  $u$  and  $v$  in  $C$ . (2) of Corollary 3 shows the known result [10] such that if  $\phi$  is continuous, then  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary weight  $W$ .

Theorem 1. Let  $\phi$  be a function in  $L^\infty$ , let  $W$  be a Helson-Szegő weight and  $t = t_W$ .



- (1)  $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W) \subseteq h^t(\mathcal{R}(\phi))$ .  
(2) if  $\phi$  is real valued,  $a = \text{ess inf } \phi$  and  $b = \text{ess sup } \phi$ , then

$$\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W) \subseteq \Delta(c, r) \cap \Delta(\bar{c}, r)$$

where  $c = \frac{a+b}{2} - i \frac{a-b}{2} \cos 2t$  and  $r = -\frac{a-b}{2 \sin 2t}$ .

- (3) Suppose  $W = e^{u+\bar{v}}$  and  $\lambda \in [a, b] \cap \mathcal{R}(\phi)^c$  in (2). Then  $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$  and  $\ell = \pi(1 - \chi_E)$  for some measurable set  $E$  in  $T$  with  $0 < m(E) < 1$ .  $\lambda \in \sigma(T_\phi^W)$  if and only if

$$\|\pi\chi_E - v\|' \geq \pi/2.$$

Proof. In (1) and (2), it is well known that  $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W)$ . Suppose  $W = e^{u+\bar{v}}$ ,  $u$  and  $v$  are functions in  $L_R^\infty$  and  $\|v\|_\infty < \pi/2$ , and  $g^2 = e^{u+\bar{v}+i(\bar{u}-v)}$ . Then  $W = |g|^2$ .

- (1) By Theorem A in Introduction, for  $\lambda \in \mathbb{C}$ ,  $T_{\phi-\lambda}^W$  is invertible if and only if

$$T_{\frac{\phi-\lambda}{|\phi-\lambda|} \frac{\bar{g}}{g}} \text{ is invertible.}$$

Suppose  $|\theta(\alpha, \beta)| = \pi - 2t$  and  $\mathcal{R}(\phi) \subseteq \mathcal{E}_{\alpha\beta}^{ij}$ . If  $\lambda \in (\mathcal{E}_{\alpha\beta}^{\ell m})^0$  with  $\ell = -i, m = -j$ , then  $T_\phi^W$  is invertible. In fact, then  $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$  where  $0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon$  a.e. or  $-\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0$  a.e. for some  $\varepsilon > 0$ . Hence  $|s_\lambda - \frac{\pi}{2} + t + \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$  a.e. or  $|s_\lambda + \frac{\pi}{2} - t - \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$  a.e.. Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{g}}{g} = e^{i(s_\lambda + v - \bar{u})}$$

and

$$\|s_\lambda + v - \bar{u}\|' \leq \frac{\pi}{2} - \varepsilon.$$

Thus  $T_{\frac{\phi-\lambda}{|\phi-\lambda|} \frac{\bar{g}}{g}}$  is invertible by Theorem C and hence  $T_{\phi-\lambda}^W$  is invertible. If  $\lambda \notin h^t(\mathcal{R}(\phi))$ , then by definition  $\lambda \in \cup\{(\mathcal{E}_{\alpha\beta}^{\ell m})^0; \mathcal{E}_{\alpha\beta}^{ij} \supseteq \mathcal{R}(\phi) \text{ and } \ell = -i, m = -j\}$  and  $|\theta(\alpha, \beta)| = \pi - 2t$ . By what was just proved,  $\lambda \notin \sigma(T_\phi^W)$ . (2) By (1),  $\sigma(T_\phi^W) \subseteq h^t(\mathcal{R}(\phi)) \subseteq h^t([a, b])$  for  $t = t_W$ . It is elementary to see that  $h^t([a, b]) \subseteq \Delta(c, r) \cap \Delta(\bar{c}, r)$ . (3) The first part is clear. The second statement is a result of Theorems A and C.

Corollary 1. Suppose  $\phi = a\chi_E + b\chi_{E^c}$  where  $a$  and  $b$  are real numbers,  $a \neq b$  and  $0 < m(E) < 1$ . Let  $W = e^{u+\bar{v}}$ , then  $\sigma(T_\phi^W) \supseteq [a, b]$  if and only if  $\|\pi\chi_E - v\|' \geq \pi/2$ .

Corollary 2. Let  $E$  be a measurable set with  $0 < m(E) < 1$ . Suppose  $W$  and  $\phi$  satisfy the following (i) and (ii) :

(i)  $W = e^{u+\tilde{v}}$  where  $u \in L_R^\infty$ ,  $\tilde{v} = d(\chi_E - \chi_{E^c}) + q$ ,  $q \in C_R$  and  $d$  is a constant with  $0 < d < \pi/2$ .

(ii)  $\phi = a\chi_E + b\chi_{E^c}$  where  $a$  and  $b$  are real numbers.

Then  $t_W = d$ ,

$$\sigma(T_\phi^W) = \{\lambda \in \mathbf{C} ; \arg \frac{a-\lambda}{b-\lambda} = \pi - 2d\}.$$

and

$$h^d(\mathcal{R}(\phi)) = \{\lambda \in \mathbf{C} ; \arg \frac{a-\lambda}{b-\lambda} = \pi - 2d \text{ or } -\pi + 2d\}.$$

Proof. Put  $v_0 = \frac{\pi}{2}(\chi_E - \chi_{E^c})$ , then  $h^2 = e^{\tilde{v}_0 - iv_0}$  and  $|h|^2/h^2 = e^{iv_0} = i(\chi_E - \chi_{E^c})$ . If  $\|\chi_E - \chi_{E^c}\|' < 1$ , then  $|h|^2 = e^{\tilde{v}_0}$  is a Helson-Szegő weight and so  $\| |h|^2/h^2 + zH^\infty \| < 1$  (see [3, Chapter IV, Theorem 3.1]). On the other hand,  $\| |h|^2/h^2 + zH^\infty \| = \| i(\chi_E - \chi_{E^c}) + zH^\infty \| = 1$ . This contradiction shows that  $\|\chi_E - \chi_{E^c}\|' = 1$ . Thus

$$\begin{aligned} t_W &= \inf \{ \|d(\chi_E - \chi_{E^c}) - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R \} \\ &= d \inf \{ \|\chi_E - \chi_{E^c} - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R \} \\ &= d. \end{aligned}$$

Put  $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$ , then  $\bar{g}/g = e^{i(\tilde{u}-v)} = \exp i\{\tilde{u} - d(\chi_E - \chi_{E^c}) - q\}$ . If  $\lambda \neq a$  and  $\lambda \neq b$ , then

$$\begin{aligned} \frac{\phi - \lambda}{|\phi - \lambda|} &= \frac{a - \lambda}{|a - \lambda|} \chi_E + \frac{b - \lambda}{|b - \lambda|} \chi_{E^c} \\ &= \exp i\{a(\lambda)\chi_E + b(\lambda)\chi_{E^c}\} \end{aligned}$$

where  $a(\lambda) = \arg(a - \lambda)$  and  $b(\lambda) = \arg(b - \lambda)$ . Thus  $(\phi - \lambda)\bar{g}/|\phi - \lambda|g = \exp i\{a(\lambda)\chi_E + b(\lambda)\chi_{E^c} + \tilde{u} - d(\chi_E - \chi_{E^c}) - q\}$ . Since  $q \in C_R$ , by the first part of the proof,

$$\begin{aligned} &\inf \{ \|a(\lambda)\chi_E + b(\lambda)\chi_{E^c} - d(\chi_E - \chi_{E^c}) + \tilde{u} - q - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R \} \\ &= \left| \frac{a(\lambda) - b(\lambda)}{2} - d \right| \inf \{ \|\chi_E - \chi_{E^c} - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R \} \\ &= \left| \frac{a(\lambda) - b(\lambda)}{2} - d \right| = \frac{1}{2} \left| \arg \frac{a - \lambda}{b - \lambda} - 2d \right|. \end{aligned}$$

Thus, by (1) of Theorem C  $\lambda \notin \sigma(T_\phi^W)$  if and only if  $\left| \arg \frac{a - \lambda}{b - \lambda} - 2d \right| \neq \pi$ . If  $\arg \frac{a - \lambda}{b - \lambda} > 0$ , then  $\left| \arg \frac{a - \lambda}{b - \lambda} - 2d \right| \neq \pi$  because  $d > 0$ , and hence  $\sigma(T_\phi^W) = \{\lambda \notin \mathbf{C} ; \arg \frac{a - \lambda}{b - \lambda} = \pi - 2d\}$ . The description of  $h^d(\mathcal{R}(\phi))$  is a result of (2) of Theorem 1.

Theorem 2. Let  $\phi$  be a function in  $L^\infty$  and let  $W$  be a Helson-Szegő weight.

- (1)  $t_W = 0$  if and only if  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary symbol  $\phi$  in  $L^\infty$ .  
(2)  $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$  for arbitrary Helson-Szegő weight  $W$  if and only if for any  $\lambda \notin \sigma(T_\phi)$ ,  $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$  and  $\|\ell\|' = 0$ .

Proof. (1) Suppose  $W = e^{u+\tilde{v}}$ ,  $t_W = 0$  and  $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$ . If  $\lambda \notin \sigma(T_\phi)$ , then by Theorem C  $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$  and  $\|\ell\|' < \pi/2$ . Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{g}}{g} = \exp i(\ell + \tilde{u} - v)$$

and since  $t_W = 0$ ,

$$\begin{aligned} & \inf \{ \|\ell + \tilde{u} - v - \tilde{s} - a\|_\infty ; s \in L_R^\infty \text{ and } a \in R \} \\ &= \inf \{ \|\ell - \tilde{s} - a\|_\infty ; s \in L_R^\infty \text{ and } a \in R \} \\ &< \frac{\pi}{2}. \end{aligned}$$

Thus  $\lambda \notin \sigma(T_\phi^W)$  by Theorems A and C. Similarly we can show that if  $\lambda \notin \sigma(T_\phi^W)$  then  $\lambda \in \sigma(T_\phi)$ . Suppose  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary symbol  $\phi$  in  $L^\infty$ . If  $t = t_W$  is nonzero and  $W = e^{u+\tilde{v}}$  is a Helson-Szegő weight, then  $T_\phi$  is invertible where  $\phi = e^{-ikv}$  and  $k = \pi/2t - 1$ . For  $\inf \{ \|kv - \tilde{s} - a\|_\infty ; s \in L_R^\infty \text{ and } a \in R \} = kt = \pi/2 - 1$ . On the other hand,  $T_\phi^W$  is not invertible. For

$$\frac{\phi}{|\phi|} \frac{\bar{g}}{g} = \exp i\{\tilde{u} - (k+1)v\}$$

and

$$\inf \{ \|\tilde{u} - (k+1)v - \tilde{s} - a\|_\infty ; s \in L_R^\infty \text{ and } a \in R \} = (k+1)t = \frac{\pi}{2}$$

where  $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$

(2) Suppose for any  $\lambda \notin \sigma(T_\phi)$ ,  $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$  and  $\inf \{ \|\ell - \tilde{s} - a\|_\infty ; s \in L_R^\infty \text{ and } a \in R \} = 0$ . We will show that  $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$  for arbitrary Helson-Szegő weight  $W$ . If  $\lambda \notin \sigma(T_\phi)$ ,  $W = e^{u+\tilde{v}}$  is a Helson-Szegő weight and  $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$ , then

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{g}}{g} = e^{i(\ell + \tilde{u} - v)}$$

and  $\inf \{ \|\ell + \tilde{u} - v - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R \} < \pi/2$  by the hypothesis. This implies that  $\sigma(T_\phi^W) \not\ni \lambda$ . Conversely suppose that  $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$  for arbitrary Helson-Szegő weight  $W$ . If  $\lambda \notin \sigma(T_\phi)$ , then  $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$  and  $b = \inf \{ \|\ell - \tilde{s} - a\|_\infty ; s \in L_R^\infty, a \in R \} < \pi/2$ . If  $b \neq 0$ , put  $W = e^{k\tilde{\ell}}$  and  $g^2 = e^{k\tilde{\ell} - ik\ell}$  where  $k = \frac{\pi}{2b} - 1$ , then  $W$  is a Helson-Szegő weight. However  $T_\phi^W$  is not invertible and so  $\lambda \in \sigma(T_\phi^W)$ . This contradiction implies that  $b = 0$ .

Corollary 3. Let  $\phi$  be a function in  $L^\infty$ .

(1) If  $W = e^{u+\bar{v}}$ ,  $u$  and  $v$  are real functions in  $L^\infty$  and  $C$  respectively, then  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary symbol  $\phi$  in  $L^\infty$ .

(2) If  $\phi$  is a function in  $C$  or  $H^\infty$ , then  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary Helson-Szegő weight  $W$ .

(3) If  $\phi = a\chi_E + b\chi_{E^c}$ ,  $0 < m(E) < 1$  and  $a, b \in \mathbb{C}$  with  $a \neq b$ , then there exists a Helson-Szegő weight  $W$  such that  $\sigma(T_\phi^W) \subsetneq \sigma(T_\phi)$ .

Proof. Since  $t_W = 0$  because  $v \in C_R$ , (1) of Theorem 2 implies (1). Suppose  $\phi$  is a function in  $C$  and  $\lambda \notin \sigma(T_\phi^{W'})$  for a Helson-Szegő weight  $W' = e^{u+\bar{v}}$ . Since  $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W'})$ ,

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{g}}{g} = z^m e^{i\ell} e^{i(\bar{u}-v)}$$

where  $m$  is an integer,  $\ell \in C_R$  and  $g^2 = e^{u+\bar{v}+i(\bar{u}-v)}$ . By Theorems A and C, we can show  $m = 0$ . As  $W' \equiv 1$ , (2) of Theorem 2 implies that  $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$  for arbitrary Helson-Szegő weight  $W$ . The converse is trivial. Suppose  $\phi$  is a function in  $H^\infty$  and  $\lambda \notin \sigma(T_\phi^{W'})$  for a Helson-Szegő weight  $W' = e^{u+\bar{v}}$ . Since  $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W'})$ ,  $\phi - \lambda$  is invertible in  $L^\infty$  and so  $\phi - \lambda = qh$  where  $q$  is inner and  $h$  is invertible in  $H^\infty$ . Since  $h = e^{\ell+i\bar{\ell}}$  and  $\ell = \log |h| \in L^\infty$ ,

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{g}}{g} = q e^{i\bar{\ell}} e^{i(\bar{u}-v)}$$

where  $g^2 = e^{u+\bar{v}+i(\bar{u}-v)}$ . By Theorems A and C, we can show that  $q$  is constant. As in case  $\phi \in C$ , we can show  $\sigma(T_\phi^W) = \sigma(T_\phi)$  for arbitrary Helson-Szegő weight  $W$ . This completes the proof of (2). Suppose  $\phi = a\chi_E + b\chi_{E^c}$ ,  $0 < m(E) < 1$  and  $a, b \in \mathbb{C}$  with  $a \neq b$ . To prove (3), without loss of generality, we may assume that  $a$  and  $b$  are real numbers. By a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]),  $\sigma(T_\phi) = [a, b]$ . If  $\lambda \notin [a, b]$ ,

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \exp i\{a(\lambda)\chi_E + b(\lambda)\chi_{E^c}\}$$

where  $a(\lambda) = \arg(a - \lambda)$  and  $b(\lambda) = \arg(b - \lambda)$ . By the proof of Corollary 1,

$$\inf\{\|a(\lambda)\chi_E + b(\lambda)\chi_{E^c} - \tilde{s} - a\|_\infty; s \in L_R^\infty \text{ and } a \in R\} = \frac{1}{2} \left| \arg \frac{a - \lambda}{b - \lambda} \right| \neq 0$$

and hence by (2) of Theorem 2, there exists a Helson-Szegő weight  $W$  such that  $\sigma(T_\phi^W) \subsetneq \sigma(T_\phi)$ .

### §3. Toeplitz operators on $H^p$

For  $1 < p < \infty$ ,  $T_\phi^p$  denotes a Toeplitz operator on  $H^p$ . We will write  $T_\phi^2 = T_\phi$ . By a theorem of Widom, Devinatz and Rochberg (cf. [8]), we know the invertibility of  $T_\phi^p$  and by a theorem of Widom (cf. [2, Corollary 7.46]),  $\sigma(T_\phi^p)$  is connected. If  $1 < q < 2 < p < \infty$ , then  $A_q \subset A_2 \subset A_p$ . It is more difficult to describe  $\sigma(T_\phi^q)$  than  $\sigma(T_\phi^p)$ . In this paper, we study only  $\sigma(T_\phi^p)$ . When  $p = 2$ , (1) of Theorem 3 is a theorem of Brown and Halmos and (2) is a theorem of Hartman and Wintner. (3) of Theorem 3 is known in [10] for arbitrary  $1 < p < \infty$ . Our proof is different from it.

Theorem 3. Suppose  $p \geq 2$  and  $t = (p-2)\pi/2p$ .

(1) If  $\phi$  is a function in  $L^\infty$ , then  $\sigma(T_\phi^p) \subseteq h^t(\mathcal{R}(\phi))$ .

(2) If  $\phi$  is a real function in  $L^\infty$ ,  $a = \text{ess inf } \phi$  and  $b = \text{ess sup } \phi$ , then

$$[a, b] \subseteq \sigma(T_\phi^p) \subseteq \Delta(c, r) \cap \Delta(\bar{c}, r)$$

where  $c = \frac{a+b}{2} - i \frac{a-b}{2} \cot 2t$  and  $r = -\frac{a-b}{2 \sin 2t}$ . In particular, if  $p = 2$ , then  $t = 0$  and hence  $\sigma(T_\phi^p) = [a, b]$ .

(3) If  $\phi$  is a function in  $C$ , then  $\sigma(T_\phi^p) = \sigma(T_\phi)$ .

Proof. (1) If  $\lambda \notin h^t(\mathcal{R}(\phi))$ , then by definition  $\lambda \in \cup\{(\mathcal{E}_{\alpha\beta}^{\ell m})^0; \mathcal{E}_{\alpha\beta}^{ij} \supseteq \mathcal{R}(\phi) \text{ and } \ell = -i, m = -j\}$  and  $|\theta(\alpha, \beta)| = \pi - 2t$ . Hence  $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$  where  $0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon$  a.e. or  $-\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0$  a.e. for some  $\varepsilon > 0$ . Put  $v_\lambda = s_\lambda - \frac{\pi}{2} + t + \varepsilon$  or  $v_\lambda = s_\lambda + \frac{\pi}{2} - t - \varepsilon$ , then  $\|v_\lambda\|_\infty \leq \frac{\pi}{2} - t - \varepsilon$ . Put  $g^2 = e^{-\bar{v}_\lambda + iv_\lambda}$ , then  $g^2$  is an outer function and  $|g|^2 = e^{-\bar{v}_\lambda}$ . Then  $\|\frac{p}{2}v_\lambda\|_\infty < \frac{\pi}{2}$  because  $\|v_\lambda\|_\infty < \frac{\pi}{2} - \frac{(p-2)\pi}{2p}$ . Hence  $|g|^p$  satisfies  $(A_2)$  condition and so  $|g|^p$  satisfies  $(A_p)$  condition by (cf. [3, Lemma 6.8]) because  $p > 2$ . Since  $(\phi - \lambda)/|\phi - \lambda| = \alpha(\bar{g}/g)$  for some constant  $\alpha$  with  $|\alpha| = 1$ , Theorem A implies (1).

(2) We may assume that  $\phi$  is not constant. By Theorem A,  $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^p)$ . Suppose  $\lambda \in [a, b]$  and  $\lambda \notin \mathcal{R}(\phi)$ , then  $(\phi - \lambda)/|\phi - \lambda| = 2\chi_E - 1$  for some measurable set  $E$  in  $T$ . If  $\lambda \notin \sigma(T_\phi^p)$ , then by Theorem A, there exists an outer function  $h_0$  in  $H^p$  such that  $2\chi_E - 1 = \bar{h}_0/h_0$ . This implies that  $h_0^2$  is a real function in  $H^1$  because  $p \geq 2$ . It is well known that only one real function in  $H^1$  is constant. Hence  $h_0$  is constant and this contradicts that  $\phi$  is not constant. Thus  $[a, b] \subseteq \sigma(T_\phi^p)$ . Now (1) implies (2).

(3) If  $\lambda \notin \mathcal{R}(\phi)$ , then  $(\phi - \lambda)/|\phi - \lambda|$  is a continuous function and hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{iv}$$

where  $\ell$  is an integer and  $v$  is a real function in  $C$ . Put  $g^2 = e^{-\bar{v} + iv}$ , then  $|g|^2 = e^{-\bar{v}}$ . Since  $v$  is continuous, for any  $\varepsilon > 0$ ,  $\bar{v} = s + \bar{t}$  where both  $s$  and  $t$  are in  $C$  and  $\|t\|_\infty < \varepsilon$ .

Suppose  $\ell = 0$ . If  $\varepsilon < \pi/p$ , then  $|g|^p = |g^2|^{\frac{p}{2}} = \exp\left(-\frac{p}{2}\tilde{v}\right) = \exp\left(-\frac{p}{2}s - \frac{p}{2}t\right)$  and  $\|\frac{p}{2}t\|_\infty < \frac{\pi}{2}$ . Hence  $|g|^p$  satisfies  $(A_2)$  condition and so  $(A_p)$ . By Theorem A,  $T_{\phi-\lambda}^p$  is invertible and so  $\lambda \notin \sigma(T_\phi^p)$ . Suppose  $\ell \neq 0$ . If  $T_{\phi-\lambda}^p$  is invertible, then by Theorem A

$$\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{i\nu} = \frac{|k|}{k} \frac{|h|^2}{h^2}$$

where  $k$  and  $k^{-1}$  are in  $H^\infty$ , and  $h$  is an outer function in  $H^p$  with  $|h|^p$  satisfying  $(A_p)$  condition. Since  $z^\ell |g|^2 / g^2 = |kh^2| / kh^2$ ,  $z^\ell f \geq 0$  a.e. where  $f = kh^2/g^2$ . If  $\ell > 0$ ,  $z^\ell f$  is a nonnegative function in  $H^{1/2}$  and hence it is constant. This contradicts that  $z^\ell$  is zero on the origin. If  $\ell < 0$ ,  $z^\ell |1 + \bar{z}^\ell| / (1 + \bar{z}^\ell)^2 \geq 0$  and so  $(1 + \bar{z}^\ell)^2 f \geq 0$  a.e.. Thus  $(1 + \bar{z}^\ell)^2 f$  is a nonnegative function in  $H^{1/2}$  and so  $f = c(1 + \bar{z}^\ell)^2$  for some constant  $c > 0$ . This contradicts that  $f^{-1} \in H^{1/2}$ .

#### §4. Singular integral operators on $L^2$

By Theorems A, B and C, we can expect that  $\sigma(S_{\alpha\beta})$  is strongly related with  $\sigma(T_\alpha)$  and  $\sigma(T_\beta)$ . (1) of Theorem 4 is an analogy of a theorem of Brown and Halmos, and (2) of Theorem 4 is an analogy of a theorem of Hartman and Wintner.

Theorem 4. Suppose  $\alpha$  and  $\beta$  are functions in  $L^\infty$ .

- (1)  $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta}) \subseteq h^t(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta))$  where  $t = \pi/4$ .
- (2) If  $\alpha$  and  $\beta$  are real functions in  $L^\infty$ ,

$$\{h(\mathcal{R}(\alpha)) \cap h(\mathcal{R}(\beta))^c\} \cup \{h(\mathcal{R}(\alpha))^c \cap h(\mathcal{R}(\beta))\} \subseteq \sigma(S_{\alpha\beta}) \subseteq \Delta(c, r) \cap \Delta(\bar{c}, r)$$

where  $a = \min\{\text{ess inf } \alpha, \text{ess inf } \beta\}$ ,  $b = \max\{\text{ess sup } \alpha, \text{ess sup } \beta\}$ ,  $c = \frac{a+b}{2} - i\frac{a-b}{2}$   
and  $r = -\frac{a-b}{2}$ .

- (3) If  $\beta$  is in  $C$ ,

$$\sigma(T_\alpha) \cap \{\lambda \in \mathbb{C} ; i_t(\beta, \lambda) = 0\} \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta}) \subseteq \sigma(T_\alpha) \cup \sigma(T_\beta).$$

- (4) If both  $\alpha$  and  $\beta$  are in  $C$ , then  $\sigma(S_{\alpha\beta}) = \{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in \mathbb{C} ; i_t(\alpha, \lambda) = i_t(\beta, \lambda) \neq 0\}$ .

- (5) Suppose both  $\alpha$  and  $\beta$  are in  $C$ . If  $\beta$  is a real function, then  $\sigma(S_{\alpha\beta}) = \sigma(T_\alpha) \cup h(\mathcal{R}(\beta))$  and hence if both  $\alpha$  and  $\beta$  are real functions, then  $\sigma(S_{\alpha\beta}) = h(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta))$ .

(6) If  $\alpha$  and  $\bar{\beta}$  are functions in  $H^\infty$ , then  $\sigma(S_{\alpha\beta}) = [\alpha(D)]^{cl} \cup \overline{[\beta(D)]^{cl}}$ .

(7) If  $\alpha$  and  $\beta$  are functions in  $H^\infty$ , then  $\sigma(S_{\alpha\beta}) = [\alpha(D)]^{cl} \cup [\beta(D)]^{cl} \setminus \{\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta); T_{q_\lambda \bar{p}_\lambda} \text{ is invertible}\}$  where  $q_\lambda$  is the inner part of  $\alpha - \lambda$  and  $p_\lambda$  is the inner part of  $\beta - \lambda$ .

(8) If  $\alpha$  and  $\beta$  are inner functions, and  $\text{sing}\alpha \neq \text{sing}\beta$ , then  $\sigma(S_{\alpha\beta}) = [D]^{cl}$ , where  $\text{sing}\alpha$  and  $\text{sing}\beta$  denote the subsets of  $\partial D$  on which  $\alpha$  and  $\beta$  can not be analytically extended, respectively.

Proof. (1) By Theorem B, it is clear that  $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$ . If  $\lambda \notin h^t(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta))$ , then  $(\alpha - \lambda)/|\alpha - \lambda| = e^{is_\lambda}$  and  $(\beta - \lambda)/|\beta - \lambda| = e^{it_\lambda}$  where  $0 \leq s_\lambda, t_\lambda \leq \frac{\pi}{2} - \varepsilon$  a.e. or  $-\frac{\pi}{2} + \varepsilon \leq s_\lambda, t_\lambda \leq 0$  a.e. for some  $\varepsilon > 0$ . Therefore

$$\frac{\alpha - \lambda}{\beta - \lambda} = \exp(U - i\tilde{V})$$

where  $U = \log|\alpha - \lambda| - \log|\beta - \lambda|$  and  $\tilde{V} = t_\lambda - s_\lambda$ . Then  $U$  is bounded and  $\exp V = \exp -(t_\lambda - s_\lambda)$  and  $\|t_\lambda - s_\lambda\|_\infty \leq \frac{\pi}{2} - \varepsilon$ . By Theorem C,  $S_{\alpha-\lambda, \beta-\lambda}$  is invertible.

(2) If  $\alpha$  and  $\beta$  are real functions and  $\lambda \in h(\mathcal{R}(\alpha)) \cap h(\mathcal{R}(\beta))^c$ , then  $\alpha - \lambda$  is a real function which is not nonnegative or nonpositive, and  $\beta - \lambda$  is a nonnegative or nonpositive function which is invertible in  $L^\infty$ .  $(\alpha - \lambda)/(\beta - \lambda)$  is a real function in  $L^\infty$  which is not nonnegative or nonpositive. If  $S_{\alpha-\lambda, \beta-\lambda}$  is invertible, then by Theorems B and C both  $\alpha - \lambda$  and  $\beta - \lambda$  are invertible in  $L^\infty$ , and

$$\frac{\alpha - \lambda}{\beta - \lambda} = \left| \frac{\alpha - \lambda}{\beta - \lambda} \right| e^{it}$$

where  $\inf\{\|t - \tilde{s} - a\|_\infty : s \in L_R^\infty \text{ and } a \in R\} < \pi/2$ . Let  $g = e^{-i+it}$ , then  $g$  is a real function in  $H^1$ . Since only one real function in  $H^1$  is constant,  $g$  is constant and so it contradicts that  $(\alpha - \lambda)/|\beta - \lambda| / (|\beta - \lambda|/|\alpha - \lambda|)$  is nonconstant. This implies that  $h(\mathcal{R}(\alpha)) \cap h(\mathcal{R}(\beta))^c \subseteq \sigma(S_{\alpha\beta})$ . The same method shows that  $h(\mathcal{R}(\alpha))^c \cap h(\mathcal{R}(\beta)) \subseteq \sigma(S_{\alpha\beta})$ . Since  $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq [a, b]$ , by (1)  $\sigma(S_{\alpha\beta}) \subseteq h^t([a, b])$  where  $t = \pi/4$ . This implies (2).

(3) Suppose  $\lambda \in \sigma(T_\alpha) \cap \{\lambda \in \mathbb{C}; i_t(\beta, \lambda) = 0\}$ . Then  $\beta - \lambda = |\beta - \lambda| e^{iv}$  and  $v \in C$  because  $\beta$  is continuous. If  $S_{\alpha-\lambda, \beta-\lambda}$  is invertible, then by Theorem B

$$\frac{\alpha - \lambda}{\beta - \lambda} = \gamma e^{(U - i\tilde{V})}$$

where  $\gamma$  is constant,  $U$  is a bounded real function,  $V$  is a real function in  $L^1$  and  $\exp V$  satisfies  $(A_2)$  condition. Hence

$$\alpha - \lambda = \gamma \exp\{U + \log|\beta - \lambda| - i(\tilde{V} - v)\}$$

,  $U + \log|\beta - \lambda|$  is in  $L^\infty$  and  $e^{V - \tilde{v}}$  satisfies  $(A_2)$  condition because  $v \in C$ . By Theorem A, this implies that  $\lambda \notin \sigma(T_\alpha)$ . This contradiction shows that  $\lambda \in \sigma(S_{\alpha\beta})$  and hence  $\sigma(T_\alpha) \cap \{\lambda \in \mathbb{C}; i_t(\beta, \lambda) = 0\} \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$ . If  $\lambda \notin \sigma(T_\alpha) \cup \sigma(T_\beta)$ , then by Theorem C and [2, Corollary 7.28]  $\alpha - \lambda = |\alpha - \lambda| e^{it}$  and  $\beta - \lambda = |\beta - \lambda| e^{i\ell}$  where  $\inf\{\|t - \tilde{s} - a\|_\infty; s \in L_R^\infty \text{ and } a \in R\} < \pi/2$  and  $\ell \in C$ . Therefore

$$\frac{\alpha - \lambda}{\beta - \lambda} = \frac{|\alpha - \lambda|}{|\beta - \lambda|} e^{i(t-\ell)}$$

and hence by Theorem C  $\lambda \notin \sigma(S_{\alpha\beta})$ .

(4) If  $\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$  and  $i_t(\alpha, \lambda) \neq i_t(\beta, \lambda)$ , then  $\alpha - \lambda = |\alpha - \lambda| z^\ell e^{i\ell u}$  and  $\beta - \lambda = |\beta - \lambda| z^\ell e^{i\ell v}$  where  $u$  and  $v$  are in  $C$ , and  $\ell$  and  $t$  are integers with  $\ell \neq t$ . Hence

$$\frac{\alpha - \lambda}{\beta - \lambda} = \frac{|\alpha - \lambda|}{|\beta - \lambda|} z^{\ell-t} e^{i(u-v)}$$

and  $\ell - t \neq 0$ . By Theorem C, we can show that  $\lambda \notin \sigma(S_{\alpha\beta})$ . This implies that  $\{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in \mathbb{C}; i_t(\alpha, \lambda) = i_t(\beta, \lambda) \neq 0\} \subseteq \sigma(S_{\alpha\beta})$ . If  $\lambda \notin \{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in \mathbb{C}; i_t(\alpha, \lambda) = i_t(\beta, \lambda) \neq 0\}$ , then  $\alpha - \lambda = |\alpha - \lambda| z^\ell e^{i\ell u}$  and  $\beta - \lambda = |\beta - \lambda| z^\ell e^{i\ell v}$  where  $u$  and  $v$  are in  $C$ , and  $\ell$  is an integer. Hence  $(\alpha - \lambda)/(\beta - \lambda) = (|\alpha - \lambda|/|\beta - \lambda|) e^{i(u-v)}$ . By Theorem C,  $\lambda \notin \sigma(S_{\alpha\beta})$ . This completes the proof of (4). (5) is a result of (4).

(6) If  $\lambda \in \alpha(D) \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$ , then  $\alpha - \lambda = qh$  and  $\beta - \lambda = \bar{p}k$  where  $q$  and  $p$  are inner, and  $h$  and  $k$  are invertible in  $H^\infty$ . Hence  $(\alpha - \lambda)/(\beta - \lambda) = qph/\bar{k}$  and so by Theorem C  $\lambda \in \sigma(S_{\alpha\beta})$ . This shows that  $\alpha(D) \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$ . By the same method we can show that  $\beta(D) \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$ . By (1),  $[\alpha(D)]^{cl} \cup [\beta(D)]^{cl} \subseteq \sigma(S_{\alpha\beta})$ . If  $\lambda \notin [\alpha(D)]^{cl} \cup [\beta(D)]^{cl}$ , then  $\alpha - \lambda = h$  and  $\beta - \lambda = \bar{k}$  where both  $h$  and  $k$  are invertible in  $H^\infty$ . By Theorem C,  $\lambda \notin \sigma(S_{\alpha\beta})$ .

(7) If  $\lambda \in [\alpha(D)]^{cl} \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$ , then  $\alpha - \lambda = q_\lambda h_\lambda$  and  $\beta - \lambda = p_\lambda k_\lambda$  where both  $q_\lambda$  and  $p_\lambda$  are inner and both  $h_\lambda$  and  $k_\lambda$  are invertible in  $H^\infty$ . Hence  $(\alpha - \lambda)/(\beta - \lambda) = q_\lambda \bar{p}_\lambda h_\lambda / k_\lambda$ . If  $T_{q_\lambda \bar{p}_\lambda}$  is not invertible, by Theorem C  $\lambda \in \sigma(S_{\alpha\beta})$ . This implies that  $\{[\alpha(D)]^{cl} \cup [\beta(D)]^{cl}\} \setminus \{\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta); T_{q_\lambda \bar{p}_\lambda} \text{ is invertible}\} \subseteq \sigma(S_{\alpha\beta})$ . If  $\lambda \notin [\alpha(D)]^{cl} \cup [\beta(D)]^{cl}$ , then  $\lambda \notin \sigma(S_{\alpha\beta})$  as in (6). If  $\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$  and  $T_{q_\lambda \bar{p}_\lambda}$  is invertible, then by Theorem C  $\lambda \notin \sigma(S_{\alpha\beta})$ .

(8)  $\sigma(S_{\alpha\beta}) \subseteq [D]^{cl}$  by (7) and so if  $\lambda \notin (\mathcal{R}(\alpha) \cup \mathcal{R}(\beta)) \cap [D]^{cl}$ , then the inner part of  $\alpha - \lambda$  is  $q_\lambda = (\alpha - \lambda)/(1 - \bar{\lambda}\alpha)$  and the inner part of  $\beta - \lambda$  is  $p_\lambda = (\beta - \lambda)/(1 - \bar{\lambda}\beta)$ . Then  $\text{sing} q_\lambda = \text{sing} q \neq \text{sing} p = \text{sing} p_\lambda$ . By [6, Theorem 1],  $T_{q_\lambda \bar{p}_\lambda}$  is not invertible. By (7), this implies that  $\sigma(S_{\alpha\beta}) = [\alpha(D)]^{cl} \cup [\beta(D)]^{cl} = [D]^{cl}$ .

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