SOLUTION SURFACES OF MONGE-AMPÈRE EQUATIONS

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0. Introduction

In this paper we examine the singularities of solution surfaces of Monge-Ampère equations and study their global and local effects on the solutions for certain kinds of equations. In particular, as a byproduct, we give a simple proof to the classical Hartman-Nirenberg's theorem by using the notion of projective duality.

Hereafter all manifolds and mappings and so on are assumed to be of class C^∞, unless otherwise stated.

A Monge-Ampère equation for a surface $z = \phi(x, y)$ in $\mathbb{R}^3$ is given by

$$Ar + 2Bs + Ct + E + F(rt - s^2) = 0,$$

where

$$p = \partial\phi/\partial x,\ q = \partial\phi/\partial y,\ r = \partial^2\phi/\partial x^2,\ s = \partial^2\phi/\partial x\partial y,\ t = \partial^2\phi/\partial y^2$$

and A, B, C, E and F are functions of x, y, z, p and q ([Go]). The Monge-Ampère equations and their generalizations may be well described in contact geometry ([M2]).

Let M be a $(2n-1)$-dimensional contact manifold with the contact distribution $D \subset TM$. A Monge-Ampère system (or simply a Monge-Ampère equation) is an exterior differential system Σ locally generated by a contact form of D (that is, a local section of the

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line bundle $D^\perp \subset T^*M$), and an $(n-1)$-form θ ([B, M1]). For example, if we take

$$\theta = Adp \wedge dy - 2Bdp \wedge dx + Cdq \wedge dx + Edx \wedge dy + Fdp \wedge dq,$$

with the contact form $dz - pdx - qdy$, then we get the Monge-Ampère equation of the form above.

Let Σ be a Monge-Ampère equation on M. A *solution* S of Σ is an integral manifold of Σ of dimension $n-1$, namely a Legendre submanifold of M where θ vanishes.

Generalizing the notion of solution, we call a mapping $f : L \longrightarrow M$ from an $(n-1)$-manifold L a *generalized solution* of Σ if the pull-back $f^*\Sigma$ of Σ by f is equal to zero. Here we do not assume f is an immersion, therefore $f(L)$ is an integral manifold possibly with singularities.

An important example of contact manifold is the projective cotangent bundle PT^*N of an n-manifold N, identified with the manifold of contact elements of N. This bundle has the natural Legendre fibring $\pi : PT^*N \longrightarrow N$. In the case $M = PT^*N$, a hypersurface $A \subset N$ is called a *geometrical solution* of Σ if the natural Legendre lift

$$\tilde{A} = \{(x, T_x N) \in PT^*N \ ; \ x \in A\}$$

of A with respect to π is a solution of Σ in the above sense.

So far we have defined the three kinds of notions for solutions of a Monge-Ampère system. When we study singularities of a solution S of a Monge-Ampère system Σ on a contact manifold M, we should distinguish the singularities of S itself, that is, the singularities of a parametrization $f : L \longrightarrow S \subset M$, and the singularities with respect to the Legendre fibring, that is, the singularities of $\pi \circ f : L \longrightarrow N$.

Among the Monge-Ampère equations, there exists a remarkable system canonically associated with projective geometry which we are going to treat: Let V be an $(n+1)$-dimensional real vector space ($n \geq 3$) and $P(V)$ the corresponding n-dimensional projective space. If we denote by V^* the dual vector space to V, then $P(V^*)$ is identified with the dual projective space to $P(V)$, that is, the totality of projective hyperplanes in $P(V)$, under the pairing $V \times V^* \longrightarrow \mathbb{R}$. Thus, for a $q \in P(V^*)$, q^*, the dual of q, stands for a hyperplane in $P(V)$. We set

$$Q = \{(p,q) \in P(V) \times P(V^*)\ ;\ p \in q^*\} = \{([x],[\xi]) \in P(V) \times P(V^*)\ ;\ \langle x, \xi \rangle = 0\}.$$

Then Q is a $(2n-1)$-dimensional manifold, endowed with the double fibrations

$$\rho : Q \longrightarrow P(V), \qquad \rho' : Q \longrightarrow P(V^*),$$

and identified with $PT^*P(V)$, the projective cotangent bundle of $P(V)$. Similarly, $Q \cong PT^*P(V^*)$. Moreover there is a canonical contact structure $D = \mathrm{K}er\rho_* \oplus \mathrm{K}er\rho'_*$ on Q such that the above isomorphisms are contact isomorphisms [A]. Then the subbundle $\mathrm{K}er\rho'_*$ of D of rank $n-1$ defines the Monge-Ampère system Σ_0 on Q whose characteristic system is $\mathrm{K}er\rho'_*$.

Let A be a hypersurface in $P(V)$. Then we regard the Legendre lift $\tilde{A} \subset PT^*P(V)$ as a Legendre submanifold in Q. Then A is a geometrical solution of Σ_0 if and only if the second projection $\rho' : \tilde{A} \to PT^*P(V^*)$ is of rank $\leq n-2$ everywhere.

In this paper we prove the following result on the global geometrical solutions of the Monge-Ampère system Σ_0:

Theorem 1. *A closed (i.e. compact without boundary) smooth hypersurface A of $P(V)$ is a geometrical solution of the Monge-Ampère system Σ_0 with rank $(\rho'|_{\tilde{A}}) \leq 1$ if and only if A is a projective hyperplane of $P(V)$.*

For the necessity of the rank condition see Example 8 of the next section.

In the complex analytic category, a similar result as Theorem 1 holds for complex analytic geometrical solutions of Σ_0 without the rank condition (Proposition 9).

Even in the real case, we have as a corollary of Theorem 1:

Corollary 2. *Let $\dim P(V) = 3$. Then a closed smooth surface of $P(V)$ is a geometric solution of the Monge-Ampère system Σ_0 if and only if it is a projective plane.*

Recall that, in the case $\dim P(V) = 3$, the system Σ_0 has $Ker\rho'_*$ as a completely integrable characteristic system, therefore Σ_0 is in the class P_0 in the sense of [M1], and locally isomorphic to the equation $rt - s^2 = 0$ ([Go, M1]). In this sense, the system Σ_0 is a global model of $rt - s^2 = 0$.

In general, a smooth surface S in Q is a solution of $\Sigma_0 : rt - s^2 = 0$ if and only if S is a Legendre submanifold of Q and the rank of $\rho' : S \to P(V^*)$ is less or equal to 1 everywhere. Therefore, by Corollary 2, if $S \subset Q$ is a solution of $rt - s^2 = 0$ and not the Legendre lift of a projective plane, then the projection $\rho : S \longrightarrow P(V)$ has always singularities.

It turns out that a geometrical solution $A \subset P(V)$ of $rt - s^2 = 0$ is a ruled surface generated by projective lines and the tangent spaces of A are constant along each generating line. For instance, the cone C defined by $z^2 = x^2 + y^2$ with affine coordinates x, y, z is a solution of Σ_0 with singularity. We note that C has the Legendre lift $\tilde{C}$, which is diffeomorphic to a torus and has no singularity, while C does.

In general, a geometrical solution in $P(V)$ of $rt - s^2 = 0$ is regarded as the ρ-projection of the projective conormal bundle of a space curve in $P(V^*)$. If the space curve is non-degenerate, that is, not contained in any plane of $P(V^*)$, the geometrical solution turns out to be a tangent developable of the dual space curve in $P(V) = P(V^{**})$ [I2]. The above example corresponds to the case the space curve in $P(V^*)$ is in fact a plane curve: The singular point corresponds to the dual point to the plane containing the plane curve. For a non-degenerate curve, the tangent developable has singularities along the dual curve in $P(V)$. Therefore the solution can be non-singular only when the space curve in $P(V^*)$ is reduced to a point. Then the corresponding solution in $P(V)$ is the dual plane to the point. Otherwise, each generating line contains a singular point. In the next section we shall clarify this argument in the proof of Theorem 1.

If a geometrical solution of $rt - s^2 = 0$ in the projective three space has singularities just on the plane at infinity, then its affine part is of constant zero curvature with respect to the induced metric of the Euclidean three space. Then, similarly to the proof of Theorem 1, we find again the classical Hartman-Nirenberg's theorem [HN, Ste, Sto]:

Theorem 3. *Let A be a properly embedded connected smooth hypersurface in $\mathbb{R}^n (\subset P(\mathbb{R}^{n+1}))$. Then A is a geometrical solution of the equation Σ_0 with rank $\rho'|_{\tilde{A}} \leq 1$ if and only if A is a cylinder generated by parallel $(n-2)$-dimensional affine subspaces of*

$\mathbb{R}^n$. *In particular, a properly embedded connected surface A in $\mathbb{R}^3$ has the zero Gaussian curvature if and only if A is a cylinder generated by parallel lines in $\mathbb{R}^3$.*

We remark here an interesting contrast between Theorem 1 or 3 and that of Bernstein and Jörgens [N, J] which asserts that a solution defined on the whole xy-plane of the equation $rt - s^2 = 1$ is a polynomial of degree 2. The latter arises from the ellipticity of the equation, but the former arises from some properties of projective geometry and holds both in the real and the complex category. Also compare with the investigations of surfaces of constant negative curvature [Mi, T2].

Besides $rt - s^2 = 0$, there is another locally trivial Monge-Ampère equation, $s = 0$. The Monge-Ampère equation $s = 0$ is given by the system

$$\alpha = dz - pdx - qdy = 0, \qquad \theta = dp \wedge dx = 0,$$

on $\mathbb{R}^5$. Remark that the equation $rt - s^2 = 0$ is given by

$$\alpha = dz - pdx - qdy = 0, \qquad \theta = dp \wedge dq = 0.$$

The type of possible singularities of generalized solutions should be an invariant of a Monge-Ampère equation. In §2, we observe that these two equations can be distinguished by their singularities of the generalized solutions (Proposition 10). There a key role is played by the open umbrella ([A, Gi, I2]).

For the first order partial differential equations, the singularities of solutions, with respect to the Legendre fibring, are well described by Legendre singularity theory. See [Iz] for instance. Remark that the open umbrella never appears as a generalized solution of a first order partial differential equation defined by a submanifold in $PT^*\mathbb{R}^2$ of a positive codimension which is transverse to the contact distribution (Lemma 11).

For other approaches to the singularities of the Monge-Ampère equations, see [K1. K2, T1]. For the classification problem of the Monge-Ampère equations, see [M1, M2, M3, LCR, Tu]. For the local classification of singular geometrical solutions of $rt - s^2 = 0$, namely, for that of tangent developables of space curves, see [Cl, S, Mo1, Mo2, I1, I2]. For the global topology of tangent developables, see [Ba] for instance.

1. Global solutions

First recall the contact distribution D of the projective cotangent bundle $M = PT^*N$ with the projection $\pi : M \longrightarrow N$ over a manifold N. A contact element $c \in M$ defines a tangent hyperplane $H \subset T_{\pi(c)}N$. Then set $D_c = \pi_*^{-1}(H) \subset T_cM$, for the differential $\pi_* : T_cM \longrightarrow T_{\pi(c)}N$. To prove Theorem 1, we will utilize the following fundamental result [A]:

Lemma 4. *Let $S \subset M = PT^*N$ be a Legendre submanifold. Assume that the restriction $\pi|_S : S \longrightarrow N$ of the projection is a submersion onto a submanifold $W \subset N$. Then S is contained in the projective conormal bundle $L = PT^*_W N$ of W in N. Moreover $S = L$ in a neighborhood of each point $s \in S$ in M.*

Proof. Let $s \in S$, and $H \subset T_{\pi(s)}N$ be the contact element defined by s. Take $v \in T_{\pi(s)}W$. Then there exists $u \in T_sS$ with $\pi_*(u) = v$. Since $T_sS \subset D_s = \pi_*^{-1}(H)$, we see $v \in H$. Thus $T_{\pi(s)}W \subset H$ and therefore $s \in L$. The second half is clear since $\dim S = \dim L$. □

If $\dim V = n+1$, then we identify $P(V)$ (resp. $P(V^*)$) with $\mathbb{R}P^n$ (resp. $\mathbb{R}P^{n*}$).

Let $A \subset \mathbb{R}P^n$ be a closed geometrical solution of Σ_0 and $(p,q) \in \tilde{A} \subset Q$. Then we see, by the definition of Σ_0, $\rho'|_{\tilde{A}}$ is of rank $\leq n-2$ at (p,q). Furthermore we observe the following:

Lemma 5. *Assume $\rho'|_{\tilde{A}}$ is of constant rank m in a neighborhood of (p,q) in $\tilde{A}$. Then there exist a submanifold W of dimension m in an open neighborhood U of q in $\mathbb{R}P^{n*}$ such that ${\rho'}^{-1}(U) \bigcap \tilde{A} = PT^*_W\mathbb{R}P^{n*}$, the projective conormal bundle of W in $\mathbb{R}P^{n*}$. In particular $\rho'|_{\tilde{A}}$ is of constant rank m on ${\rho'}^{-1}(U)$.*

Proof. There exists an open neighborhood V of (p,q) in $\tilde{A}$ (resp. U of q in $\mathbb{R}P^{n*}$) such that $\rho'|_{\tilde{A}}$ is of constant rank m on V and $W = \rho'(V)$ is a submanifold of dimension m in U. Set $L = PT^*_W\mathbb{R}P^{n*}$. Then, by Lemma 4, we see $L = \tilde{A}$ in V.

Let $q' \in W$. On the fibre $(\rho'|_L)^{-1}(q')$, the subset

$$L_{q'} = \{(p', q')\ ;\ L = \tilde{A} \text{ in a neighborhood of } (p', q')\},$$

is open and non-empty. Moreover $L_{q'}$ is also closed. In fact, let (p', q') be on the closure of $L_{q'}$. Then $(p', q') \in L \bigcap \tilde{A}$ and $T_{(p',q')}\tilde{A} = T_{(p',q')}L$. Therefore the rank of $\rho'|_{\tilde{A}}$ at (p', q') is equal to m. This indicates also that the rank is constant m in a neighborhood of (p', q') in $\tilde{A}$: If not, then we have $T_{(p',q')}\tilde{A} = T_{(p',q')}L'$ for the projective conormal bundle L' of a submanifold W' in $\mathbb{R}P^{n*}$ of dimension greater than m, which leads a contradiction. By Lemma 4 again, we know, near (p', q'), $\tilde{A}$ is a projective conormal bundle of a submanifold of dimension m, which coincides with W in a neighborhood of q'. This shows $(p', q') \in L_{q'}$.

From the connectivity of $(\rho'|_L)^{-1}(q')$, we see that $L_{q'} = (\rho'|_L)^{-1}(q')$ and that $L = \tilde{A}$ on a neighborhood of $(\rho'|_L)^{-1}(q')$. □

Let (W, q) be a submanifold-germ of $(\mathbb{R}P^{n*}, q)$. Then set $\mathcal{F} = Q \cap (\mathbb{R}P^n \times W) \cong PT^*\mathbb{R}P^{n*}|_W$. The critical locus of $\rho|_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathbb{R}P^n$ is equal to the projective conormal bundle L of W. If W is, as above, a local image of the germ $\rho'|_{\tilde{A}}$ at (p, q) where $\rho'|_{\tilde{A}}$ is of constant rank, then, by Lemma 5, $\rho|_L : L \longrightarrow \mathbb{R}P^n$ is a parametrization of A near ${\rho'}^{-1}(q) \bigcap \tilde{A} = {\rho'}^{-1}(q) \bigcap L$. We are going to study the singularity of this map-germ.

Take an affine coordinate $(y_1, \dots, y_n)$ of $\mathbb{R}P^{n*}$ centered at q such that W is locally represented by

$$y_{m+1} = \varphi_{m+1}(y_1, \dots, y_m), \quad \dots, \quad y_n = \varphi_n(y_1, \dots, y_m),$$

for some functions $\varphi_j, m + 1 \leq j \leq n$.

Denote by $H(\varphi_j)$ the Hesse matrix $(\partial^2\varphi_j/\partial y_k \partial y_\ell)_{1 \leq k,\ell \leq m}$ of φ_j.

For an homogeneous coordinate $(X_0, X_1, \dots, X_n)$, $\mathcal{F}$ is defined by

$$F(X; y_1, \dots, y_m) = X_0\varphi_n + \cdots + X_{n-m-1}\varphi_{m+1} + X_{n-m}y_m + \cdots + X_n = 0,$$

and L is defined by

$$F = \partial F/\partial y_1 = \cdots = \partial F/\partial y_m = 0.$$

Then we have

Lemma 6. *The projection $\rho|_L$ is not an immersion at a point $(p, q) \in L$ if and only if the symmetric matrix*

$$H = \sum_{k=0}^{n-m-1} X_k H(\varphi_{n-k})(0)$$

is degenerate, where $(X_0, \ldots, X_{n-m-1}, X_{n-m}, \ldots, X_n)$ is the homogeneous coordinate of $p \in \mathbb{R}P^n$. In particular, $\rho|_L$ is an immersion along the fibre $(\rho'|_L)^{-1}(q)$ if and only if the linear family $H = \sum_{k=0}^{n-m-1} X_k H(\varphi_{n-k})(0)$ of symmetric matrices represents non-degenerate matrices for any $(X_0, \ldots, X_{n-m-1}) \neq (0, \ldots, 0)$.

Proof. $\rho|_L$ is an immersion at $(p,q) \in L$ if and only if

$$(F, \partial F/\partial y_1, \ldots, \partial F/\partial y_m, X_0, \ldots, X_n)$$

is an immersion at (p,q). This is equivalent to that rank$H = m$. □

As a consequence we have

Lemma 7. *If $n > (1/2)m(m+3)$ or m is odd, then $\rho|_L$ is not an immersion along $(\rho'|_L)^{-1}(q)$ for any $q \in W$.*

Proof. Consider the vector space $S(m)$ of real symmetric $m \times m$ matrices and the variety $\Sigma \subset S(m)$ of degenerate matrices.

If $n - m > \dim S(m) = (1/2)m(m+1)$, then $H(\varphi_{m+1})(0), \ldots, H(\varphi_n)(0)$ are linearly dependent in $S(m)$. Thus H represents the zero matrix O for some $(X_0, \ldots, X_{n-m-1}) \neq (0, \ldots, 0)$.

In the case $H(\varphi_{m+1})(0), \ldots, H(\varphi_n)(0)$ are linearly independent, we take the $(n-m)$-plane H spanned by $H(\varphi_{m+1})(0), \ldots, H(\varphi_n)(0)$ in $S(m)$. Since the projectification $P\Sigma \subset PS(m)$ is of degree m, if m is odd, then the projective $(n-m-1)$-plane PH intersects to $P\Sigma$ in $PS(m)$. □

Proof of Theorem 1. Let $A \subset \mathbb{R}P^n = P(V)$ be a closed smooth geometrical solution of Σ_0 with rank $\rho'|_{\tilde{A}} \leq 1$. Suppose that, for a point $p \in A$, the rank of $\rho'|_{\tilde{A}}$ at (p,q), $q = (T_pA) \in \mathbb{R}P^{n*}$, is equal to 1, which is the maximal rank in this case. Then $\rho'|_{\tilde{A}}$ is of constant rank 1 near (p,q). Consider the projective conormal bundle L of the local image of $\tilde{A}$ near (p,q). Then, by Lemma 5, $L = \tilde{A}$ in a neighborhood of $(\rho'|_L)^{-1}(q)$. By Lemma 7 with $m = 1$, we see that $\rho|_{\tilde{A}}$ is not an immersion, which leads to a contradiction.

Consequently we see that the rank of $\rho'|_{\tilde{A}}$ should be identically zero. This means A is a part of a projective hyperplane. Since A is closed as a subspace of $\mathbb{R}P^n$, A is a union of hyperplanes: But A being smooth, we conclude that A is a projective hyperplane. $\square$

Remark: By the same proof based on Lemma 7, we see that if rank $\rho'|_{\tilde{A}} = m > 0$, and if m is odd or satisfies $(1/2)m(m+3) < n$, then A necessarily has singularities.

Proof of Theorem 3. Let A be a properly embedded geometrical solution of Σ_0 in $\mathbb{R}^n$ with rank $\rho'|_{\tilde{A}} \leq 1$. Regard $\tilde{A}$ as a (non-properly embedded) hypersurface in $PT^*(\mathbb{R}P^n) \hookrightarrow \mathbb{R}P^n \times \mathbb{R}P^{n*}$. Let $(p,q) \in \tilde{A}$. First consider, as in the proof of Theorem 1, the case that the rank of $\rho'|_{\tilde{A}}$ at (p,q) is equal to 1 and denote by L the projective conormal bundle of the local image W of $\tilde{A}$ by ρ'. Then, by Lemma 7, $\rho|_L$ is not immersive along $(\rho'|_L)^{-1}(q)$. By the assumption that A is properly embedded in $\mathbb{R}^n$, we see that $\tilde{A}$ is properly embedded in $PT^*\mathbb{R}^n (\subset PT^*(\mathbb{R}P^n))$. Moreover by Lemma 4, we see, for each $q' \in W$ near q, that any singular point (p', q') of $\rho|_L$ on the fibre $(\rho'|_L)^{-1}(q')$ belongs to $\rho'^{-1}(q') \bigcap (L - \tilde{A})$, and that p' is on the hyperplane at infinity $\mathbb{R}P^n - \mathbb{R}^n$.

Let $y_1 = t,\ y_2 = \varphi_2(t), \ldots,\ y_n = \varphi_n(t)$ be a parametrization at $t = 0$ of the curve W for some affine coordinates centered at q. By Lemma 6, we have

$$\left(\frac{d^2\varphi_2}{dt^2}(0), \ldots, \frac{d^2\varphi_n}{dt^2}(0)\right) \neq \mathbf{0}:$$

If not, $\rho|_L$ has a singular point on $(\rho'|_L)^{-1}(q) \cap \tilde{A}$.

Consider the families of $3 \times (n+1)$ matrices

$$S = \begin{pmatrix} 1 & t & \varphi_2 & \ldots & \varphi_n \\ 0 & 1 & d\varphi_2/dt & \ldots & d\varphi_n/dt \\ 0 & 0 & d^2\varphi_2/dt^2 & \ldots & d^2\varphi_n/dt^2 \end{pmatrix},$$

with the parameter t. By Lemma 6 again, the coordinates $X = {}^t(X_n, \ldots, X_1, X_0)$ of singular values of $\rho|_L$ satisfies the linear equation $SX = \mathbf{0}$. Since all singular points are on the hyperplane at infinity, we see that the rank of S is equal to the rank of the $4 \times (n+1)$ matrix

$$\tilde{S} = \begin{pmatrix} 1 & t & \varphi_2 & \ldots & \varphi_n \\ 0 & 1 & d\varphi_2/dt & \ldots & d\varphi_n/dt \\ 0 & 0 & d^2\varphi_2/dt^2 & \ldots & d^2\varphi_n/dt^2 \\ a_0 & a_1 & a_2 & \ldots & a_n \end{pmatrix},$$

for each t near 0, where $a_0X_n+\cdots+a_nX_0=0$ is the equation of the hyperplane at infinity. Therefore the fourth row of $\tilde{S}$ is a functional linear combination of other three rows of $\tilde{S}$. Thus we easily see that

$$\operatorname{rank}\begin{pmatrix} d^2\varphi_2/dt^2 & \dots & d^2\varphi_n/dt^2 \\ d^3\varphi_2/dt^3 & \dots & d^3\varphi_n/dt^3 \end{pmatrix} = 1.$$

This means that W is not contained in any line and is contained in a plane Π. Hence, in a neighborhood of $(\rho'|_L)^{-1}(q)$, L projects to a hypersurface ruled by $(n-2)$-dimensional planes intersecting along the common $(n-3)$-dimensional plane Π^*, the projective dual of Π, which lies on the hyperplane at infinity. In particular, these $(n-2)$-dimensional planes are parallel to each other.

Decompose $A = A_1 \bigcup A_0$, where A_1 (resp. A_0) is the open subset (resp. closed subset) of A consisting of the points p such that the rank of $\rho'|_{\tilde{A}}$ at (p,q) is equal to 1 (resp. 0). By Lemma 5, we see that each connected components of A_1 is the union of parallel $(n-2)$-dimensional planes, while the interior of A_0 is a union of parts of hyperplanes.

If the boundary of a component of the interior of A_0 is non-empty in $\mathbb{R}^n$, then it consists of two $(n-2)$-dimensional planes on the boundaries of possibly different components of A_1. Since these non-intersecting two $(n-2)$-dimensional planes are on one hyperplane, we see they are parallel. Also the closure of A_1 minus the closure of the interior of A_0 consists of parallel $(n-2)$-dimensional planes by continuity. Thus, by the connectivity of A, we conclude that A is ruled by parallel $(n-2)$-dimensional planes. $\square$

Example 8. Consider a surface-germ W, q in $\mathbb{R}P^{4*}$ locally defined by

$$y_3 = 2y_1y_2, \qquad y_4 = y_1^2 - y_2^2,$$

with respect to some affine coordinates y_1, y_2, y_3, y_4 with center q. Then the pencil

$$H = X_0\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} + X_2\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

of symmetric matrices represents, for any $(X_0, X_1) \neq (0,0)$, non-degenerate matrices. Therefore in this case $\rho|_L$ is in fact an immersion along $(\rho'|_L)^{-1}(q)$.

More strictly, we see that there exists a closed smooth geometrical solution $A \subset \mathbb{R}P^4$ of the Monge-Ampère system Σ_0 which is different from the projective hyperplane: Consider the mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^{5*}$ defined by

$$\varphi(u,v,w) = (u^2 - v^2,\ v^2 - w^2,\ w^2 - 2uv,\ uv - vw,\ vw - wu).$$

Then $\varphi^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ and φ induces a mapping $\bar{\varphi} : \mathbb{R}P^2 \to \mathbb{R}P^{4*}$, which is a linear projection of the Veronese surface in $\mathbb{R}P^{5*}$. Since the Jacobi matrix of φ,

$$J(u,v,w) = \begin{pmatrix} 2u & 0 & -2v & v & -w \\ -2v & 2v & -2u & u-w & w \\ 0 & -2w & 2w & -v & v-u \end{pmatrix},$$

is of rank 3, provided $(u,v,w) \neq \mathbf{0}$, we see that $\bar{\varphi}$ is an immersion. Set

$$F(X;u,v,w) = X_0(vw - wu) + X_1(uv - vw) + X_2(w^2 - 2uv) + X_3(v^2 - w^2) + X_4(u^2 - v^2),$$

$X = (X_0, X_1, \ldots, X_4)$. Then we see that $\bar{\varphi}$ induces a diffeomorphism between

$$L = \{(X;u,v,w) \in \mathbb{R}P^4 \times \mathbb{R}P^2 \mid F = \partial F/\partial u = \partial F/\partial v = \partial F/\partial w = 0\}$$

and the projective conormal bundle of the immersed manifold $\bar{\varphi}(\mathbb{R}P^2)$, and that $\rho : L \to \mathbb{R}P^4$ is an embedding. These follow from the fact that F defines a linear family of real plane quadratic curves without degenerate singular points, and that the 6×5 matrix

$$\begin{pmatrix} J(u,v,w) \\ \\ J(u',v',w') \end{pmatrix} = \begin{pmatrix} 2u & 0 & -2v & v & -w \\ -2v & 2v & -2u & u-w & w \\ 0 & -2w & 2w & -v & v-u \\ 2u' & 0 & -2v' & v' & -w' \\ -2v' & 2v' & -2u' & u'-w' & w' \\ 0 & -2w' & 2w' & -v' & v'-u' \end{pmatrix}$$

is of rank 5, provided $[u,v,w] \neq [u',v',w'] \in \mathbb{R}P^2$; the set of common zeros of its 5-minors in $\mathbb{R}^6$ coincides with the zero set of $(a+b)^2 + (b+c)^2 + (c+a)^2$, and therefore with the set $\{a = b = c = 0\}$, where a, b, c are 2-minors of

$$\begin{pmatrix} u & v & w \\ u' & v' & w' \end{pmatrix}.$$

Thus $A = \rho(L) \subset \mathbb{R}P^4$ is a closed smooth geometrical solution of Σ_0 with $\operatorname{rank}(\rho'|_{\tilde{A}}) = 2$, which is diffeomorphic to an $\mathbb{R}P^1$-bundle over $\mathbb{R}P^2$. $\square$

In the complex analytic case, a similar result as Theorems 1 and 3 holds without the rank condition:

Proposition 9. *Let $A \subset \mathbb{C}P^n$ be a complex closed (i.e. algebraic) hypersurface. Then A is a geometrical solution of the system Σ_0 if and only if A is a projective hyperplane in $\mathbb{C}P^n$.*

Proof. First remark that Lemma 6 is valid also in the complex analytic case. Now assume that the maximal rank m of $\rho'|_{\tilde{A}}$ is positive: $0 < m < n-1$. In the complex analytic case, by the similar argument as in Lemma 7, the family H represents a degenerate matrix for some $(X_0, \dots, X_{n-m-1}) \neq (0, \dots, 0)$. Then, similarly to the proof of Theorem 1, we see $\rho|_{\tilde{A}}$ is not an immersion, which leads to a contradiction. Thus we see that $m = 0$ and that A is a projective hyperplane. □

2. Singularities

Consider a map-germ $f : \mathbb{R}^2, 0 \to \mathbb{R}^5, 0$ defined by

$$(x, y, z; p, q) \circ f = (u, v^2, uv^3; v^3, \frac{3}{2}uv),$$

which is an integral mapping with respect to the contact form $\alpha = dz - pdx - qdy$ on $\mathbb{R}^5$, that is, $f^*\alpha = 0$.

Let M^5 be a contact manifold, and $g : \mathbb{R}^2, 0 \longrightarrow M$ an isotropic map-germ. Then g is called an *open umbrella* if g is transformed to the above map-germ f by a contact diffeomorphism-germ $(M, g(0)) \to (\mathbb{R}^5, 0)$ up to parametrization [A].

An open umbrella f appears as a parametrization of the projective conormal bundle of a space curve-germ of type $(2,3,4)$, and it has the isolated singular point and kernel rank one there. The corresponding geometric solution in $\mathbb{R}P^3$ is the developable surface of a curve of type $(1,2,4)$. Recall that, for a space curve-germ in $\mathbb{R}P^3$ we indicate the triplet of natural numbers (a_1, a_2, a_3), $1 \leq a_1 < a_2 < a_3$. Then the Legendre lift of the developable surface of the curve is non-singular if and only if $a_3 = a_2 + 1$ [I1, I2]. Therefore the case $(1,2,4)$ is the simplest among the case when the corresponding solutions of $rt - s^2 = 0$ have singularities.

Then we have the following:

Proposition 10. *An open umbrella can be a generalized solution of $rt - s^2 = 0$ but cannot be a generalized solution of $s = 0$.*

The following shows a significant characteristic of the open umbrella:

Lemma 11. *Let $f : \mathbb{R}^2, 0 \longrightarrow M, f(0)$ be an open umbrella and $H : M, f(0) \longrightarrow \mathbb{R}, 0$ be a C^∞ function-germ with $dH(f(0)) \neq 0$ and $H \circ f = 0$. Then $dH(f(0))$ determines the contact hyperplane in $T_{f(0)}M$.*

Proof. From the contact invariance, it suffices to check for the local model of the open umbrella. If we set $H(u, v^2, uv^3, v^3, (3/2)uv) \equiv 0$, then we have $(\partial H/\partial x)(0) = (\partial H/\partial y)(0) = (\partial H/\partial p)(0) = (\partial H/\partial q)(0) = 0$. $\square$

The Monge-Ampère equation $s = 0$ is given by

$$\alpha = dz - pdx - qdy = 0, \qquad \theta = dp \wedge dx = 0,$$

on $\mathbb{R}^5$. The following gives the crucial property of generalized solutions for $s = 0$:

Lemma 12. *Let $f : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}^5, 0$ be an isotropic map-germ. If f is a generalized solution of $s = 0$, (namely $f^*\alpha = f^*\theta = 0$), with $\mathrm{rank} T_0 f \geq 1$, then there exists a C^∞ function-germ $H : \mathbb{R}^5, 0 \longrightarrow \mathbb{R}, 0$ such that $H \circ f = 0$ and $dH(0)$ restricted to the contact hyperplane is not equal to zero.*

Proof. Take $v \in T_0 f(T_0\mathbb{R}^2)$, $v \neq 0$. Consider the projection $(x, p) : \mathbb{R}^5, 0 \to \mathbb{R}^2, 0$ and set $f' = (x, p) \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Then $\mathrm{rank} T_0 f' \leq 1$. Similarly setting $f'' = (y, q) \circ f$, we get $\mathrm{rank} T_0 f'' \leq 1$. Since $\langle v, dx \rangle, \langle v, dy \rangle, \langle v, dp \rangle$ and $\langle v, dq \rangle$ are not all zero, we see $\mathrm{rank} T_0 f' \geq 1$ or $\mathrm{rank} T_0 f'' \geq 1$. Therefore either f' or f'' is of constant rank one. If we choose a submersion $h : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}, 0$ with $h \circ f' = 0$ or $h \circ f'' = 0$, then it suffices to set $H = h(x, p)$ or $H = h(y, q)$ to see Lemma 12. $\square$

Proof of Proposition 10. Since an open umbrella is realized as the projective conormal bundle of a space curve, it is a solution of $rt - s^2 = 0$. On the other hand, by Lemmas 9 and 10, we see any open umbrella is not a solution of $s = 0$. $\square$

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