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Series #377. April 1997

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A relative transversality theorem and its applications

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Dedicated to Professor Tzee-Char Kuo on his 60th birthday.

1991 *Mathematics Subject Classification*: Primary 58C27; Secondary 58D10

Keywords: relative transversality theorem, relative immersion, relative embedding, generic collaring theorem.

1 Introduction.

The transversality theorems, Thom's transversality theorem ([21][22]) and Mather's multi-transversality theorem ([14]), are basic tools of the singularity theory of differentiable mappings: They are indispensable to clarify genericity conditions of mappings, in terms of jets. When we treat various spaces of differentiable mappings with several constraints, depending on the situations, we need to utilize appropriate "transversality theorems", whose validity depends, as a rule, on the given mapping spaces.

In the present paper we treat the space of differentiable mappings between manifolds with the constraint that a fixed submanifold is mapped into another fixed submanifold, or, with the constraint that the boundary is mapped into the boundary. Then naturally we encounter the need of a "relative transversality theorem".

There are several theorems known as relative transversality theorems, for instance, in [1][6][19]. However the general result seems to have not been explicitly published yet which is directly applicable to the approximation problem in the space of relative mappings. The purpose of the present paper is to formulate explicitly the relative transversality theorem as well as the relative multi-transversality theorem and to prove them. Moreover several applications are given, for instance, the relative immersion (resp. embedding) theorem, the collaring of generic mappings along boundaries.

Let N be an n-dimensional C^∞ manifold and P a p-dimensional C^∞ manifold. Furthermore fix an m-dimensional (resp. q-dimensional) properly embedded submanifold M of N (resp. Q of P). Then a *relative mapping* $f : (N, M) \to (P, Q)$ is a C^∞ mapping from N to P with $f(M) \subset Q$. Denote simply by $C^\infty(N, M; P, Q)$ the space of relative mappings $f : (N, M) \to (P, Q)$ such that $f : N \to P$ is proper, namely the inverse images by f of compact subsets of P are compact in N. We endow $C^\infty(N, M; P, Q)$ with the induced topology from the C^∞ Whitney topology on $C^\infty(N, P)$. Then $C^\infty(N, M; P, Q)$ is a Baire space. See Lemma 2.1. Thus a countable intersection of open dense subsets, namely a residual subset of $C^\infty(N, M; P, Q)$ is dense in $C^\infty(N, M; P, Q)$.

Let r be a non-negative integer. In the jet bundle $J^r(N, P)$, we consider the submanifold $J^r(N, M; P, Q)$ of r-jets along M of relative mappings $(N, M) \to (P, Q)$. Then $J^r(N, M; P, Q)$ is a fibration over $M \times Q$ with fiber $J^r(n, m; p, q)$, which is identified with the set of r-jets of germs of relative mappings $h : (\mathbf{R}^n, \mathbf{R}^m) \to (\mathbf{R}^p, \mathbf{R}^q)$ at $0 \in \mathbf{R}^m \subset \mathbf{R}^n$ with $h(0) = 0$. Here $\mathbf{R}^m = \{x_1 \dots, x_m, 0, \dots, 0\}$ and $\mathbf{R}^q = \{y_1, \dots, y_q, 0, \dots, 0\}$. For a relative mapping $f : (N, M) \to (P, Q)$, the jet section $j^r f : N \to J^r(N, P)$ maps M to $J^r(N, M; P, Q)$.

Then we have

Theorem 1.1 (Relative transversality theorem; single version):*For a given* *countable family S of submanifolds of $J^r(N,P)$, T of $J^r(N,M;P,Q)$ and U of $J^r(M,Q)$, there exists a residual subset R in $C^\infty(N,M;P,Q)$ such that any $f \in R$ satisfies that $j^r f : N \to J^r(N,P)$ is transverse to S on $N - M$, $(j^r f)|_M : M \to J^r(N,M;P,Q)$ is transverse to T and $j^r(f|_M) : M \to J^r(M,Q)$ is transverse to U.*

The relative multi-transversality theorem is also formulated naturally: Fix a positive integer s. For each pair (k,ℓ) of non-negative integer with $k+\ell \le s$, we set ${}_{k,\ell}J^r(N,M;P,Q) = J^r(N,M;P,Q)^k \times J^r(N,P)^\ell$ and denote by $M^{(k)} \times (N-M)^{(\ell)}$ the set of $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_k;\boldsymbol{b}_1,\dots,\boldsymbol{b}_\ell) \in N^{k+\ell}$ such that $\boldsymbol{a}_1,\dots,\boldsymbol{a}_k$ (resp. $\boldsymbol{b}_1,\dots,\boldsymbol{b}_\ell$) are disjoint points in M (resp. in $N-M$).

For a relative mapping $f : (N,M) \to (P,Q)$, we define the relative multi-jet section ${}_{k,\ell}j^r f : M^{(k)} \times (N-M)^{(\ell)} \to {}_{k,\ell}J^r(N,M;P,Q)$, by

$$ {}_{k,\ell}j^r f(\boldsymbol{a}_1,\dots,\boldsymbol{a}_k;\boldsymbol{b}_1,\dots,\boldsymbol{b}_\ell) = (j^r f(\boldsymbol{a}_1),\dots,j^r f(\boldsymbol{a}_k); j^r f(\boldsymbol{b}_1),\dots,j^r f(\boldsymbol{b}_\ell)), $$

and ${}_k j^r(f|_M) : M^{(k)} \to J^r(M,Q)^k$ by, as usual,

$$ {}_k j^r(f|_M)(\boldsymbol{a}_1,\dots,\boldsymbol{a}_k) = (j^r(f|_M)(\boldsymbol{a}_1),\dots,j^r(f|_M)(\boldsymbol{a}_k)). $$

Then we have

Theorem 1.2 (Relative multi-transversality theorem):
Let r be a non-negative integer and s a positive integer. For a given countable family $S_{k,\ell}$ of submanifolds of ${}_{k,\ell}J^r(N,M;P,Q)$, $k+\ell \le s$, and U_k of $J^r(M,Q)^k$, $k \le r$, there exists a residual subset R in $C^\infty(N,M;P,Q)$ such that any $f \in R$ satisfies that ${}_{k,\ell}j^r f : M^{(k)} \times (N-M)^{(\ell)} \to {}_{k,\ell}J^r(N,M;P,Q)$ is transverse to $S_{k,\ell}$, $k+\ell \le s$, and that ${}_k j^r(f|_M) : M^{(k)} \to J^r(M,Q)^k$ is transverse to U_k, $k \le r$.

Remark that Theorem 1.1 follows from Theorem 1.2 ($s = 1$).

As an easy consequence of Theorem 1.1 and Theorem 1.2, we have

Corollary 1.3 (Relative immersion (embedding) theorem):
Any relative mapping $f : (N, M) \to (\mathbf{R}^p, \mathbf{R}^q)$ *is approximated by a relative immersion (resp. embedding)* $g : (N, M) \to (\mathbf{R}^p, \mathbf{R}^q)$*, namely an immersion (resp. an embedding)* $g : N \to \mathbf{R}^p$*,* $g|_M : M \to \mathbf{R}^q$ *being an immersion (resp. an embedding) as well, for the* C^∞ *Whitney topology, if* $2n \le p$*,* $2m \le q$ *(resp.* $2n + 1 \le p$*,* $2m + 1 \le q$ *).*

Similarly to Theorem 1.2, we treat the case of manifolds with boundary: Let N (resp. P) be a manifold of dimension n (resp. p) with boundary ∂N (resp. ∂P). Denote by $C^\infty(N, \partial N; P, \partial P)$ the space of proper C^∞ mappings $f : N \to P$ with $f(\partial N) \subset \partial P$. Actually we extend N (resp. P) to an open manifold $\tilde{N}$ (resp. $\tilde{P}$) such that ∂N (resp. ∂P) is a C^∞ hypersurface of $\tilde{N}$ (resp. $\tilde{P}$), and regard $C^\infty(N, \partial N; P, \partial P)$ as a subspace of $C^\infty(\tilde{N}, \partial N; \tilde{P}, \partial P) \subset C^\infty(\tilde{N}, \tilde{P})$. Then $C^\infty(N, \partial N; P, \partial P)$ endowed with the C^∞ Whitney topology is a Baire space. See Lemma 2.1. We define ${}_{k,\ell}J^r(N, \partial N; P, \partial P)$ just as the relative jet bundle defined above, and define the jet sections ${}_{k,\ell}j^r f$ similarly as above. Then we have the following, word for word:

Theorem 1.4 (Transversality theorem; the case with boundary): *Let* r *be a non-negative integer and* s *a positive integer. For a given countable family of manifolds* $S_{k,\ell}$ *of* ${}_{k,\ell}J^r(N, \partial N; P, \partial P)$*,* $k + \ell \le s$*, and a countable family of manifolds* U_k *of the jet bundle over the boundary* $J^r(\partial N, \partial P)^k$*,* $k \le r$*, there exists a residual subset* R *of* $C^\infty(N, \partial N; P, \partial P)$ *such that any* $f \in R$ *satisfies that* ${}_{k,\ell}j^r f : M^{(k)} \times (N - \partial N)^{(\ell)} \to {}_{k,\ell}J^r(N, \partial N; P, \partial P)$ *is transverse to* $S_{k,\ell}$*,* $k+\ell \le s$*, and* ${}_k j^r(f|_{\partial N}) : (\partial N)^{(k)} \to J^r(\partial N, \partial P)^k$ *is transverse to* U_k*,* $k \le r$*.*

As a remarkable consequence of Theorem 1.4, we have the topological collaring theorem of generic mappings along boundaries, in the framework of Mather [16][17]:

Theorem 1.5 (Generic topological collaring theorem): *Assume ∂N and ∂P are compact. Then there exists a residual subset $R \subset C^\infty(N, \partial N; P, \partial P)$ such that any $f : (N, \partial N) \to (P, \partial P)$ belonging to R satisfies the following conditions:* (0) $f^{-1}(\partial P) = \partial N$, (1) $f|_{N-\partial N} : N - \partial N \to P - \partial P$ *is topologically stable,* (2) $f|_{\partial N} : \partial N \to \partial P$ *is topologically stable, and* (3) *There exist a topological collaring $\varphi : \partial N \times [0, \varepsilon) \to N$ of N near ∂N and $\psi : \partial P \times [0, \varepsilon) \to P$ of P near ∂P, for some $\varepsilon > 0$, with the properties that $f(\varphi(\partial N \times [0, \varepsilon))) \subset \psi(\partial P \times [0, \varepsilon))$ and that*

$$\psi^{-1} \circ f \circ \varphi = (f|_{\partial N}) \times 1 : \partial N \times [0, \varepsilon) \to \partial P \times [0, \varepsilon).$$

Remark that Theorem 1.5 does not hold in the relative case: For instance, the relative map-germ $f : (\mathbf{R}^2, \mathbf{R}, 0) \to (\mathbf{R}^2, \mathbf{R}, 0)$ defined by $f(x, u) = (x, u^2 + xu)$ generically appears, which does not admit any collaring.

We also see that,

Theorem 1.6 (Generic differentiable collaring theorem): *In Theorem 1.5, if $(n-1, p-1)$ belongs to the nice range in the sense of Mather, then φ and ψ can be taken to be C^∞ collarings.*

This result is originally suggested by O. Saeki, in the case $n = 2, p = 3$, which relates to a generalization of results in the paper [2]:

Theorem 1.7 *Let N be a compact surface with boundary ∂N, P a compact 3-manifold with boundary ∂P, and $f : (N, \partial N) \to (P, \partial P)$ a relative mapping. Then f is approximated by a relative mapping $g : (N, \partial N) \to (P, \partial P)$ via the C^∞-topology such that* (0) $g^{-1}(\partial P) = \partial N$, (1) $g|_{N-\partial N} : N - \partial N \to P - \partial P$ *has only the cross-caps, the transverse self-intersections and the generic triple-points as singularities,* (2) $g|_{\partial N} : \partial N \to \partial P$ *is an immersion with transverse self-intersections, and* (3) *the germ of g along ∂N is*

C^∞ equivalent to that of $(g|_{\partial N}) \times 1 : (\partial N) \times [0, \varepsilon) \to (\partial P) \times [0, \varepsilon)$ *along* $(\partial N) \times \{0\}$.

In this paper we use the following terminology: Let $f : (N, M, \boldsymbol{a}) \to (P, Q, \boldsymbol{b})$ and $g : (N', M', \boldsymbol{a}') \to (P', Q', \boldsymbol{b}')$ be relative map-germs. Then f and g are called *equivalent* (resp. *topologically equivalent*) if there exist relative C^∞ diffeomorphism-germs (resp. homeomorphism-germs) $\sigma : (N, M, \boldsymbol{a}) \to (N', M', \boldsymbol{a}')$ and $\tau : (P, Q, \boldsymbol{b}) \to (P', Q', \boldsymbol{b}')$ such that $\tau \circ f = g \circ \sigma$. We use the same terminology also for relative mappings and for relative map-germs along subsets.

In the next section we give proofs of all results of this paper.

We will show Theorems 1.5, 1.6, 1.7, using Theorem 1.4. Alternatively we can prove these results directly from the method of proof of Theorem 1.4 given in the next section.

Transversality in the equivariant case is closely related to that in the relative case; see [7][3][24][19]. Several other notions of the singularity theory, for instance, finite determinacy, stability and versality are studied, in the relative case, by several authors; see [5][4]. In geometrical applications of singularity theory, there appear various variants of transversality theorems. See [23]. In the previous paper [12], it is given the transversality theorem for isotropic mappings. We also remark that some transversality argument with constraint is powerful for the local study of differentiable mappings. See [9].

The author is grateful to O. Saeki for the question which motivates the present paper, to S. Izumiya for valuable suggestion on the references, and to the organizers of the Kuo symposium for the continuous effort.

2 Proofs of Results

First of all, we start with the following:

Lemma 2.1 *$C^\infty(N, M; P, Q)$ (resp. $C^\infty(N, \partial N; P, \partial P)$) is a Baire space.*

Proof: This lemma is similarly proved as in [18] or Proposition 3.1 of [14]: In fact it is sufficient, for the proof of Lemma 2.1, to remark that, if $f_i \in C^\infty(N, M; P, Q)$ (resp. $C^\infty(N, \partial N; P, \partial P)$), then $f = \lim_{i\to\infty} f_i$ belongs to $C^\infty(N, M; P, Q)$ (resp. $C^\infty(N, \partial N; P, \partial P)$). □

Since Theorem 1.1 follows from Theorem 1.2, we will show Theorem 1.2:

Proof of Theorem 1.2 : We denote by $P(n; p; r)$ the affine space of polynomial mappings $h : \mathbf{R}^n \to \mathbf{R}^p$ of degree $\leq r$; $\dim P(n; p; r) = p \cdot {}_{n+r}C_n$. Taking coordinates $x_1, \ldots, x_n$ of $\mathbf{R}^n$ and $y_1, \ldots, y_p$ of $\mathbf{R}^p$, We define coordinate functions $c_{i\alpha} : P(n; p; r) \to \mathbf{R}$, $1 \leq i \leq p, \alpha \in \mathbf{Z}^n_+, |\alpha| \leq r$, by $c_{i\alpha}(h) = (1/\alpha!)(\partial^{|\alpha|}/\partial x^\alpha)(0)$, $(h \in P(n; p; r))$.

We set $\mathbf{R}^m = \{x_{m+1} = 0, \ldots, x_n = 0\}$ (resp. $\mathbf{R}^q = \{y_{q+1} = 0, \ldots, y_p = 0\}$). Consider the set $P(n, m; p, q; r)$ of relative polynomial mappings $h : (\mathbf{R}^n, \mathbf{R}^m) \to (\mathbf{R}^p, \mathbf{R}^q)$ of degree $\leq r$. Then $P(n, m; p, q; r)$ is an affine subspace of $P(n; p; r)$ of codimension $(p-q) \cdot {}_{m+r}C_m$ defined by $c_{i\alpha} = 0$, $(q+1 \leq i \leq p,\ \alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0), |\alpha| \leq r)$. Moreover $J^r(n, m; p, q)$ is (non-canonically) identified with the affine subspace in $P(n, m; p, q; r)$ of codimension q defined by $c_{i\mathbf{0}} = 0, (1 \leq i \leq q)$.

Now, the proof of relative transversality theorem is achieved by replacing $P(n; p; r)$ by $P(n, m; p, q; r)$ in the proof of ordinary transversality theorem (see [14], see also [11][20][13]).

To prove Theorem 1.2, it suffices to show that, for each relative mapping $f : (N, M) \to (P, Q)$ and for each point $(\boldsymbol{a}_0, \boldsymbol{b}_0) \in M^{(k)} \times (N - M)^{(\ell)}$, there exist a manifold E, $e_0 \in E$, and a continuous mapping $\varphi : (E, e_0) \to (C^\infty(N, M; P, Q), f)$ such that the induced mappings

$$\Phi : E \times M^{(k)} \times (N - M)^{(\ell)} \to {}_{k,\ell}J^r(N, M; P, Q),$$

defined by $\Phi(e, \boldsymbol{a}, \boldsymbol{b}) = {}_{k,\ell}j^r\varphi(e)(\boldsymbol{a}, \boldsymbol{b})$ is submersive at $(e_0, \boldsymbol{a}_0, \boldsymbol{b}_0)$ and

$$\Psi : E \times M^{(k)} \to J^r(M, Q)^k$$

defined by $\Psi(e, \boldsymbol{a}) = {}_kj^r(\varphi(e)|_M)(\boldsymbol{a})$ is also a submersion. (See Lemma 3.2 and the argument in Proposition 3.3 of [14].) Then, the key point is the following, which is easy to show:

Lemma 2.2 *Let $U \subset \mathbf{R}^n$ be an open subset. Set $V = U \cap \mathbf{R}^m$. Let $f : (U, V) \to (\mathbf{R}^p, \mathbf{R}^q)$ be a relative mapping. Then the mapping*

$$F : P(n, m; p, q; r)^k \times P(n; p; r)^\ell \times V^{(k)} \times (U - V)^{(\ell)} \to {}_{k,\ell}J^r(U, V; \mathbf{R}^p, \mathbf{R}^q)$$

defined by

$$F(h, g; \boldsymbol{a}, \boldsymbol{b}) = (j^r(f + h)(\boldsymbol{a}), j^r(f + g)(\boldsymbol{b})),$$

is a submersion. Furthermore $G : P(n, m; p, q; r)^k \times V^{(k)} \to {}_kJ^r(V, \mathbf{R}^q)$ defined by $G(h; \boldsymbol{a}) = j^r((f + h)|_M)(\boldsymbol{a})$ is also a submersion.

Then, for the proof of Theorem 1.2, it is sufficient to set E as an open neighborhood in $P(n, m; p, q)^k \times P(n, p)^\ell$ of $e_0 = \mathbf{0}$, (resp. in $P(n, m; p, q)^k$ of $e_0 = \mathbf{0}$) and construct Φ (resp. Ψ) from F (resp. G) of Lemma 2.2, using partitions of unity.

In fact we follow Mather's argument (Proposition 3.3 of [14]) as follows: Let S (resp. U) be submanifold of ${}_{k,\ell}J^r(N, M; P, Q)$ (resp. $J^r(M, Q)^k$). Choose a countable covering $\{S_\alpha\}$ of S (resp. $\{U_\beta\}$ of U) by compact submanifolds with corners such that, for each α (resp. β),

$$\pi(S_\alpha \cap \pi^{-1}(M^{(k)} \times Q^k \times (N - M)^{(\ell)} \times P^\ell)) \subset \prod V_j \times \prod W_j \times \prod V'_i \times \prod W'_i,$$

for some disjoint coordinate charts $V_1, \ldots, V_k$ in M, $V'_1, \ldots, V'_\ell$ in $N - M$, $W_1, \ldots, W_k$ in Q, and $W'_1, \ldots, W'_\ell$ in P, respectively, (resp.

$$\pi(U_\beta \cap \pi^{-1}(M^{(k)} \times Q^k)) \subset \prod V_j \times \prod W_j),$$

where

$$\pi : \ {}_{k,\ell}J^r(N, M; P, Q) \to (M \times Q)^k \times (N \times P)^\ell,$$

(resp.

$$\pi : J^r(M, Q)^k \to (M \times Q)^k),$$

denotes the natural projection to the base manifold. Moreover we assume $V'_1, \ldots, V'_\ell$ are relatively compact in $N - M$, taking a refinement of $\{S_\alpha\}$ if necessary.

Then we shall show that

$$T_\alpha = \{f \in C^\infty(N, M; P, Q) \mid \ {}_{k,\ell}j^r f \text{ is transverse to } S_\alpha\}$$

(resp.

$$T_\beta = \{f \in C^\infty(N, M; P, Q) \mid \ {}_k j^r(f|_M) \text{ is transverse to } U_\beta\}),$$

is open and dense. Then it suffices to set R as the countable intersection of all T_α and all T_β for all submanifolds belonging the countable family $S_{k,\ell}$ and U_k.

It is clear that T_α and T_β are open. Therefore we will show that they are dense, according to Lemma 3.2 of [14]. For this, fix $f \in C^\infty(N, M; P, Q)$.

Now take coordinate charts $\tilde{V}_j$ of N and $\tilde{W}_j$ of P, for each j, such that $\tilde{V}_j \cap M = V_j$ and $\tilde{W}_j \cap Q = W_j$, for each j, and that $\tilde{V}_1, \ldots, \tilde{V}_k, V'_1, \ldots, V'_\ell$ are disjoint.

Denote by $S_{\alpha,j}$ (resp. $S''_{\alpha,i}$, $S'_{\alpha,j}$, $S'''_{\alpha,i}$, $U_{\beta,j}$, $U'_{\beta,j}$) the image of $S_\alpha \cap \pi^{-1}(M^{(k)} \times Q^k \times (N-M)^{(\ell)} \times P^\ell)$ or $U_\beta \cap \pi^{-1}(M^{(k)} \times Q^k)$ under the projection to V_j (resp. V'_i, W_j, W'_i, V_j, W_j). Choose a C^∞ function ρ_j (resp. ρ''_i, ρ'_j,

ρ_i''', κ_j, κ_j') on $\tilde{V}_j$ (resp. V_i', $\tilde{W}_j$, W_i', $\tilde{V}_j$, $\tilde{W}_j$) with values in $[0,1]$ and with compact support, being identically one in a neighborhood of $S_{\alpha,j}$ (resp. $S''_{\alpha,i}$, $S'_{\alpha,j}$, $S'''_{\alpha,i}$, $U_{\beta,j}$, $U'_{\beta,j}$). Denote charts we have took by $\xi_j : \tilde{V}_j \to \mathbf{R}^n$ (resp. $\xi_i'' : V_i' \to \mathbf{R}^n$, $\xi_j'' : \tilde{W}_j \to \mathbf{R}^p$, $\xi_i''' : W_i' \to \mathbf{R}^p$).

Taking a sufficiently small open neighbourhood E in $P(n,m;p,q)^k \times P(n,p)^\ell$ of $e_0 = \mathbf{0}$, (resp. in $P(n,m;p,q)^k$ of $e_0 = \mathbf{0}$), we define $\varphi : E \to C^\infty(N,M;P,Q)$ (resp. $\psi : E \to C^\infty(N,M;P,Q)$) as follows: If $\boldsymbol{a} \in N - ((\cup\tilde{V}_j) \cup (\cup V_i'))$, or $\boldsymbol{a}$ belongs to one of $\tilde{V}_j$ but $f(\boldsymbol{a})$ does not belongs to $\tilde{W}_j$, or $\boldsymbol{a}$ belongs to one of $\tilde{V}_i'$ but $f(\boldsymbol{a})$ does not belongs to $\tilde{W}_i'$, then set $\varphi(e)(\boldsymbol{a}) = f(\boldsymbol{a})$. If $\boldsymbol{a} \in \tilde{V}_j$ and $f(\boldsymbol{a}) \in \tilde{W}_j$ then we set

$$\varphi(e)(\boldsymbol{a}) = \xi_j''^{-1}(\rho_j(\boldsymbol{a})\rho_j''(f(\boldsymbol{a}))e_j(\xi_j(\boldsymbol{a})) + \xi_j''(f(\boldsymbol{a})).$$

If $\boldsymbol{a} \in V_i'$ and $f(\boldsymbol{a}) \in W_i'$ then we set

$$\varphi(e)(\boldsymbol{a}) = \xi_i'''^{-1}(\rho_i'(\boldsymbol{a})\rho_i'''(f(\boldsymbol{a}))e_i'(\xi_i'(\boldsymbol{a})) + \xi_i'''(f(\boldsymbol{a})).$$

Similarly, we define ψ as follows: If $\boldsymbol{a} \in N - (\cup\tilde{V}_j)$, or $\boldsymbol{a}$ belongs to one of $\tilde{V}_j$ but $f(\boldsymbol{a})$ does not belongs to $\tilde{W}_j$, then set $\psi(e)(\boldsymbol{a}) = f(\boldsymbol{a})$. If $\boldsymbol{a} \in \tilde{V}_j$ and $f(\boldsymbol{a}) \in \tilde{W}_j$ then we set

$$\psi(e)(\boldsymbol{a}) = \xi_j''^{-1}(\kappa_j(\boldsymbol{a})\kappa_j'(f(\boldsymbol{a}))e_j(\xi_j(\boldsymbol{a})) + \xi_j''(f(\boldsymbol{a})).$$

Here we have set

$$e = (e_1, \ldots, e_k; e_1', \ldots, e_\ell') \in E \subset P(n,m;p,q)^k \times P(n,p)^\ell,$$

(resp.

$$e = (e_1, \ldots, e_k) \in E \subset P(n,m;p,q)^k.)$$

Then $\varphi(e)$ and $\psi(e)$ are well-defined and C^∞. Also, by the same argument of p.312 of [14], it is easy to see that φ and ψ are continuous.

Set $F = C^\infty(N,M;P,Q)$ and denote by

$$j : F \to C^\infty(M^{(k)} \times (N-M)^{(\ell)},\ {}_{k,\ell}J^r(N,M;P,Q)),$$

(resp.

$$j : F \to C^\infty(M^{(k)}, J^r(M,Q)^k)),$$

the mapping defined by $j(f) = {}_{k,\ell}j^r f$ (resp. $j(f) = {}_kj^r(f|_M)$). Then the induced mapping

$$\Phi : E \times M^{(k)} \times (N-M)^{(\ell)} \to {}_{k,\ell}J^r(N,M;P,Q)$$

(resp.

$$\Psi : E \times M^{(k)} \to J^r(M,Q)^k),$$

defined by $\Phi(e,v) = j\varphi(e)(v)$ (resp. $\Psi(e,v) = j\psi(e)(v)$), is C^∞ and transverse to S_α (resp. U_β), by Lemma 2.2. This completes the proof of Theorem 1.2. □

Proof of Corollary 1.3: Let $m \le q$ and $n \le p$. Let $\Sigma \subset J^1(n,m;p,q)$ be the set of linear mappings $h : \mathbf{R}^n \to \mathbf{R}^p$ with $h(\mathbf{R}^m) \subset \mathbf{R}^q$ with non-trivial kernel. Then Σ is an algebraic set of $J^1(n,m;p,q)$ of codimension $\ge \min\{q-m+1, p-n+1\}$.

In fact, $\Sigma = \Sigma' \cap J^1(n,m;p,q)$, where $\Sigma' \subset J^1(n,p)$ is the set of linear mappings $\mathbf{R}^n \to \mathbf{R}^p$ with non-trivial kernel, and Σ' is an algebraic subset of $J^1(n,p)$ of codimension $p-n+1$ as well-known. Moreover Σ' is transversal to $J^1(n,m;p,q)$ along $J^1(n,m;p,q) - \pi^{-1}(\Sigma'')$, where $\pi : J^1(n,m;p,q) \to J^1(m,q)$ is the projection defined by the restriction, and $\Sigma'' \subset J^1(m,q)$ is the set of linear mappings $\mathbf{R}^m \to \mathbf{R}^q$ with non-trivial kernel. Since $\pi^{-1}(\Sigma'')$ is of codimension $q-m+1$ in $J^1(n,m;p,q)$, we have the required result.

Moreover Σ is invariant under the natural equivalence relation of relative-jets. Thus we get $\Sigma(N,M;P,Q)$ as the set of 1-jets $j^1f(\boldsymbol{a})$ of relative non-immersive map-germs $f : (N,M,\boldsymbol{a}) \to (P,Q,f(\boldsymbol{a}))$.

Now, by Theorem 1.1, any relative mapping $(N,M) \to (P,Q)$ is approximated by a relative mapping $g : (N,M) \to (P,Q)$ such that $(j^1g)|_{N-M} : N-M \to J^1(N,P)$ is transverse to $\Sigma'(N,P)$, $j^1(g|_M) : M \to J^1(M,Q)$ is transverse to $\Sigma''(M, Q)$ and that $(j^1 g)|_M : M \to J^1(N, M; P, Q)$ is transverse to $\Sigma(N, M; P, Q)$ (as a stratified set). If $2m \leq n$ and $2q \leq p$, then we see that $g|_{N-M}$ is an immersion, $g|_M$ is an immersion and that g is an immersion along M; therefore also g itself, as well as $g|_M$, is an immersion.

Consider $\Delta \subset P^2$ (resp. $\Delta' \subset Q^2$, $\Delta'' \subset Q \times P$), the set of disjoint pairs of points, and $\pi^{-1}\Delta \subset {}_{0,2}J^0(N, M; P, Q) = N^2 \times P^2$ (resp. $\pi^{-1}\Delta' \subset {}_{2,0}J^0(N, M; P, Q) = M^2 \times Q^2$, $\pi^{-1}\Delta'' \subset {}_{1,1}J^0(N, M; P, Q) = (M \times Q) \times (N \times P)$), where $\pi : {}_{0,2}J^0(N, M; P, Q) \to P^2$ (resp. $\pi : {}_{2,0}J^0(N, M; P, Q) \to Q^2$, $\pi : {}_{1,1}J^0(N, M; P, Q) \to Q \times P$) is the natural projection. Then $\pi^{-1}\Delta$ (resp. $\pi^{-1}\Delta'$, $\pi^{-1}\Delta''$) is of codimension p (resp. q, p). By Theorem 1.2, we can pose to g further conditions that ${}_{0,2}j^0 g$ is transverse to $\pi^{-1}\Delta$ on $(N - M)^2$, that ${}_{2,0}j^0 g$ is transverse to $\pi^{-1}\Delta'$ on M^2 and that ${}_{1,1}j^0 g$ is transverse to $\pi^{-1}\Delta''$ on $M \times (N - M)$. If $2m+1 \leq q, 2n+1 \leq p$, then clearly $m+n+1 \leq p$. Therefore the transversalities imply that g is injective on the whole N. Hence g is a relative embedding. $\square$

Proof of Theorem 1.4: In the case N and P are manifolds with boundary $M = \partial N$ and $Q = \partial P$, the proof of the transversality theorem can be modified appropriately: Let $V \subset \mathbf{R}^n$ be a compact neighbourhood of 0. Then, in the manifold $P(n, n-1; p, p-1; r)$ (resp. $P(n; p; r)$), the set of polynomial mappings h with $h(\mathbf{R}^n_+ \cap V) \subset \mathbf{R}^p_+$, (resp. $h(V) \subset \mathbf{R}^p_+$), contains an open subset with $0 \in P(n, n-1; p, p-1; r)$ (resp. $0 \in P(n; p; r)$) in its closure. Here $\mathbf{R}^n_+ = \{x_n \geq 0\}$ and $\mathbf{R}^p_+ = \{y_p \geq 0\}$. In fact, set

$$P_+ = \{h \in P(n, n-1; p, p-1; r) \mid y_p \circ h = x_n \theta(x_1, \ldots, x_n) \text{ with } \theta|_V > 0\},$$

(resp.

$$P'_+ = \{h \in P(n; p; r) \mid y_p \circ h|_V > 0\}).$$

Then P_+ (resp. P'_+) is an open sebset of $P(n, n-1; p, p-1; r)$ (resp. $P(n; p; r)$). If $h \in P_+$, then $h(\mathbf{R}^n_+ \cap V) \subset \mathbf{R}^p_+$. If $h \in P'_+$, then $h(V) \subset \mathbf{R}^p_+$. Moreover, for each $\varepsilon > 0$, the polynomial $h \in P(n, n-1; p, p-1; r)$ (resp.

$P(n;p;r)$) defined by $y_j \circ h = 0, 1 \leq j \leq p-1$, and $y_p \circ h = \varepsilon x_n$, (resp. $y_p \circ h = \varepsilon$), belongs to P_+ (resp. P'_+).

Then we follow the same argument of the proof of Theorem 1.2, for the space of relative mappings $C^\infty(\tilde{N}, \partial N; \tilde{P}, \partial P)$, and regard $C^\infty(N, \partial N; P, \partial P)$ as a subspace of $C^\infty(\tilde{N}, \partial N; \tilde{P}, \partial P)$.

In the situation of Lemma 3.2 of [14], there exists an open submanifold $E_+ \subset E$ such that $\varphi(E_+) \subset C^\infty(N, \partial N; P, \partial P)$ and that $e_0 \in \overline{E_+}$. Then, in the argument based on Sard theorem, it suffices to take a regular value $e \in E$ belonging to E_+ arbitrarily close to e_0. □

Proof of Theorems 1.5 1.6 1.7: Consider $N \times \partial P \subset J^0(N, P)$. Assume j^0 is transverse to $N \times \partial P$ on $N - \partial N$. If $(j^0 f^{-1}(N \times \partial P)) \cap (N - \partial N) \neq \emptyset$, then there exists a point $\boldsymbol{a} \in N - \partial N$ such that f is transverse to ∂P. This contradicts to that $f(N) \subset P$. Thus we see that $f(N - \partial N) \subset P - \partial P$, and equivalently, $f^{-1}(\partial P) \subset \partial N$.

Mather's topological stability theorem [16][17][10][8] implies that, if $f|_{\partial N}$ (resp. $f|_{N-\partial N}$) satisfies certain multi-transversality conditions, then $f|_{\partial N}$ (resp. $f|_{N-\partial N}$) is topologically stable.

Define the subset $\Sigma_{\mathrm{rel}}(N, \partial N; P, \partial P)$ of $J^1(N, \partial N; P, \partial P)$ as the collection of 1-jets $j^1 f(\boldsymbol{a})$ of $f : (N, \partial N) \to (P, \partial P)$ at $\boldsymbol{a} \in \partial N$ satisfying $\partial(y_p \circ f)/\partial x_n = 0$, for some (hence any) coordinates $x_1, \ldots, x_{n-1}, x_n$ of N centered at $\boldsymbol{a}$, N being defined by $x_n \geq 0$, and $y_1, \ldots, y_{p-1}, y_p$ of P centered at $f(\boldsymbol{a})$, P being defined by $y_p \geq 0$. Then $\Sigma_{\mathrm{rel}}(N, \partial N; P, \partial P)$ is a smooth hypersurface of $J^1(N, \partial N; P, \partial P)$. Assume that the relative jet section $(j^1 f)|_{\partial N} : \partial N \to J^1(N, \partial N; P, \partial P)$ is transverse to $\Sigma_{\mathrm{rel}}(N, \partial N; P, \partial P)$. Then we claim that the inverse image

$$\Sigma_{\mathrm{rel}}(f) = ((j^1 f)|_{\partial N})^{-1}(\Sigma_{\mathrm{rel}}(N, \partial N; P, \partial P)) = \emptyset.$$

In fact, by the transversality, if $\boldsymbol{a} \in \Sigma_{\mathrm{rel}}(f)$, then, near $\boldsymbol{a}$, there exists a point $\boldsymbol{a}' \in \partial N$ with $\partial(y_p \circ f)/\partial x_n < 0$. This contradicts to the fact that always $\partial(y_p \circ f)/\partial x_n \geq 0$ for $f : (N, \partial N) \to (P, \partial P)$. Thus we see that f is an unfolding of $f|_{\partial N}$. Since $f|_{\partial N}$ is topologically stable, we see that f is a topological suspension of $f|_{\partial N}$ near ∂N.

If f satisfies all required transversality conditions, then f satisfies conditions (0), (1), (2), (3). Therefore Theorem 1.4 implies Theorem 1.5.

If $(n-1, p-1)$ belongs to the nice range in the sense of Mather [14][15], then "the codimension of K-moduli" $\sigma(n-1, p-1) > n-1$, and the multi-transversality of $f|_{\partial N}$ to the stratification by K-orbits implies that $f|_{\partial N}$ is C^∞ stable [14]. Moreover, if $(j^1 f)|_{\partial N}$ is transverse to $\Sigma_{\text{rel}}(N, \partial N; P, \partial P) (\subset J^1(N, \partial N; P, \partial P))$, then f is a one-parameter unfolding of $f|_{\partial N}$ near ∂N, therefore f is a C^∞ suspension of $f|_{\partial N}$ near ∂N. This proves Theorem 1.6.

Theorem 1.7 follows directly from Theorem 1.6, if we set $n = 2, p = 3$. □

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