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# On the Zeroes of Solutions of an Extremal Problem in $H^1$

Jyunji Inoue and Takahiko Nakazi

To the memory of Professor Katsutoshi Takahashi

Abstract. For a non-zero function $f$ in $H^1$, the classical Hardy space on the unit disc, we put

$$\mathcal{S}^f = \{g \in H^1 : \arg f(e^{i\theta}) = \arg g(e^{i\theta}) \ \ a.e. \ \ \theta\}.$$

The intersection of $\mathcal{S}^f$ and the unit sphere in $H^1$ is just a set of solutions of some extremal problem in $H^1$. It is known that $\mathcal{S}^f$ can be represented in the form $\mathcal{S}^f = \mathcal{S}^{\mathcal{B}} \times g_0$, where $\mathcal{B}$ is a Blaschke product and $g_0$ is a function in $H^1$ with $\mathcal{S}^{g_0} = \{\lambda \cdot g_0 : \lambda > 0\}$. Also it is known that the linear span of $\mathcal{S}^f$ is of finite dimensional if and only if $\mathcal{B}$ is a finite Blaschke product, and when $\mathcal{B}$ is a finite Blaschke product, we can describe completely the set $\mathcal{S}^{\mathcal{B}}$ and the zeros of functions in $\mathcal{S}^{\mathcal{B}}$.

In this paper, we study the set of zeros of functions in $\mathcal{S}^{\mathcal{B}}$ when $\mathcal{B}$ is an infinite Blaschke product whose set of singularities is not the whole circle. Especially we study the behavior of zeros of functions in $\mathcal{S}^{\mathcal{B}}$ in the sectors of the form: $\Delta = \{re^{i\theta} : 0 < r \leq 1, c_1 < \theta < c_2\}$ on which the zeros of $\mathcal{B}$ has no accumulation points, and establish a convergence order theorem of zeros in $\Delta$ of functions in $\mathcal{S}^{\mathcal{B}}$.

## 1. Introduction and preliminary results

$N_+$ and $H^p$ for $0 < p \leq \infty$ denote the Smirnov class and the Hardy spaces on the open unit disc $D$ in the complex plane $C$, respectively. A function $h$ in $N_+$ is called outer if it is invertible in $N_+$. A function $q$ in $N_+$ is called inner if $|\, q \,| = 1$ a.e. on $\partial D$.

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For a non-zero function $f$ in $H^1$, put

$$\mathcal{S}^f = \{g \in H^1 : \arg f(e^{i\theta}) = \arg g(e^{i\theta}) \text{ a. e. on } \partial D\}$$

The intersection of $\mathcal{S}^f$ and the unit sphere in $H^1$ is just a set of solutions of an extremal problem about a continuous linear functional on $H^1$ (cf. [1]). The set $\mathcal{S}^f$ were studied by several authors (cf. [1], [3], [4], [6] and [7]). Hayashi [3], [4] showed that there exists a Blaschke product $\mathcal{B}$ and an outer function $g_0$ in $H^1$ such that

$$\mathcal{S}^f = \mathcal{S}^{\mathcal{B}} \times g_0 \quad \text{and} \quad \mathcal{S}^{g_0} = \{\lambda \cdot g_0 : \lambda > 0\}.$$

Hence it is important to study $\mathcal{S}^f$ when $f$ is a Blaschke product.

If $\mathcal{B}$ is a finite Blaschke product, each function $f$ in $\mathcal{S}^{\mathcal{B}}$ is analytic on $D \cup \partial D$ and so the set of zeros of $f$ in $D \cup \partial D$ consists of finite points, and we can describe completely the set $\mathcal{S}^{\mathcal{B}}$ and the zeros of functions in $\mathcal{S}^{\mathcal{B}}$ (cf. [7]). When $\mathcal{B}$ is an infinite Blaschke product, we need further considerations and new ideas to study the zeros of functions in $\mathcal{S}^{\mathcal{B}}$.

For each $f$ in $H^1$, sing $(f)$ denotes the set of points of $\partial D$ on which $f$ cannot be analytically extended.

In this paper, we consider the case in which Blaschke products $\mathcal{B}$ have the property $\text{sing}(\mathcal{B}) \neq \partial D$.

The following is the first elementary result we need, which is probably known.

**Proposition 1.** *If $Q$ is an inner function and if $f$ is a function in $\mathcal{S}^Q$, we have*

$$\text{sing}(Q) = \text{sing}(f).$$

Proof. Since $f \in \mathcal{S}^{\mathcal{B}}, f/Q$ is nonnegative a.e. on $\partial D$. By [2]; Lemma 4.2, $f/Q$ extends analytically across any open arc $J$ such that $J \subset \partial D \setminus \text{sing}(Q)$, and since $Q$ is analytic on $\partial D \setminus \text{sing}Q$ we have $\text{sing}(f) \subseteq \text{sing}(Q)$. By the same method, if $\beta \in \partial D \setminus \text{sing}(f)$, $Q/f$ has a meromorphic extension to a neighborhood $V$ of $\beta$, and so $Q$ also has a meromorphic extension on $V$. But, since $Q$ is bounded on $V \cap D, Q$ is in fact analytic on $V$, that is $\beta \notin \text{sing}(Q)$. Q.E.D.

**Definition 1.** For a function $f$ in $H^1$ and a positive integer $n$, we say that $\alpha \in \partial D$

is a zero of order $n$ of $f$ if

$$\frac{f(z)}{(z-\beta)^{2n}} \in H^1 \text{ and } \frac{f(z)}{(z-\beta)^{2n+2}} \notin H^1$$

hold. The set of all the zeros of $f$ on $\partial D$ counted according to its order is denoted by $Z(f;\partial D)$. $Z(f;D)$ denotes the usual zeros in $D$ of $f$ and put

$$Z(f;\overline{D}) = Z(f;D) \cup Z(f;\partial D).$$

## 2. Lemmas to the main theorem.

In this section, we collect lemmas which we use in the proof of the main theorem in the next section.

**Lemma 1** *Let $b$ be a Blaschke product :*

$$b(z) = \prod_n \frac{-\overline{z_n}}{\mid z_n \mid} \frac{z-z_n}{1-\overline{z_n}z} \quad z_n = r_n e^{ix_n} \quad (n = 1, 2, ...).$$

*We put*

$$g_b(z) := \prod_n \frac{(1-e^{-ix_n}z)^2}{(1-r_n e^{-ix_n}z)^2}. \tag{1}$$

*Then the right hand side product in* (1) *converges uniformly on each compact set of D, and we have*

(i) $g_b \in H^\infty(D)$.

(ii) *$g_b$ is outer, and we have*

$$\mid g_b(e^{ix}) \mid = \prod_n \left| \frac{1-e^{-i(x_n-x)}}{1-r_n e^{-i(x_n-x)}} \right|^2 \text{ a. e. } x \text{ on } [0, 2\pi]. \tag{2}$$

(iii) $g_b(e^{ix}) = \mid g_b(e^{ix}) \mid b(e^{ix})$ *a.e. $x$ on* $[0, 2\pi]$, *and hence* $g_b \in \mathcal{S}^b$.

Proof. If if $b$ is a finite Blaschke product, the lemma follows by easy calculus, and so we consider the case that $b$ is an infinite Blaschke product.

Let us define outer functions $g_n(z) = (1-e^{-ix_n}z)/(1-r_n e^{-ix_n}z)$ $n = 1, 2, \ldots.$ Then we have

$$|1-g_n(z)| = \left|1-\frac{1-e^{-ix_n}z}{1-r_n e^{-ix_n}z}\right| = \frac{|z|(1-r_n)}{|1-\overline{z_n}z|},$$

and for each $\epsilon > 0$, if we put $K_\varepsilon = \{z \in D : |z| \leq 1-\varepsilon\}$, we have

$$\sup\{|1-g_n(z)| : z \in K_\epsilon\} \leq (1-r_n)/\varepsilon.$$

Therefore, we have

$$\sum_{n=1}^{\infty} \sup\{|1-g_n(z)| : z \in K_\varepsilon\} \leq \frac{1}{\varepsilon}\sum_{n=1}^{\infty}(1-r_n) < \infty,$$

and the right hand side of (1) converges to a continuous function $g_b(z)$ which is holomorphic on $D$.

(i) To show that $g_b \in H^\infty$, we examine the modulus of $g_n$ ;

$$\begin{aligned} |g_n(e^{ix})|^2 &= \frac{(1-e^{i(x-x_n)})(1-e^{-i(x-x_n)})}{(1-r_n e^{i(x-x_n)})(1-r_n e^{-i(x-x_n)})} \\ &= \frac{2(1-\cos(x-x_n))}{1-2r_n\cos(x-x_n)+r_n^2}. \end{aligned}$$

If we put, $p(x) = 2(1-\cos x)/(1-2r\cos x + r^2)$, we have

$$\max_{0 \leq x \leq 2\pi} |p(x)| = |p(\pi)| = \left(\frac{2}{1+r}\right)^2$$

by elementary calculus, and hence

$$|g_n(e^{ix})|^2 \leq \frac{4}{(1+r_n)^2} \leq e^{2(1-r_n)} \qquad (0 \leq x \leq 2\pi).$$

By the maximal principle, $|g_n(z)|^2 \leq e^{2(1-r_n)}$ $(z \in D)$, and

$$|g_b(z)| \leq \prod_{n=1}^{\infty} e^{2(1-r_n)} = e^{2\sum_{n=1}^{\infty} 1-r_n} < \infty \qquad (z \in D),$$

that is , $g_b \in H^\infty$.

(ii) Since $|g_n(e^{ix})| \leq \frac{2}{1+r_n}$, we have

$$\frac{1+r_n}{2}|g_n(e^{ix})| \leq 1 \qquad (0 \leq x \leq 2\pi),$$

and we can apply Lebesgue's monotone convergence theorem in the following calucu-lations:

$$\begin{aligned}\left|g_b(0)\prod_{n=1}^{\infty}\left(\frac{1+r_n}{2}\right)^2\right| &= \prod_{n=1}^{\infty}\left|\left(\frac{1+r_n}{2}\right)g_n(0)\right|^2 \\ &= \prod_{n=1}^{\infty}\exp\int_0^{2\pi}\log\left|\frac{1+r_n}{2}g_n(e^{ix})\right|^2 dx/2\pi \\ &= \exp\int_0^{2\pi}\sum_{n=1}^{\infty}\log\left|\frac{1+r_n}{2}g_n(e^{ix})\right|^2 dx/2\pi. \qquad (3)\end{aligned}$$

By (3), we have $\sum_{n=1}^{\infty}\log|\frac{1+r_n}{2}g_n(e^{ix})|^2$ converges a. e. $x$ to an integrable function, say $\phi(x)$. But since $\sum_{n=1}^{\infty}\log|\frac{1+r_n}{2}|^2$ converges to a finite negative constant, it follows that $\sum_{n=1}^{\infty}\log\mid g_n(e^{ix})\mid^2$ converges a. e. $x$ to an integrable function. In this case, $\{|\sum_{n=1}^{N}\log|g_n(e^{ix})|^2|: N=1,2,...\}$ is bounded by an integrable function $\max\{\mid\phi(x)\mid, 2\sum_{n=1}^{\infty}(1-r_n)\}$, and hence we can apply Lebesgue's dominated convergence theorem in the following calculation of modulus of $\mid g_b(z)\mid$ :

$$\begin{aligned}\mid g_b(z)\mid &= \prod_{n=1}^{\infty}\mid g_n(z)\mid^2 \\ &= \prod_{n=1}^{\infty}\exp\int_0^{2\pi}\Re\left[\frac{e^{ix}+z}{e^{ix}-z}\right]\log\mid g_n(e^{ix})\mid^2 dx/2\pi \\ &= \exp\int_0^{2\pi}\Re\left[\frac{e^{it}+z}{e^{ix}-z}\right]\log\prod_{n=1}^{\infty}\mid g_n(e^{ix})\mid^2 dx/2\pi. \\ &= \left|\exp\int_0^{2\pi}\frac{e^{ix}+z}{e^{ix}-z}\log\prod_{n=1}^{\infty}\mid g_n(e^{ix})\mid^2 dx/2\pi\right| \quad (z\in D). \qquad (4)\end{aligned}$$

(4) implies that $g_b$ is outer and that (2) holds.

(iii) Since $b_N(e^{ix}) := \prod_{n=1}^{N}\{(-\overline{z_n})/(\mid z_n\mid)\cdot(e^{ix}-z_n)/(1-\overline{z_n}e^{ix})\}$ converges to $b(e^{ix})$ in $H^2$ (cf. [5] p.65), we can choose a subsequence $\{b_{N_k}\}_{k=1}^{\infty}$ such that

$$b(e^{ix}) = \lim_{k\to\infty}b_{N_k}(e^{ix}) \text{ a. e.} x \text{ on } [0,2\pi]. \qquad (5)$$

From the relations

$$\left(-\frac{\overline{z_n}}{\mid z_n\mid}\frac{e^{ix}-z_n}{1-\overline{z_n}e^{ix}}\right)/g_n^2(e^{ix}) \geq 0 \quad (x\in[0,2\pi]),\ n=1,2,....,$$

it follows that

$$\left|\prod_{n=1}^{N_k} g_n^2(e^{ix})\right| = \left(\prod_{n=1}^{N_k} g_n^2(e^{ix})\right)/b_{N_k}(e^{ix}) \quad (x \in [0,2\pi]), \ \ k = 1,2,\ldots.. \tag{6}$$

If we multiply the both side of (6) by $b_{N_k}(e^{ix})$ and take the limit in $k$, we have by (2) and (5),

$$b(e^{ix}) \mid g_b(e^{ix}) \mid = \lim_{k\to\infty} \prod_{n=1}^{N_k} g_n^2(e^{ix}) \ \text{ a. e. } x \text{ on } [0,2\pi]. \tag{7}$$

Since the pointwise convergence in (7) is bounded by $\prod_{n=1}^{\infty}(\frac{2}{1+r_n})^2 < \infty$, we can apply Lebesgue's dominated convergence theorem in the following caluculations :

$$\begin{aligned} g_b(z) &= \lim_{k\to\infty} \prod_{n=1}^{N_k} g_n^2(z) \\ &= \lim_{k\to\infty} \int_0^{2\pi} P_r(\theta - x)\left(\prod_{n=1}^{N_k} g_n^2(e^{ix})\right)dx/2\pi \\ &= \int_0^{2\pi} P_r(\theta - x)\left(\lim_{k\to\infty}\prod_{n=1}^{N_k} g_n^2(e^{ix})\right)dx/2\pi \\ &= \int_0^{2\pi} P_r(\theta - x) b(e^{ix}) \mid g_b(e^{ix}) \mid dx/2\pi, \end{aligned} \tag{8}$$

where $z = re^{i\theta} \in D$ and $P_r(x)$ denote the Poisson kernel. (8) implies that (iii) holds. Q.E.D.

**Lemma 2.** Let $c_0, c_1$ be real numbers such that $0 \le c_0 < 2\pi$, $c_0 < c_1 \le c_0 + 2\pi$, and let $s(z)$ be a singular inner function such that

$$s(z) = \exp\Big(-\int \frac{e^{it} + z}{e^{it} - z} d\mu(t)\Big),$$

where $\mu \neq 0$ is a nonnegative singular measure whose support (supp($\mu$)) is contained in $[c_0, c_0 + 2\pi] \setminus (c_0, c_1)$. We denote by $\Delta_j$ $(j = 0,1)$ the sectors of the form:

$$\Delta_0 = \{re^{i\theta} : 0 < r \le 1,\ c_0 < \theta < \frac{c_0 + c_1}{2}\},\ \ \Delta_1 = \{re^{i\theta} : 0 < r \le 1,\ \frac{c_0 + c_1}{2} < \theta < c_1\}.$$

Then $(1 + s(z))^2$ is an outer function in $\mathcal{S}^s$ which satisfies

$$\sum_{z \in Z((1+s)^2;\overline{D})\cap\Delta_0} \mid e^{ic_0} - z \mid^{\rho} + \sum_{z \in Z((1+s)^2;\overline{D})\cap\Delta_1} \mid e^{ic_1} - z \mid^{\rho} < \infty \qquad (\rho > 1). \tag{9}$$

Proof. Let us define two auxiliary singular inner functions:

$$s_j(z) = \exp\Big(-\|\mu\|\frac{e^{ic_j}+z}{e^{ic_j}-z}\Big) \quad j=0,1. \tag{10}$$

For a fixed $\theta(c_0 < \theta < c_1)$, we have

$$i\frac{e^{ic_0}+e^{i\theta}}{e^{ic_0}-e^{i\theta}} \leq i\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}} \leq i\frac{e^{ic_1}+e^{i\theta}}{e^{ic_1}-e^{i\theta}} \quad (t \in \mathrm{supp}(\mu)),$$

and hence

$$i\|\mu\|\frac{e^{ic_0}+e^{i\theta}}{e^{ic_0}-e^{i\theta}} \leq \int i\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}d\mu(t) \leq i\|\mu\|\frac{e^{ic_1}+e^{i\theta}}{e^{ic_1}-e^{i\theta}} \qquad (c_0 < \theta < c_1). \tag{11}$$

As we can see easily, $(1+s(z))^2$ has a zero at $e^{i\theta}(c_0 < \theta < c_1)$ if and only if $\int -(e^{it}+e^{i\theta})/(e^{it}-e^{i\theta})d\mu(t) = i\pi(2k+1)$ for some $k \in Z$. Moreover, each of zeros of $(1+s(z))^2$ in $\{e^{i\theta} : c_0 < \theta < c_1\}$ is of order 1 (cf. Definition 1) since

$$\frac{d}{dz}(1+s(z))\Big|_{z=e^{i\theta}} = s(e^{i\theta})\int \frac{2e^{-i\theta}}{|\, e^{it}-e^{i\theta}\,|^2}d\mu(t) \neq 0, \quad (c_0 < \theta < c_1).$$

Thus if we define $\theta_n \in (c_0, c_1)$ (if such a $n \in Z$ exists)by

$$i\int \frac{e^{it}+e^{\theta_n}}{e^{it}-e^{\theta_n}}d\mu(t) = \pi(2n+1), c_0 < \theta_n < c_1, \tag{12}$$

we have $Z((1+s(z))^2; \overline{D}) \cap \{re^{i\theta} : 0 < r \leq 1, c_0 < \theta < c_1\} = \{e^{i\theta_n}\}_{n\in\Gamma}$ for some $\Gamma \subseteq Z$.

By the same reason, each of the zeros of $(1+s_j(z))^2$ has a zero of order 1, and if we put

$$Z((1+s_0(z))^2; \overline{D}) \cap \{re^{i\theta} : 0 < r \leq 1, c_0 < \theta < c_0 + 2\pi\} = \{e^{ix_n}\}_{n\in Z},$$

$$c_0 < \dots < x_n < x_{n+1} < \dots, < c_0 + 2\pi,$$

$$Z((1+s_1(z))^2; \overline{D}) \cap \{re^{i\theta} : 0 < r \leq 1, c_1 - 2\pi < \theta < c_1\} = \{e^{iy_n}\}_{n\in Z},$$

$$c_1 - 2\pi < \dots < y_n < y_{n+1} < \dots, < c_1,$$

$x_n(n < 0)$ and $y_n(n \geq 0)$ are characterized by the equations

$$i\|\mu\|\frac{e^{ic_0}+e^{ix_n}}{e^{ic_0}-e^{ix_n}} = \pi(2n+1), \quad c_0 < x_n < c_0 + \pi \quad (n < 0), \tag{13}$$

$$i\|\mu\|\frac{e^{ic_1}+e^{iy_n}}{e^{ic_1}-e^{iy_n}} = \pi(2n+1), \quad c_1 - \pi < y_n < c_1 \quad (n \geq 0). \tag{14}$$

Thus, by (12), (13) and (14) considering the relation (11), we have

$$c_0+\pi > x_n \geq \theta_n \quad n \in \Gamma, n<0 \text{ and } \theta_n > y_n > c_1-\pi \quad n\in\Gamma, n\geq 0. \tag{15}$$

By (15), we have

$$\sum_{n\in\Gamma,n<0} | e^{ic_0}-e^{i\theta_n} |^\rho + \sum_{n\in\Gamma,n\geq 0} | e^{ic_1}-e^{i\theta_n} |^\rho$$

$$\leq \sum_{n=1}^{\infty} | e^{ic_0}-e^{ix_{-n}} |^\rho + \sum_{n=0}^{\infty} | e^{ic_1}-e^{iy_n} |^\rho \quad (\rho>1). \tag{16}$$

On the other hand, it follows from (13) and (14) that

$$| e^{ic_0}-e^{ix_{-n}} | \sim \frac{\|\mu\|}{\pi n} \quad (n\to\infty), \quad | e^{ic_1}-e^{iy_n} | \sim \frac{\|\mu\|}{\pi n} \quad (n\to\infty). \tag{17}$$

By (16) and (17), we get

$$\sum_{n\in\Gamma,n<0} | e^{ic_0}-e^{i\theta_n} |^\rho + \sum_{n\in\Gamma,n\geq 0} | e^{ic_1}-e^{i\theta_n} |^\rho < \infty \quad (\rho>1). \tag{18}$$

(18) is equivalent to (9). Q.E.D.

**Lemma 3.** *Let $\{x_n\}_{n\in Z}$ be an infinite sequence in the interval $(0,2\pi)$ such that*

$$2\pi > \ldots \geq x_1 \geq x_0 (\geq \pi) > x_{-1} \geq \ldots > 0, \quad \lim_{n\to\infty} x_n = 2\pi, \quad \textit{and} \lim_{n\to\infty} x_{-n} = 0,$$

*and let $\sigma \geq 1$. Then the following* (a) *and* (b) *are equivalent.*

$$\text{(a)} \quad \sum_{n\in Z} | n |^p (x_n - x_{n-1}) < \infty \quad (0<p<1/\sigma).$$

$$\text{(b)} \quad \sum_{n=0}^{\infty} | 2\pi - x_n |^\rho + \sum_{n=1}^{\infty} x_{-n}^\rho < \infty \quad (\sigma<\rho<\infty).$$

Proof. (a) implies (b): Let $\rho \in (\sigma,\infty)$ , and let $\phi = \sum_{n\in Z} n\mathcal{X}_{(x_{n-1},x_n]}$ , where $\mathcal{X}_E$ denotes the characteristic function of $E$. Then by (a), $\phi \in L^p([0,2\pi]) \subset weak\ L^p([0,2\pi])$ $(0<p<1/\sigma)$. Therefore, for each $p(0<p<1/\sigma)$, there exists $C_p>0$ satisfying

$$| \left\{ x : | \phi(x) | \geq n \right\} | = (2\pi - x_{n-1}) + x_{-n} \leq \frac{C_p}{n^p} \quad (n\geq 1), \tag{19}$$

where $\mid E \mid$ denotes the Lebesgue measure of $E$. If we choose $p_0 (0 < p_0 < 1/\sigma)$ so that $p_0 \rho > 1$, we have by (19),

$$\sum_{n=1}^{\infty}(2\pi - x_{n-1})^{\rho} + \sum_{n=1}^{\infty} x_{-n}{}^{\rho} \le 2C_{p_0}^{\rho} \sum_{n=1}^{\infty} \frac{1}{n^{\rho p_0}} < \infty.$$

(b) implies (a): Let $p \in (0, 1/\sigma)$ be arbitrary. For each positive integer $n$, there exists a $\theta \in (0,1)$ such that

$$\frac{n^p - (n-1)^p}{n^{p-1}} = \frac{1-(1-1/n)^p}{1/n} = p(1-\frac{\theta}{n})^{p-1},$$

by the mean value theorem. So, there exists $C_p > 0$ such that

$$n^p - (n-1)^p \le C_p n^{p-1} \quad (n \ge 2), \tag{20}$$

Since $0 < p < 1/\sigma$, we can choose $\rho > \sigma$ and $q$ such that $q(p-1) < -1,\ 1/\rho + 1/q = 1$. In fact, we can simply set $\rho$ so that $\rho = 1/p - \varepsilon > \sigma$ for some small $\epsilon > 0$. Then by (20) and Hörder's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^p (x_n - x_{n-1}) + \sum_{n=1}^{\infty} (-n)^p (x_{-n} - x_{-n-1}) \\
=\ & \sum_{n=1}^{\infty} (n^p - (n-1)^p)(2\pi - x_{n-1}) + \sum_{n=1}^{\infty} (n^p - (n-1)^p) x_{-n} \\
\le\ & 2\pi - x_0 + x_{-1} + \sum_{n=2}^{\infty} C_p n^{p-1} \{(2\pi - x_{n-1}) + x_{-n}\} \\
&\le 2\pi - x_0 + x_{-1} + C_p \left( \sum_{n=2}^{\infty} n^{(p-1)q} \right)^{1/q} \left( \sum_{n=2}^{\infty} (2\pi - x_{n-1})^{\rho} \right)^{1/\rho} \\
&\quad + C_p \left( \sum_{n=2}^{\infty} n^{(p-1)q} \right)^{1/q} \left( \sum_{n=2}^{\infty} (x_{-n})^{\rho} \right)^{1/\rho}.
\end{aligned} \tag{21}$$

By (21), we can conclude that (b) implies (a). Q.E.D.

## 3. The statement and the proof of the main theorem.

**Theorem 1.** *Let $\mathcal{B}$ be an infinite Blaschke product such that there exist $c_0, c_1 \in R$ which satisfy*

$$0 \le c_0 < 2\pi, c_0 < c_1 \le c_0 + 2\pi, \ \{e^{i\theta} : c_0 \le \theta \le c_1\} \cap \mathrm{sing}(\mathcal{B}) = \{e^{ic_0}, e^{ic_1}\}.$$

*We denote by $\Delta_j$ $(j = 0, 1)$ the sectors of the form*

$$\Delta_0 = \{re^{i\theta} : 0 < r \le 1, \ c_0 < \theta < \frac{c_0 + c_1}{2}\}, \ \Delta_1 = \{re^{i\theta} : 0 < r \le 1, \ \frac{c_0 + c_1}{2} < \theta < c_1\}.$$

*For $f \in \mathcal{S}^{\mathcal{B}}$, we define the order of convergence of $Z(f; \overline{D}) \cap \Delta_j$ to $e^{ic_j}$ $j = 0, 1$ by*

$$\mathbf{Ord}[e^{ic_j}; Z(f; \overline{D}) \cap \Delta_j] = \inf\{\rho > 0 : \sum_{z \in Z(f;\overline{D}) \cap \Delta_j} | z - e^{ic_j} |^{\rho} < \infty\}.$$

*Then the following* (i) *and* (ii) *hold.*

(i) *If* $\mathbf{Ord}[e^{ic_j}; Z(f; \overline{D}) \cap \Delta_j] = \sigma > 1$ *for some $f \in \mathcal{S}^{\mathcal{B}}$, then we have*

$$\mathbf{Ord}[e^{ic_j}; Z(g; \overline{D}) \cap \Delta_j] = \sigma \quad (g \in \mathcal{S}^{\mathcal{B}}).$$

(ii) *If* $\mathbf{Ord}[e^{ic_j}; Z(f; \overline{D}) \cap \Delta_j] \le 1$ *for some $f \in \mathcal{S}^{\mathcal{B}}$, then we have*

$$\mathbf{Ord}[e^{ic_j}; Z(g; \overline{D}) \cap \Delta_j] \le 1 \quad (g \in \mathcal{S}^{\mathcal{B}}).$$

Proof. Let $f, g \in \mathcal{S}^{\mathcal{B}}$ and $j \in \{0, 1\}$ be arbitrary, and suppose that the following inequality

$$\sum_{z \in Z(f;\overline{D}) \cap \Delta_j} | e^{ic_j} - z |^{\sigma} < \infty \tag{22}$$

holds for some $\sigma > 1$. We will deduce the following inequalities

$$\sum_{z \in Z(g;\overline{D}) \cap \Delta_j} | e^{ic_j} - z |^{\rho} < \infty \quad (\rho > \sigma), \tag{23}$$

from (22), and it is easy to see that this is sufficient to prove (i) and (ii).

Let $f = hbs$ be the canonical decomposition of $f$, where $h$ is the outer part, $b$ is the Blaschke part and $s$ is the singular inner part of $f$, respectively. If we define the outer function $g_b \in \mathcal{S}^b \cap H^\infty$ by the method of Lemma 1, then it follows from Lemma 1 and Lemma 2 that $\tilde{f} := hg_b(1 + s)^2$ is an outer function in $\mathcal{S}^{\mathcal{B}}$ which satisfies

$$\sum_{z\in Z(\tilde{f};\overline{D})\cap\Delta_j} | e^{ic_j} - z |^{\sigma} < \infty.$$

In the same way, we construct an outer function $\tilde{g} \in \mathcal{S}^{\mathcal{B}}$ from $g$. If we can show that the relation (23) is true for $\tilde{g}$, then it follows that the relation (23) is also true for $g$ by the definition of $\tilde{g}$ , Proposition 1, Lemma 1 and Lemma 2. Therefore, to prove (23) from (22), we have only to prove (23) in case $f$ and $g$ in (22) are both outer.

We prove (23) only in case $j = 0$ since the proof in case $j = 1$ is almost the same with that of the case $j$=0. Further, we can assume (without loss of generality) $c_0 = 0$.

Let us assume that $f$ and $g$ in (22) are both outer and denote the zeros of $f$ and $g$ in $\Delta_0$ in the form :

$$Z(f;\overline{D}) \cap \Delta_0 \quad = \quad \{e^{ix_n}\}_{n<0};\ c_1/2 > x_{-1} \geq x_{-2} \geq \ldots > 0, \tag{24}$$

$$Z(g;\overline{D}) \cap \Delta_0 \quad = \quad \{e^{iy_n}\}_{n<0};\ c_1/2 > y_{-1} \geq y_{-2} \geq \ldots > 0, \tag{25}$$

then the following inequality holds by (22)(note that $e^{ic_0} = 1$ in our present situations).

$$\sum_n x_n^{\sigma} < \infty. \tag{26}$$

We define functions $\varphi_f(\theta), \varphi_g(\theta)$ and $\varphi_{g/f}(\theta)$ on $(0, 2\pi)$ as follows:

$$\varphi_f(\theta) = \begin{cases} -2\pi\#\{n : x_n \geq \theta\} & 0 < \theta < c_1/2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varphi_g(\theta) = \begin{cases} -2\pi\#\{n : y_n \geq \theta\} & 0 < \theta < c_1/2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varphi_{g/f}(\theta) = \begin{cases} -2\pi\Big(\#\{n : y_n \geq \theta\} - \#\{n : x_n \geq \theta\}\Big) & 0 < \theta < c_1/2, \\ 0 & \text{otherwise}, \end{cases}$$

where $\#A$ denotes the number of elements of A. We can deduce from (26) and Lemma 3 that,

$$\varphi_f \in L^p([0, 2\pi]) \quad (0 < p < 1/\sigma). \tag{27}$$

If we define $\log(g/f)$ so that $\lim_{x\uparrow c_0/2} \Im[\log(g/f)(e^{ix})] = 0$, a moment's thought reveals that

$$\mathcal{X}_{(0,c_1/2)}(\theta)\Im[\log(g/f)(e^{i\theta})] = \varphi_{g/f}(\theta) \ \text{ a.e. } \theta. \tag{28}$$

Since $g/f \in N_+$, it follows from Kolmogorov's theorem that

$$\Im[\log(g/f)(e^{i\theta})] \in L^p([0, 2\pi]) \ \ (0 < p < 1). \tag{29}$$

By (27), (28) and (29) we have

$$\varphi_g = \varphi_{g/f} + \varphi_f \in L^p([0, 2\pi]) \ \ (0 < p < 1/\sigma). \tag{30}$$

From (30) and Lemma 3, we get

$$\sum_n y_n^\rho < \infty \quad (\sigma < \rho). \tag{31}$$

(31) is equivalent to the desired inequalities (23) for an outer $g \in \mathcal{S}^{\mathcal{B}}$ in case $j = 0$ and $c_0 = 0$. Q.E.D.

**Example 1.** Let $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2\mathcal{B}_3$ , where $\mathcal{B}_j : \ j = 1, 2, 3$ are Blaschke products such that:

(i) sing $(\mathcal{B}_1) \subseteq \{e^{i\theta} : \pi < \theta < 2\pi\}$,

(ii) $\mathcal{B}_2$ is the Blaschke product with the zeros $\{\alpha_{-n}\}_{n=1}^{\infty}$ with

$$\alpha_{-n} = (1 - (\frac{1}{n})^{p_0})e^{i(\frac{1}{n})^{1/\sigma_0}}, \quad p_0 > 1, \ \sigma_0 > 1.$$

(iii) $\mathcal{B}_3$ is the Blaschke product with the zeros $\{\alpha_n\}_{n=1}^{\infty}$, where

$$\alpha_n = (1 - (\frac{1}{n})^{p_1})e^{i(\pi - (\frac{1}{n})^{1/\sigma_1})}, \quad p_1 > 1, \ \sigma_1 > 1.$$

We put

$$\Delta_0 = \{re^{i\theta}; 0 < r \leq 1, \ 0 < \theta < \pi/2\}, \ \ \Delta_1 = \{re^{i\theta}; 0 < r \leq 1, \ \pi/2 < \theta < \pi\},$$

Then we have for $j \in \{0, 1\}$

$$\mathbf{Ord}[(-1)^j; Z(\mathcal{B}; \overline{D}) \cap \Delta_j] = \sigma_j > 1,$$

and hence we have by Theorem 1

$$\mathbf{Ord}[(-1)^j; Z(g; \overline{D}) \cap \Delta_j)] = \sigma_j > 1 \quad (g \in \mathcal{S}^{\mathcal{B}}).$$

**Remark 1.** In Theorem 1 (i), we can not replace $\sigma > 1$ by $\sigma \geq 1$ as the following example shows.

**Example 2.** Let $\mathcal{B}$ be an infinite Blaschke product with the zeros $\{\alpha_n\}_{n=1}^{\infty}$, where

$$\alpha_n = \Big(1 - \frac{1}{n(\log(n+1))^2}\Big)e^{i/n^2}, \quad n = 1, 2, \ldots$$

Then , $\mathbf{Ord}[1; Z(\mathcal{B}; \overline{D}) \cap \Delta_0] = 1$, where $\Delta_0 = \{re^{i\theta} : 0 < r \leq 1, 0 < \theta < \pi\}$. On the other hand we have by Lemma 1 that

$$g_{\mathcal{B}}(z) := \prod_{n=1}^{\infty} \frac{(1 - e^{i/n^2}z)^2}{(1 - \overline{\alpha_n}z)^2} \in \mathcal{S}^{\mathcal{B}} \cap H^{\infty},$$

and $\mathbf{Ord}[1; Z(g_{\mathcal{B}}; \overline{D}) \cap \Delta_0)] = 1/2$.

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