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**J. Inoue and T. Nakazi**

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# On the Zeroes of Solutions of an Extremal Problem in $H^1$

Jyunji Inoue and Takahiko Nakazi

To the memory of Professor Katsutoshi Takahashi

Abstract. For a non-zero function  $f$  in  $H^1$ , the classical Hardy space on the unit disc, we put

$$\mathcal{S}^f = \{g \in H^1 : \arg f(e^{i\theta}) = \arg g(e^{i\theta}) \text{ a.e. } \theta\}.$$

The intersection of  $\mathcal{S}^f$  and the unit sphere in  $H^1$  is just a set of solutions of some extremal problem in  $H^1$ . It is known that  $\mathcal{S}^f$  can be represented in the form  $\mathcal{S}^f = \mathcal{S}^{\mathcal{B}} \times g_0$ , where  $\mathcal{B}$  is a Blaschke product and  $g_0$  is a function in  $H^1$  with  $\mathcal{S}^{g_0} = \{\lambda \cdot g_0 : \lambda > 0\}$ . Also it is known that the linear span of  $\mathcal{S}^f$  is of finite dimensional if and only if  $\mathcal{B}$  is a finite Blaschke product, and when  $\mathcal{B}$  is a finite Blaschke product, we can describe completely the set  $\mathcal{S}^{\mathcal{B}}$  and the zeros of functions in  $\mathcal{S}^{\mathcal{B}}$ .

In this paper, we study the set of zeros of functions in  $\mathcal{S}^{\mathcal{B}}$  when  $\mathcal{B}$  is an infinite Blaschke product whose set of singularities is not the whole circle. Especially we study the behavior of zeros of functions in  $\mathcal{S}^{\mathcal{B}}$  in the sectors of the form:  $\Delta = \{re^{i\theta} : 0 < r \leq 1, c_1 < \theta < c_2\}$  on which the zeros of  $\mathcal{B}$  has no accumulation points, and establish a convergence order theorem of zeros in  $\Delta$  of functions in  $\mathcal{S}^{\mathcal{B}}$ .

## 1. Introduction and preliminary results

$N_+$  and  $H^p$  for  $0 < p \leq \infty$  denote the Smirnov class and the Hardy spaces on the open unit disc  $D$  in the complex plane  $C$ , respectively. A function  $h$  in  $N_+$  is called outer if it is invertible in  $N_+$ . A function  $q$  in  $N_+$  is called inner if  $|q| = 1$  a.e. on  $\partial D$ .

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For a non-zero function  $f$  in  $H^1$ , put

$$\mathcal{S}^f = \{g \in H^1 : \arg f(e^{i\theta}) = \arg g(e^{i\theta}) \text{ a. e. on } \partial D\}$$

The intersection of  $\mathcal{S}^f$  and the unit sphere in  $H^1$  is just a set of solutions of an extremal problem about a continuous linear functional on  $H^1$  (cf. [1]). The set  $\mathcal{S}^f$  were studied by several authors (cf. [1], [3], [4], [6] and [7]). Hayashi [3], [4] showed that there exists a Blaschke product  $\mathcal{B}$  and an outer function  $g_0$  in  $H^1$  such that

$$\mathcal{S}^f = \mathcal{S}^{\mathcal{B}} \times g_0 \text{ and } \mathcal{S}^{g_0} = \{\lambda \cdot g_0 : \lambda > 0\}.$$

Hence it is important to study  $\mathcal{S}^f$  when  $f$  is a Blaschke product.

If  $\mathcal{B}$  is a finite Blaschke product, each function  $f$  in  $\mathcal{S}^{\mathcal{B}}$  is analytic on  $D \cup \partial D$  and so the set of zeros of  $f$  in  $D \cup \partial D$  consists of finite points, and we can describe completely the set  $\mathcal{S}^{\mathcal{B}}$  and the zeros of functions in  $\mathcal{S}^{\mathcal{B}}$  (cf. [7]). When  $\mathcal{B}$  is an infinite Blaschke product, we need further considerations and new ideas to study the zeros of functions in  $\mathcal{S}^{\mathcal{B}}$ .

For each  $f$  in  $H^1$ ,  $\text{sing}(f)$  denotes the set of points of  $\partial D$  on which  $f$  cannot be analytically extended.

In this paper, we consider the case in which Blaschke products  $\mathcal{B}$  have the property  $\text{sing}(\mathcal{B}) \neq \partial D$ .

The following is the first elementary result we need, which is probably known.

**Proposition 1.** *If  $Q$  is an inner function and if  $f$  is a function in  $\mathcal{S}^Q$ , we have*

$$\text{sing}(Q) = \text{sing}(f).$$

*Proof.* Since  $f \in \mathcal{S}^Q$ ,  $f/Q$  is nonnegative a.e. on  $\partial D$ . By [2]; Lemma 4.2,  $f/Q$  extends analytically across any open arc  $J$  such that  $J \subset \partial D \setminus \text{sing}(Q)$ , and since  $Q$  is analytic on  $\partial D \setminus \text{sing}Q$  we have  $\text{sing}(f) \subseteq \text{sing}(Q)$ . By the same method, if  $\beta \in \partial D \setminus \text{sing}(f)$ ,  $Q/f$  has a meromorphic extension to a neighborhood  $V$  of  $\beta$ , and so  $Q$  also has a meromorphic extension on  $V$ . But, since  $Q$  is bounded on  $V \cap D$ ,  $Q$  is in fact analytic on  $V$ , that is  $\beta \notin \text{sing}(Q)$ . Q.E.D.

**Definition 1.** For a function  $f$  in  $H^1$  and a positive integer  $n$ , we say that  $\alpha \in \partial D$

is a zero of order  $n$  of  $f$  if

$$\frac{f(z)}{(z-\beta)^{2n}} \in H^1 \text{ and } \frac{f(z)}{(z-\beta)^{2n+2}} \notin H^1$$

hold. The set of all the zeros of  $f$  on  $\partial D$  counted according to its order is denoted by  $Z(f; \partial D)$ .  $Z(f; D)$  denotes the usual zeros in  $D$  of  $f$  and put

$$Z(f; \overline{D}) = Z(f; D) \cup Z(f; \partial D).$$

## 2. Lemmas to the main theorem.

In this section, we collect lemmas which we use in the proof of the main theorem in the next section.

**Lemma 1** *Let  $b$  be a Blaschke product :*

$$b(z) = \prod_n \frac{-\overline{z_n}}{|z_n|} \frac{z - z_n}{1 - \overline{z_n}z} \quad z_n = r_n e^{ix_n} \quad (n = 1, 2, \dots).$$

*We put*

$$g_b(z) := \prod_n \frac{(1 - e^{-ix_n z})^2}{(1 - r_n e^{-ix_n z})^2}. \quad (1)$$

*Then the right hand side product in (1) converges uniformly on each compact set of  $D$ , and we have*

- (i)  $g_b \in H^\infty(D)$ .
- (ii)  $g_b$  is outer, and we have

$$|g_b(e^{ix})| = \prod_n \left| \frac{1 - e^{-i(x_n - x)}}{1 - r_n e^{-i(x_n - x)}} \right|^2 \quad \text{a. e. } x \text{ on } [0, 2\pi]. \quad (2)$$

- (iii)  $g_b(e^{ix}) = |g_b(e^{ix})| b(e^{ix})$  a.e.  $x$  on  $[0, 2\pi]$ , and hence  $g_b \in S^b$ .

**Proof.** If  $b$  is a finite Blaschke product, the lemma follows by easy calculus, and so we consider the case that  $b$  is an infinite Blaschke product.

Let us define outer functions  $g_n(z) = (1 - e^{-ix_n z}) / (1 - r_n e^{-ix_n z})$   $n = 1, 2, \dots$ . Then we have

$$|1 - g_n(z)| = \left| 1 - \frac{1 - e^{-ix_n z}}{1 - r_n e^{-ix_n z}} \right| = \frac{|z| (1 - r_n)}{|1 - \bar{z}_n z|},$$

and for each  $\epsilon > 0$ , if we put  $K_\epsilon = \{z \in D : |z| \leq 1 - \epsilon\}$ , we have

$$\sup\{|1 - g_n(z)| : z \in K_\epsilon\} \leq (1 - r_n) / \epsilon.$$

Therefore, we have

$$\sum_{n=1}^{\infty} \sup\{|1 - g_n(z)| : z \in K_\epsilon\} \leq \frac{1}{\epsilon} \sum_{n=1}^{\infty} (1 - r_n) < \infty,$$

and the right hand side of (1) converges to a continuous function  $g_b(z)$  which is holomorphic on  $D$ .

(i) To show that  $g_b \in H^\infty$ , we examine the modulus of  $g_n$  ;

$$\begin{aligned} |g_n(e^{ix})|^2 &= \frac{(1 - e^{i(x-x_n)})(1 - e^{-i(x-x_n)})}{(1 - r_n e^{i(x-x_n)})(1 - r_n e^{-i(x-x_n)})} \\ &= \frac{2(1 - \cos(x - x_n))}{1 - 2r_n \cos(x - x_n) + r_n^2}. \end{aligned}$$

If we put,  $p(x) = 2(1 - \cos x) / (1 - 2r \cos x + r^2)$ , we have

$$\max_{0 \leq x \leq 2\pi} |p(x)| = |p(\pi)| = \left( \frac{2}{1+r} \right)^2$$

by elementary calculus, and hence

$$|g_n(e^{ix})|^2 \leq \frac{4}{(1+r_n)^2} \leq e^{2(1-r_n)} \quad (0 \leq x \leq 2\pi).$$

By the maximal principle,  $|g_n(z)|^2 \leq e^{2(1-r_n)}$  ( $z \in D$ ), and

$$|g_b(z)| \leq \prod_{n=1}^{\infty} e^{2(1-r_n)} = e^{2 \sum_{n=1}^{\infty} (1-r_n)} < \infty \quad (z \in D),$$

that is,  $g_b \in H^\infty$ .

(ii) Since  $|g_n(e^{ix})| \leq \frac{2}{1+r_n}$ , we have

$$\frac{1+r_n}{2} |g_n(e^{ix})| \leq 1 \quad (0 \leq x \leq 2\pi),$$

and we can apply Lebesgue's monotone convergence theorem in the following calculations:

$$\begin{aligned}
\left| g_b(0) \prod_{n=1}^{\infty} \left( \frac{1+r_n}{2} \right)^2 \right| &= \prod_{n=1}^{\infty} \left| \left( \frac{1+r_n}{2} \right) g_n(0) \right|^2 \\
&= \prod_{n=1}^{\infty} \exp \int_0^{2\pi} \log \left| \frac{1+r_n}{2} g_n(e^{ix}) \right|^2 dx / 2\pi \\
&= \exp \int_0^{2\pi} \sum_{n=1}^{\infty} \log \left| \frac{1+r_n}{2} g_n(e^{ix}) \right|^2 dx / 2\pi. \tag{3}
\end{aligned}$$

By (3), we have  $\sum_{n=1}^{\infty} \log \left| \frac{1+r_n}{2} g_n(e^{ix}) \right|^2$  converges a. e.  $x$  to an integrable function, say  $\phi(x)$ . But since  $\sum_{n=1}^{\infty} \log \left| \frac{1+r_n}{2} \right|^2$  converges to a finite negative constant, it follows that  $\sum_{n=1}^{\infty} \log |g_n(e^{ix})|^2$  converges a. e.  $x$  to an integrable function. In this case,  $\{|\sum_{n=1}^N \log |g_n(e^{ix})|^2| : N = 1, 2, \dots\}$  is bounded by an integrable function  $\max\{|\phi(x)|, 2\sum_{n=1}^{\infty} (1-r_n)\}$ , and hence we can apply Lebesgue's dominated convergence theorem in the following calculation of modulus of  $|g_b(z)|$ :

$$\begin{aligned}
|g_b(z)| &= \prod_{n=1}^{\infty} |g_n(z)|^2 \\
&= \prod_{n=1}^{\infty} \exp \int_0^{2\pi} \Re \left[ \frac{e^{ix} + z}{e^{ix} - z} \right] \log |g_n(e^{ix})|^2 dx / 2\pi \\
&= \exp \int_0^{2\pi} \Re \left[ \frac{e^{ix} + z}{e^{ix} - z} \right] \log \prod_{n=1}^{\infty} |g_n(e^{ix})|^2 dx / 2\pi. \\
&= \left| \exp \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} \log \prod_{n=1}^{\infty} |g_n(e^{ix})|^2 dx / 2\pi \right| \quad (z \in D). \tag{4}
\end{aligned}$$

(4) implies that  $g_b$  is outer and that (2) holds.

(iii) Since  $b_N(e^{ix}) := \prod_{n=1}^N \{(-\bar{z}_n)/(|z_n|) \cdot (e^{ix} - z_n)/(1 - \bar{z}_n e^{ix})\}$  converges to  $b(e^{ix})$  in  $H^2$  (cf. [5] p.65), we can choose a subsequence  $\{b_{N_k}\}_{k=1}^{\infty}$  such that

$$b(e^{ix}) = \lim_{k \rightarrow \infty} b_{N_k}(e^{ix}) \text{ a. e. } x \text{ on } [0, 2\pi]. \tag{5}$$

From the relations

$$\left( -\frac{\bar{z}_n}{|z_n|} \frac{e^{ix} - z_n}{1 - \bar{z}_n e^{ix}} \right) / g_n^2(e^{ix}) \geq 0 \quad (x \in [0, 2\pi]), \quad n = 1, 2, \dots,$$



it follows that

$$\left| \prod_{n=1}^{N_k} g_n^2(e^{ix}) \right| = \left( \prod_{n=1}^{N_k} g_n^2(e^{ix}) \right) / b_{N_k}(e^{ix}) \quad (x \in [0, 2\pi]), \quad k = 1, 2, \dots \quad (6)$$

If we multiply the both side of (6) by  $b_{N_k}(e^{ix})$  and take the limit in  $k$ , we have by (2) and (5),

$$b(e^{ix}) |g_b(e^{ix})| = \lim_{k \rightarrow \infty} \prod_{n=1}^{N_k} g_n^2(e^{ix}) \quad \text{a. e. } x \text{ on } [0, 2\pi]. \quad (7)$$

Since the pointwise convergence in (7) is bounded by  $\prod_{n=1}^{\infty} (\frac{2}{1+r_n})^2 < \infty$ , we can apply Lebesgue's dominated convergence theorem in the following calculations :

$$\begin{aligned} g_b(z) &= \lim_{k \rightarrow \infty} \prod_{n=1}^{N_k} g_n^2(z) \\ &= \lim_{k \rightarrow \infty} \int_0^{2\pi} P_r(\theta - x) \left( \prod_{n=1}^{N_k} g_n^2(e^{ix}) \right) dx / 2\pi \\ &= \int_0^{2\pi} P_r(\theta - x) \left( \lim_{k \rightarrow \infty} \prod_{n=1}^{N_k} g_n^2(e^{ix}) \right) dx / 2\pi \\ &= \int_0^{2\pi} P_r(\theta - x) b(e^{ix}) |g_b(e^{ix})| dx / 2\pi, \end{aligned} \quad (8)$$

where  $z = re^{i\theta} \in D$  and  $P_r(x)$  denote the Poisson kernel. (8) implies that (iii) holds. Q.E.D.

**Lemma 2.** Let  $c_0, c_1$  be real numbers such that  $0 \leq c_0 < 2\pi$ ,  $c_0 < c_1 \leq c_0 + 2\pi$ , and let  $s(z)$  be a singular inner function such that

$$s(z) = \exp\left(-\int \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right),$$

where  $\mu \neq 0$  is a nonnegative singular measure whose support ( $\text{supp}(\mu)$ ) is contained in  $[c_0, c_0 + 2\pi] \setminus (c_0, c_1)$ . We denote by  $\Delta_j$  ( $j = 0, 1$ ) the sectors of the form:

$$\Delta_0 = \{re^{i\theta} : 0 < r \leq 1, c_0 < \theta < \frac{c_0 + c_1}{2}\}, \quad \Delta_1 = \{re^{i\theta} : 0 < r \leq 1, \frac{c_0 + c_1}{2} < \theta < c_1\}.$$

Then  $(1 + s(z))^2$  is an outer function in  $\mathcal{S}^s$  which satisfies

$$\sum_{z \in \mathcal{Z}((1+s)^2; \bar{D}) \cap \Delta_0} |e^{ic_0} - z|^\rho + \sum_{z \in \mathcal{Z}((1+s)^2; \bar{D}) \cap \Delta_1} |e^{ic_1} - z|^\rho < \infty \quad (\rho > 1). \quad (9)$$

Proof. Let us define two auxiliary singular inner functions:

$$s_j(z) = \exp\left(-\|\mu\| \frac{e^{ic_j} + z}{e^{ic_j} - z}\right) \quad j = 0, 1. \quad (10)$$

For a fixed  $\theta (c_0 < \theta < c_1)$ , we have

$$i \frac{e^{ic_0} + e^{i\theta}}{e^{ic_0} - e^{i\theta}} \leq i \frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} \leq i \frac{e^{ic_1} + e^{i\theta}}{e^{ic_1} - e^{i\theta}} \quad (t \in \text{supp}(\mu)),$$

and hence

$$i \|\mu\| \frac{e^{ic_0} + e^{i\theta}}{e^{ic_0} - e^{i\theta}} \leq \int i \frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} d\mu(t) \leq i \|\mu\| \frac{e^{ic_1} + e^{i\theta}}{e^{ic_1} - e^{i\theta}} \quad (c_0 < \theta < c_1). \quad (11)$$

As we can see easily,  $(1 + s(z))^2$  has a zero at  $e^{i\theta} (c_0 < \theta < c_1)$  if and only if  $\int -\frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} d\mu(t) = i\pi(2k + 1)$  for some  $k \in Z$ . Moreover, each of zeros of  $(1 + s(z))^2$  in  $\{e^{i\theta} : c_0 < \theta < c_1\}$  is of order 1 (cf. Definition 1) since

$$\frac{d}{dz}(1 + s(z)) \Big|_{z=e^{i\theta}} = s(e^{i\theta}) \int \frac{2e^{-i\theta}}{|e^{it} - e^{i\theta}|^2} d\mu(t) \neq 0, \quad (c_0 < \theta < c_1).$$

Thus if we define  $\theta_n \in (c_0, c_1)$  (if such a  $n \in Z$  exists) by

$$i \int \frac{e^{it} + e^{i\theta_n}}{e^{it} - e^{i\theta_n}} d\mu(t) = \pi(2n + 1), \quad c_0 < \theta_n < c_1, \quad (12)$$

we have  $Z((1 + s(z))^2; \overline{D}) \cap \{re^{i\theta} : 0 < r \leq 1, c_0 < \theta < c_1\} = \{e^{i\theta_n}\}_{n \in \Gamma}$  for some  $\Gamma \subseteq Z$ .

By the same reason, each of the zeros of  $(1 + s_j(z))^2$  has a zero of order 1, and if we put

$$Z((1 + s_0(z))^2; \overline{D}) \cap \{re^{i\theta} : 0 < r \leq 1, c_0 < \theta < c_0 + 2\pi\} = \{e^{ix_n}\}_{n \in Z},$$

$$c_0 < \dots < x_n < x_{n+1} < \dots < c_0 + 2\pi,$$

$$Z((1 + s_1(z))^2; \overline{D}) \cap \{re^{i\theta} : 0 < r \leq 1, c_1 - 2\pi < \theta < c_1\} = \{e^{iy_n}\}_{n \in Z},$$

$$c_1 - 2\pi < \dots < y_n < y_{n+1} < \dots < c_1,$$

$x_n (n < 0)$  and  $y_n (n \geq 0)$  are characterized by the equations

$$i \|\mu\| \frac{e^{ic_0} + e^{ix_n}}{e^{ic_0} - e^{ix_n}} = \pi(2n + 1), \quad c_0 < x_n < c_0 + \pi \quad (n < 0), \quad (13)$$

$$i \|\mu\| \frac{e^{ic_1} + e^{iy_n}}{e^{ic_1} - e^{iy_n}} = \pi(2n + 1), \quad c_1 - \pi < y_n < c_1 \quad (n \geq 0). \quad (14)$$

Thus, by (12), (13) and (14) considering the relation (11), we have

$$c_0 + \pi > x_n \geq \theta_n \quad n \in \Gamma, n < 0 \quad \text{and} \quad \theta_n > y_n > c_1 - \pi \quad n \in \Gamma, n \geq 0. \quad (15)$$

By (15), we have

$$\begin{aligned} & \sum_{n \in \Gamma, n < 0} |e^{ic_0} - e^{i\theta_n}|^\rho + \sum_{n \in \Gamma, n \geq 0} |e^{ic_1} - e^{i\theta_n}|^\rho \\ & \leq \sum_{n=1}^{\infty} |e^{ic_0} - e^{ix_{-n}}|^\rho + \sum_{n=0}^{\infty} |e^{ic_1} - e^{iy_n}|^\rho \quad (\rho > 1). \end{aligned} \quad (16)$$

On the other hand, it follows from (13) and (14) that

$$|e^{ic_0} - e^{ix_{-n}}| \sim \frac{\|\mu\|}{\pi n} \quad (n \rightarrow \infty), \quad |e^{ic_1} - e^{iy_n}| \sim \frac{\|\mu\|}{\pi n} \quad (n \rightarrow \infty). \quad (17)$$

By (16) and (17), we get

$$\sum_{n \in \Gamma, n < 0} |e^{ic_0} - e^{i\theta_n}|^\rho + \sum_{n \in \Gamma, n \geq 0} |e^{ic_1} - e^{i\theta_n}|^\rho < \infty \quad (\rho > 1). \quad (18)$$

(18) is equivalent to (9). Q.E.D.

**Lemma 3.** *Let  $\{x_n\}_{n \in \mathbb{Z}}$  be an infinite sequence in the interval  $(0, 2\pi)$  such that*

$$2\pi > \dots \geq x_1 \geq x_0 (\geq \pi) > x_{-1} \geq \dots > 0, \quad \lim_{n \rightarrow \infty} x_n = 2\pi, \quad \text{and} \quad \lim_{n \rightarrow \infty} x_{-n} = 0,$$

*and let  $\sigma \geq 1$ . Then the following (a) and (b) are equivalent.*

$$(a) \quad \sum_{n \in \mathbb{Z}} |n|^p (x_n - x_{n-1}) < \infty \quad (0 < p < 1/\sigma).$$

$$(b) \quad \sum_{n=0}^{\infty} |2\pi - x_n|^\rho + \sum_{n=1}^{\infty} x_{-n}^\rho < \infty \quad (\sigma < \rho < \infty).$$

*Proof.* (a) implies (b): Let  $\rho \in (\sigma, \infty)$ , and let  $\phi = \sum_{n \in \mathbb{Z}} n \chi_{(x_{n-1}, x_n]}$ , where  $\chi_E$  denotes the characteristic function of  $E$ . Then by (a),  $\phi \in L^p([0, 2\pi]) \subset \text{weak } L^p([0, 2\pi])$  ( $0 < p < 1/\sigma$ ). Therefore, for each  $p$  ( $0 < p < 1/\sigma$ ), there exists  $C_p > 0$  satisfying

$$|\{x : |\phi(x)| \geq n\}| = (2\pi - x_{n-1}) + x_{-n} \leq \frac{C_p}{n^p} \quad (n \geq 1), \quad (19)$$

where  $|E|$  denotes the Lebesgue measure of  $E$ . If we choose  $p_0 (0 < p_0 < 1/\sigma)$  so that  $p_0\rho > 1$ , we have by (19),

$$\sum_{n=1}^{\infty} (2\pi - x_{n-1})^\rho + \sum_{n=1}^{\infty} x_{-n}^\rho \leq 2C_{p_0}^\rho \sum_{n=1}^{\infty} \frac{1}{n^{\rho p_0}} < \infty.$$

(b) implies (a): Let  $p \in (0, 1/\sigma)$  be arbitrary. For each positive integer  $n$ , there exists a  $\theta \in (0, 1)$  such that

$$\frac{n^p - (n-1)^p}{n^{p-1}} = \frac{1 - (1 - 1/n)^p}{1/n} = p\left(1 - \frac{\theta}{n}\right)^{p-1},$$

by the mean value theorem. So, there exists  $C_p > 0$  such that

$$n^p - (n-1)^p \leq C_p n^{p-1} \quad (n \geq 2), \quad (20)$$

Since  $0 < p < 1/\sigma$ , we can choose  $\rho > \sigma$  and  $q$  such that  $q(p-1) < -1$ ,  $1/\rho + 1/q = 1$ . In fact, we can simply set  $\rho = 1/p - \varepsilon > \sigma$  for some small  $\varepsilon > 0$ . Then by (20) and Hölder's inequality, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^p (x_n - x_{n-1}) + \sum_{n=1}^{\infty} (-n)^p (x_{-n} - x_{-n-1}) \\ &= \sum_{n=1}^{\infty} (n^p - (n-1)^p) (2\pi - x_{n-1}) + \sum_{n=1}^{\infty} (n^p - (n-1)^p) x_{-n} \\ &\leq 2\pi - x_0 + x_{-1} + \sum_{n=2}^{\infty} C_p n^{p-1} \{(2\pi - x_{n-1}) + x_{-n}\} \\ &\leq 2\pi - x_0 + x_{-1} + C_p \left( \sum_{n=2}^{\infty} n^{(p-1)q} \right)^{1/q} \left( \sum_{n=2}^{\infty} (2\pi - x_{n-1})^\rho \right)^{1/\rho} \\ &\quad + C_p \left( \sum_{n=2}^{\infty} n^{(p-1)q} \right)^{1/q} \left( \sum_{n=2}^{\infty} (x_{-n})^\rho \right)^{1/\rho}. \end{aligned} \quad (21)$$

By (21), we can conclude that (b) implies (a). Q.E.D.

### 3. The statement and the proof of the main theorem.

**Theorem 1.** *Let  $\mathcal{B}$  be an infinite Blaschke product such that there exist  $c_0, c_1 \in \mathbb{R}$  which satisfy*

$$0 \leq c_0 < 2\pi, c_0 < c_1 \leq c_0 + 2\pi, \{e^{i\theta} : c_0 \leq \theta \leq c_1\} \cap \text{sing}(\mathcal{B}) = \{e^{ic_0}, e^{ic_1}\}.$$

We denote by  $\Delta_j$  ( $j = 0, 1$ ) the sectors of the form

$$\Delta_0 = \{re^{i\theta} : 0 < r \leq 1, c_0 < \theta < \frac{c_0 + c_1}{2}\}, \quad \Delta_1 = \{re^{i\theta} : 0 < r \leq 1, \frac{c_0 + c_1}{2} < \theta < c_1\}.$$

For  $f \in \mathcal{S}^{\mathcal{B}}$ , we define the order of convergence of  $Z(f; \overline{D}) \cap \Delta_j$  to  $e^{ic_j}$   $j = 0, 1$  by

$$\text{Ord}[e^{ic_j}; Z(f; \overline{D}) \cap \Delta_j] = \inf\{\rho > 0 : \sum_{z \in Z(f; \overline{D}) \cap \Delta_j} |z - e^{ic_j}|^\rho < \infty\}.$$

Then the following (i) and (ii) hold.

(i) *If  $\text{Ord}[e^{ic_j}; Z(f; \overline{D}) \cap \Delta_j] = \sigma > 1$  for some  $f \in \mathcal{S}^{\mathcal{B}}$ , then we have*

$$\text{Ord}[e^{ic_j}; Z(g; \overline{D}) \cap \Delta_j] = \sigma \quad (g \in \mathcal{S}^{\mathcal{B}}).$$

(ii) *If  $\text{Ord}[e^{ic_j}; Z(f; \overline{D}) \cap \Delta_j] \leq 1$  for some  $f \in \mathcal{S}^{\mathcal{B}}$ , then we have*

$$\text{Ord}[e^{ic_j}; Z(g; \overline{D}) \cap \Delta_j] \leq 1 \quad (g \in \mathcal{S}^{\mathcal{B}}).$$

*Proof.* Let  $f, g \in \mathcal{S}^{\mathcal{B}}$  and  $j \in \{0, 1\}$  be arbitrary, and suppose that the following inequality

$$\sum_{z \in Z(f; \overline{D}) \cap \Delta_j} |e^{ic_j} - z|^\sigma < \infty \tag{22}$$

holds for some  $\sigma > 1$ . We will deduce the following inequalities

$$\sum_{z \in Z(g; \overline{D}) \cap \Delta_j} |e^{ic_j} - z|^\rho < \infty \quad (\rho > \sigma), \tag{23}$$

from (22), and it is easy to see that this is sufficient to prove (i) and (ii).

Let  $f = hbs$  be the canonical decomposition of  $f$ , where  $h$  is the outer part,  $b$  is the Blaschke part and  $s$  is the singular inner part of  $f$ , respectively. If we define the outer function  $g_b \in \mathcal{S}^b \cap H^\infty$  by the method of Lemma 1, then it follows from Lemma 1 and Lemma 2 that  $\tilde{f} := hg_b(1 + s)^2$  is an outer function in  $\mathcal{S}^{\mathcal{B}}$  which satisfies

$$\sum_{z \in Z(f; \bar{D}) \cap \Delta_j} |e^{ic_j} - z|^\sigma < \infty.$$

In the same way, we construct an outer function  $\tilde{g} \in \mathcal{S}^B$  from  $g$ . If we can show that the relation (23) is true for  $\tilde{g}$ , then it follows that the relation (23) is also true for  $g$  by the definition of  $\tilde{g}$ , Proposition 1, Lemma 1 and Lemma 2. Therefore, to prove (23) from (22), we have only to prove (23) in case  $f$  and  $g$  in (22) are both outer.

We prove (23) only in case  $j = 0$  since the proof in case  $j = 1$  is almost the same with that of the case  $j=0$ . Further, we can assume (without loss of generality)  $c_0 = 0$ .

Let us assume that  $f$  and  $g$  in (22) are both outer and denote the zeros of  $f$  and  $g$  in  $\Delta_0$  in the form :

$$Z(f; \bar{D}) \cap \Delta_0 = \{e^{ix_n}\}_{n < 0; c_1/2 > x_{-1} \geq x_{-2} \geq \dots > 0, \quad (24)$$

$$Z(g; \bar{D}) \cap \Delta_0 = \{e^{iy_n}\}_{n < 0; c_1/2 > y_{-1} \geq y_{-2} \geq \dots > 0, \quad (25)$$

then the following inequality holds by (22) (note that  $e^{ic_0} = 1$  in our present situations).

$$\sum_n x_n^\sigma < \infty. \quad (26)$$

We define functions  $\varphi_f(\theta)$ ,  $\varphi_g(\theta)$  and  $\varphi_{g/f}(\theta)$  on  $(0, 2\pi)$  as follows:

$$\begin{aligned} \varphi_f(\theta) &= \begin{cases} -2\pi \#\{n : x_n \geq \theta\} & 0 < \theta < c_1/2, \\ 0 & \text{otherwise,} \end{cases} \\ \varphi_g(\theta) &= \begin{cases} -2\pi \#\{n : y_n \geq \theta\} & 0 < \theta < c_1/2, \\ 0 & \text{otherwise,} \end{cases} \\ \varphi_{g/f}(\theta) &= \begin{cases} -2\pi \left( \#\{n : y_n \geq \theta\} - \#\{n : x_n \geq \theta\} \right) & 0 < \theta < c_1/2, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where  $\#A$  denotes the number of elements of  $A$ . We can deduce from (26) and Lemma 3 that,

$$\varphi_f \in L^p([0, 2\pi]) \quad (0 < p < 1/\sigma). \quad (27)$$

If we define  $\log(g/f)$  so that  $\lim_{x \uparrow c_0/2} \Im[\log(g/f)(e^{ix})] = 0$ , a moment's thought reveals that

$$\mathcal{X}_{(0, c_1/2)}(\theta) \Im[\log(g/f)(e^{i\theta})] = \varphi_{g/f}(\theta) \text{ a.e. } \theta. \quad (28)$$

Since  $g/f \in N_+$ , it follows from Kolmogorov's theorem that

$$\Im[\log(g/f)(e^{i\theta})] \in L^p([0, 2\pi]) \quad (0 < p < 1). \quad (29)$$

By (27), (28) and (29) we have

$$\varphi_g = \varphi_{g/f} + \varphi_f \in L^p([0, 2\pi]) \quad (0 < p < 1/\sigma). \quad (30)$$

From (30) and Lemma 3, we get

$$\sum_n y_n^p < \infty \quad (\sigma < \rho). \quad (31)$$

(31) is equivalent to the desired inequalities (23) for an outer  $g \in \mathcal{S}^B$  in case  $j = 0$  and  $c_0 = 0$ . Q.E.D.

**Example 1.** Let  $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2\mathcal{B}_3$ , where  $\mathcal{B}_j : j = 1, 2, 3$  are Blaschke products such that:

(i)  $\text{sing}(\mathcal{B}_1) \subseteq \{e^{i\theta} : \pi < \theta < 2\pi\}$ ,

(ii)  $\mathcal{B}_2$  is the Blaschke product with the zeros  $\{\alpha_{-n}\}_{n=1}^{\infty}$  with

$$\alpha_{-n} = \left(1 - \left(\frac{1}{n}\right)^{p_0}\right) e^{i\left(\frac{1}{n}\right)^{1/\sigma_0}}, \quad p_0 > 1, \sigma_0 > 1.$$

(iii)  $\mathcal{B}_3$  is the Blaschke product with the zeros  $\{\alpha_n\}_{n=1}^{\infty}$ , where

$$\alpha_n = \left(1 - \left(\frac{1}{n}\right)^{p_1}\right) e^{i\left(\pi - \left(\frac{1}{n}\right)^{1/\sigma_1}\right)}, \quad p_1 > 1, \sigma_1 > 1.$$

We put

$$\Delta_0 = \{re^{i\theta}; 0 < r \leq 1, 0 < \theta < \pi/2\}, \quad \Delta_1 = \{re^{i\theta}; 0 < r \leq 1, \pi/2 < \theta < \pi\},$$

Then we have for  $j \in \{0, 1\}$

$$\text{Ord}[(-1)^j; Z(\mathcal{B}; \overline{D}) \cap \Delta_j] = \sigma_j > 1,$$

and hence we have by Theorem 1

$$\text{Ord}[(-1)^j; Z(g; \overline{D}) \cap \Delta_j] = \sigma_j > 1 \quad (g \in \mathcal{S}^B).$$

**Remark 1.** In Theorem 1 (i), we can not replace  $\sigma > 1$  by  $\sigma \geq 1$  as the following example shows.

**Example 2.** Let  $\mathcal{B}$  be an infinite Blaschke product with the zeros  $\{\alpha_n\}_{n=1}^{\infty}$ , where

$$\alpha_n = \left(1 - \frac{1}{n(\log(n+1))^2}\right) e^{i/n^2}, \quad n = 1, 2, \dots$$

Then,  $\text{Ord}[1; Z(\mathcal{B}; \overline{D}) \cap \Delta_0] = 1$ , where  $\Delta_0 = \{re^{i\theta} : 0 < r \leq 1, 0 < \theta < \pi\}$ . On the other hand we have by Lemma 1 that

$$g_{\mathcal{B}}(z) := \prod_{n=1}^{\infty} \frac{(1 - e^{i/n^2} z)^2}{(1 - \overline{\alpha_n} z)^2} \in \mathcal{S}^{\mathcal{B}} \cap H^{\infty},$$

and  $\text{Ord}[1; Z(g_{\mathcal{B}}; \overline{D}) \cap \Delta_0] = 1/2$ .

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