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On Pinching of Curves Moved by Surface Diffusion

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Abstract

We give a rigorous proof for formation of pinching of evolving curves moved by surface diffusion.

Keywords: pinching, surface diffusion, evolving curves, unique local existence.

1 Introduction

We study motion by surface diffusion which was first derived by Mullins [8].

Let $\Gamma_t \subset \mathbf{R}^2$ be a closed evolving curve depending on time t with initial data $\Gamma_t|_{t=0} = \Gamma_0$. The governing equation for evolving curves by surface diffusion is of the form

$$V = -\kappa_{ss}. \tag{1}$$

Here V denotes the outward normal velocity and κ denotes the outward curvature; s denotes the arclength parameter of Γ_t. There are several derivations of this equation other than Mullins [8]. See for example Cahn and Taylor [2] and Cahn, Elliott and Novick-Cohen [3]. In the latter paper, (1) is obtained as some formal limit of Cahn-Hilliard equations. A typical feature of Γ_t moved by (1) is that the area enclosed by Γ_t is preserved. Related equations to (1) are well explained in Elliott and Garcke [4] and Cahn and Taylor [2]. For physical background of these equations, see [2], [4] and references cited there.

In [4] local existence of solution for (1) was proved without uniqueness as well as for other equations. They proved that if initial data is close to a circle, then Γ_t exists globally in time and it converges to a circle with the same area enclosed by Γ_0 as t tends to

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infinity. They also conjectured that Γ_t moved by (1) may cease to be embedded for some embedded smooth initial data. In this paper we give a rigorous proof to their conjecture.

Let us explain our idea. We consider a smooth closed embedded curve Γ_0 which is symmetric with respect to x-axis and y-axis. We assume that Γ_0 is of the form

$$\Gamma_0 = \{(x, y); y = \pm u_0(x)\},$$

where $u_0(x)$ is even and $u_0(x)$ takes the only local minimum at $x = 0$. If Γ_t is represented by $y = u(t, x)$, then (1) becomes a fourth order equation of $u(t, x)$. If we linearize (1) around $u = 0$, we obtain

$$u_t = -u_{xxxx}.$$

If we consider the Cauchy problem for this equation with $u_0(x) \geq 0$ and $u_0(x) = x^4 + \delta$ for small $\delta > 0$ near $x = 0$, then $u(t, 0)$ would be negative in a short time. In other words, the comparison principle does not hold. It is easy to guess this phenomenon since $u(t, x) = x^4 - 4!t + \delta$ solves $u_t = -u_{xxxx}$. In this paper we shall rigorously prove that for a good choice of $u_0(x)$, $u(t, 0)$ becomes negative in short time during the period that solution Γ_t: $y = u(t, x)$ of (1) exists as immersed curves. Since Γ_t is represented by $y = u(t, x)$, and symmetric with respect to $y = 0$, this means that Γ_t ceases to be embedded in short time even if Γ_0 is embedded. This is a rough idea of our proof. For this purpose we shall review unique local existence theorem for immersed curves with estimates of existence time interval as well as of solutions. Of course there are several versions of unique existence theorems which apply in this setting e.g. by Lunardi [7] but we presented a simple version with the aid of the classical Lax-Milgram type abstract existence theorem due to J. L. Lions [5] (see e.g. [6], [9]) which just needs Hilbert spaces. Since we proved the uniqueness of solutions as well as higher derivative estimate near $t = 0$, our unique local existence theorem is not included in Elliott and Garcke [4] which is based on the method of X. Chen [1].

Our paper is organized as follows. In Section 2, we introduce a parametrized equation for (1). After showing a unique local existence theorem for this equation, we rigorously prove formation of pinching of evolving curves moved by surface diffusion. In Section 3, we briefly prove the unique local existence theorem in a general framework of the parametrized equation for (1).

2 Pintching of evolving closed curves

2.1 Parametrization

We summarize here a parametrization of (1) by following Elliott and Garcke [4],

Let M^0 be a fixed reference immersed closed curve with arclength $2L$. For $\mathbf{T} = \mathbf{R}/(2L\mathbf{Z})$, let

$$\begin{aligned} X^0 : \quad & \mathbf{T} \to M^0, \\ & \eta \mapsto X^0(\eta) \end{aligned}$$

be an arclength parametrization of M^0. Then, $\tau^0(\eta) = X^0_\eta(\eta)$ is the unit tangent vector of M^0 and the Frenet formula gives

$$\begin{aligned} \tau^0_\eta(\eta) &= \kappa^0(\eta) n^0(\eta), \\ n^0_\eta(\eta) &= -\kappa^0(\eta) \tau^0(\eta), \end{aligned}$$

where $n^0(\eta)$ is the unit normal vector and $\kappa^0(\eta)$ is the curvature of M^0 with the sign convention that the curvature of a circle is negative.

Let $\Gamma_t \subset \mathbf{R}^2$ be a closed curve moved by surface diffusion law with respect to time $t \geq 0$ starting from initial closed curve Γ_0. For small $T > 0$ we expect that Γ_t is parametrized by

$$\begin{array}{rccl} X: & [0,T) \times \mathbf{T} & \to & \Gamma_t, \\ & (t,\eta) & \mapsto & X(t,\eta), \end{array}$$

$$X(t,\eta) = X^0(\eta) + d(t,\eta)n^0(\eta)$$

with some $d(t,\eta)$ defined on $[0,T) \times \mathbf{T}$. If Γ_0 is embedded and Γ_t is close to Γ_0, then $d(t,\eta)$ is the distance function from M^0. By this parametrization, (1) is equivalent to

$$\frac{1 - d\kappa^0}{J} d_t = -\frac{1}{J}\partial_\eta\left(\frac{1}{J}\partial_\eta \kappa\right),$$

where $J = |X_\eta| = \partial s/\partial \eta$ is the Jacobian and $\kappa(t,\eta)$ is the curvature of Γ_t in the direction of n^0. Their explicit forms are

$$J = J(\eta, \alpha_0, \alpha_1)|_{(\alpha_0,\alpha_1)=(d,d_\eta)} = (d_\eta^2 + (1 - d\kappa^0)^2)^{1/2},$$

$$\kappa = \frac{1}{J^3}\{(1 - d\kappa^0)d_{\eta\eta} + 2\kappa^0 d_\eta^2 + \kappa_\eta^0 d d_\eta + \kappa^0(1 - d\kappa^0)^2\}.$$

Thus, the equation (1) for $d(t,\eta)$ with initial data $\Gamma_t|_{t=0} = \Gamma_0$ is of the form:

$$\begin{cases} d_t + J^{-4} d_{\eta\eta\eta\eta} + P d_{\eta\eta\eta} + Q = 0, \ 0 < t < T, \ \eta \in \mathbf{T}, \\ d(0,\eta) = d_0(\eta), \ \eta \in \mathbf{T}, \end{cases} \tag{2}$$

where P and Q are polynomials with arguments $(1 - \kappa^0 d)^{-1}$, J^{-1}, κ^0, κ_η^0, $\kappa_{\eta\eta}^0$, $\kappa_{\eta\eta\eta}^0$, d, d_η and $d_{\eta\eta}$.

2.2 Local existence

We state here a result of the unique local existence of smooth solutions of (2). To do this, we first treat a general framework.

We consider the equation:

$$\begin{cases} u_t + a(x,u,u_x)u_{xxxx} + b(x,u,u_x,u_{xx})u_{xxx} + c(x,u,u_x,u_{xx}) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3}$$

for $t > 0$ and $x \in \mathbf{T} = \mathbf{R}/(\omega\mathbf{Z})$ with $\omega > 0$. For (3), we assume:

(a) The function $a(x,\alpha_0,\alpha_1)$ is positive.

(b) Let $M > 0$ be given. The functions $a(x,\alpha_0,\alpha_1)$, $b(x,\alpha_0,\alpha_1,\alpha_2)$ and $c(x,\alpha_0,\alpha_1,\alpha_2)$ are smooth in their all arguments but restricted for $|\alpha_0| \leq 2\mu M$ and ω-periodic in x, where $\mu = \mu(\mathbf{T}) > 0$ denotes a number in Sobolev inequality:

$$\|f\|_{L^\infty(\mathbf{T})} \leq \mu \|f\|_{H^1(\mathbf{T})} \quad \text{for } f \in H^1(\mathbf{T}). \tag{4}$$

Theorem 1 *(Local existence for (3)). Let $M > 0$. Assume (a)-(b). Then, for any $u_0 \in H^4(\mathbf{T})$ with $\|u_0\|_{H^4(\mathbf{T})} \le M$, there is a $T_0(M) > 0$ such that there exists a unique solution $u(t,x)$ of (3) satisfying*

$$u \in L^2(0, T_0(M); H^6(\mathbf{T})), \quad u_t \in L^2(0, T_0(M); H^2(\mathbf{T})), \tag{5}$$

$$\|u\|_{H^4(\mathbf{T})}(t) \le 2M \quad \text{for } t \in [0, T_0(M)]. \tag{6}$$

Corollary 2 *Let $m \ge 4$ be integers and let $N \in (0, M]$. Then, for any $u_0 \in H^m(\mathbf{T})$ with $\|u_0\|_{H^m(\mathbf{T})} \le N$, there is a $T_1(N) > 0$ such that there exists a unique solution $u(t,x)$ of (3) satisfying*

$$u \in L^2(0, T_1(N); H^{m+2}(\mathbf{T})), \quad u_t \in L^2(0, T_1(N); H^{m-2}(\mathbf{T})), \tag{7}$$

$$\|u\|_{H^m(\mathbf{T})}(t) \le 2N \quad \text{for } t \in [0, T_1(N)]. \tag{8}$$

An outline of Theorem 1 is given in Section 3. From Theorem 1 and Corollary 2, we obtain

Theorem 3 *(Local existence for (2)). Let ω and δ_0 be as*

$$\omega = 2L, \quad 4\delta_0 \|\kappa^0\|_{L^\infty(\mathbf{T})} \le 1. \tag{9}$$

(i) Let $M \in (0, \delta_0/\mu)$ where μ is in (4). Then, for any $d_0 \in H^4(\mathbf{T})$ with $\|d_0\|_{H^4(\mathbf{T})} \le M$, there is a $T_0(M) > 0$, which is nonincreasing in M, such that there exists a unique solution $d(t,\eta)$ of (2) satisfying

$$d \in L^2(0, T_0(M); H^6(\mathbf{T})), \quad d_t \in L^2(0, T_0(M); H^2(\mathbf{T})), \tag{10}$$

$$\|d\|_{H^4(\mathbf{T})}(t) \le 2M \quad \text{for } t \in [0, T_0(M)]. \tag{11}$$

(ii) Let $m \ge 4$ be integers and $N \in (0, \delta_0/\mu)$. Then, for any $d_0 \in H^m(\mathbf{T})$ with $\|d_0\|_{H^m(\mathbf{T})} \le N$, there is a $T_1(N) > 0$, which is nonincreasing in N, such that there exists a unique solution $d(t,\eta)$ of (2) satisfying

$$d \in L^2(0, T_1(N); H^{m+2}(\mathbf{T})), \quad d_t \in L^2(0, T_1(N); H^{m-2}(\mathbf{T})), \tag{12}$$

$$\|d\|_{H^m(\mathbf{T})}(t) \le 2N \quad \text{for } t \in [0, T_1(N)]. \tag{13}$$

Remark 1. Our local existence theorem in particular implies that for any immersed smooth curve Γ_0, there exists a unique local-in-time solution of (1) by taking Γ_0 as a reference curve M^0 with $d_0 = 0$.

Remark 2. It is easy to see from the above construction manner of solutions that solution curves Γ_t are uniquely determined by Γ_0 not depending on parametrizations.

Remark 3. Since our main conceren in this paper is pintching, we do not pursue regularity property like higher regularity away from $t = 0$.

Proof of Theorem 3 admitting Theorem 1 and Corollary 2. It suffices to prove (b). For $|\alpha_0| \le 2\mu M$, $J(\eta, \alpha_0, \alpha_1)^{-4}$ has no singularities: in fact, it follows from (9) that

$$\begin{aligned} &\alpha_1^2 + (1 - \alpha_0 \kappa^0(\eta))^2 \ge (1 - 2\mu M \|\kappa^0\|_{L^\infty(\mathbf{T})})^2 \\ &\ge (1 - 2\delta_0 \|\kappa^0\|_{L^\infty(\mathbf{T})})^2 \ge (\frac{1}{2})^2. \end{aligned}$$

Similarly, we can show that P and Q have no singularities on the place where we are concerned. This gives (b). Applying Theorem 1 to (2), we obtain (i). The proof of (ii) is essentially the same with M replaced by N, so we omit the proof of (ii). $\square$

2.3 Pinching of evolving closed curves

We show that there is an evolving closed curve which ceases to be embedded in finite time, even if initial curve is embedded.

To do this, we take a special reference curve M^0 immersed in $\mathbf{R}^2$. This is parametrized by

$$X^0(\eta) = (X_1^0(\eta), X_2^0(\eta)) \quad \text{for } \eta \in \mathbf{T} = \mathbf{R}/(2L\mathbf{Z})$$

satisfying

$$\begin{cases} X^0(\eta) = X^0(-\eta), & 0 \le \eta \le L, \\ X^0(\eta) = (\eta, 0), & 0 \le \eta \le L/4, \\ (X_1^0)_\eta(\eta) > 0, & 0 \le \eta \le L/2, \\ X_1^0(L/2+\eta) = X_1^0(L/2-\eta), & 0 \le \eta \le L/2, \\ X_2^0(\eta) > 0, & L/4 < \eta < L/2, \\ X_2^0(L/2+\eta) = -X_2^0(L/2-\eta), & 0 \le \eta \le L/2, \end{cases}$$

where η is an arclength parameter. We define two sets of functions in $\mathbf{T}$ and in $(0,T)\times\mathbf{T}$ depending on positive parameters N, ε and T:

$$\begin{aligned} D_0(N,\varepsilon) = \; & \{d_0 \in H^9(\mathbf{T}); d_0(-\eta) = d_0(\eta) = d_0(L-\eta), \quad d_0(\eta) > 0 \quad (\forall \eta \in \mathbf{T}), \\ & \|d_0\|_{H^9(\mathbf{T})} \le N, \quad d_0(0) < \varepsilon, \quad -3d_0''(0)^3 + d_0^{(4)}(0) > 0, \\ & d_0(\eta) \text{ attains its global minimum at } \eta = 0\}, \end{aligned}$$

$$\begin{aligned} D_T(N) = \; & \{d \in L^2(0,T; H^9(\mathbf{T})); d_t \in L^2(0,T; H^5(\mathbf{T})), \\ & \|d\|_{H^9(\mathbf{T})}(t) \le 2N \quad (t \in [0,T])\}. \end{aligned}$$

Note that closed curves Γ_0 parametrized by $X(0,\eta) = X^0(\eta) + d_0(\eta)n^0(\eta)$ with $d_0 \in D_0(N,\varepsilon)$ are embedded in $\mathbf{R}^2$. Then, our main result is stated as follows.

Theorem 4 *(Pinching of evolving closed curves). For any $N \in (0, \delta_0/\mu)$, there is an $\varepsilon_0 > 0$; for any $\varepsilon \in (0,\varepsilon_0)$, there are $d_0 \in D_0(N,\varepsilon)$, $t_0 \in (0, T_1(N))$ (where $T_1(N)$ is in Theorem 3 (ii)) and $t_1 (> t_0)$ such that for initial embedded closed curve Γ_0 with parametrization*

$$\Gamma_0 = \{X(0,\eta) = X^0(\eta) + d_0(\eta)n^0(\eta); \eta \in \mathbf{T}\},$$

the solution curve Γ_t with parametrization

$$\Gamma_t = \{X(t,\eta) = X^0(\eta) + d(t,\eta)n^0(\eta); \eta \in \mathbf{T}\}, \quad t \in [0, T_1(N)],$$

where $d \in D_{T_1(N)}(N)$ is the unique solution of (2) established in Theorem 3 (ii), ceases to be embedded for at least $t_0 < t < \min(t_1, T_1(N))$.

Proof. Take a $\tilde{d}_0 = \tilde{d}_0 \in D_0(N, \tilde{d}_0(0))$. For this $\tilde{d}_0$, Theorem 3 (ii) implies that there are $T_1(N) > 0$ and a unique solution $\tilde{d} \in D_{T_1(N)}(N)$ of (2) with initial data $\tilde{d}_0(\eta)$. Then, there is a $K(N) > 0$ such that

$$|-\partial_t\partial_\eta^4\tilde{d}(t,0) + 3\partial_t(\partial_\eta^2\tilde{d}(t,0))^3| \le K(N) \quad \text{for } t \in [0, T_1(N)].$$

Put

$$\sigma(\tilde{d}_0) = -3\tilde{d}_0''(0)^3 + \tilde{d}_0^{(4)}(0) > 0.$$

Take $\varepsilon_0 > 0$ as

$$\sigma(\tilde{d}_0)^2 - 4\varepsilon_0 K(N) > 0,$$
$$\frac{\sigma(\tilde{d}_0) - \sqrt{\sigma(\tilde{d}_0)^2 - 4\varepsilon_0 K(N)}}{2K(N)} < T_1(N).$$

Then, it holds

$$\sigma(\tilde{d}_0)^2 - 4\varepsilon K(N) > 0,$$
$$\frac{\sigma(\tilde{d}_0) - \sqrt{\sigma(\tilde{d}_0)^2 - 4\varepsilon K(N)}}{2K(N)} < T_1(N) \tag{14}$$

for $0 < \varepsilon < \varepsilon_0$. Take θ as

$$\max\{0, \tilde{d}_0(0) - \frac{\sigma(\tilde{d}_0)^2}{4K(N)}, \tilde{d}_0(0) - \varepsilon\} < \theta < \tilde{d}_0(0).$$

Put $d_0(\eta) = \tilde{d}_0(\eta) - \theta$. Then, it is easy to check that $d_0 \in D_0(N, \varepsilon)$. Thus, it follows from Theorem 3 (ii) that there exists a unique solution $d \in D_{T_1(N)}(N)$ of (2) with initial data $d_0(\eta)$. It follows from the uniqueness that

$$d_0(-\eta) = d_0(\eta) = d_0(L - \eta)$$

implies

$$d(t, -\eta) = d(t, \eta) = d(t, L - \eta). \tag{15}$$

Furthermore,

$$\sigma \equiv \sigma(d_0) = \sigma(\tilde{d}_0),$$
$$|-\partial_t \partial_\eta^4 d(t, 0) + 3\partial_t(\partial_\eta^2 d(t, 0))^3| \leq K(N) \quad \text{for } t \in [0, T_1(N)].$$

Then, from (14),

$$t_0 \equiv \frac{\sigma - \sqrt{\sigma^2 - 4\varepsilon K(N)}}{2K(N)} < T_1(N)$$

for $0 < \varepsilon < \varepsilon_0$. Finally, from the first equality of (15),

$$\begin{aligned}
d(t, 0) &= d_0(0) + \int_0^t d_\tau(\tau, 0) d\tau \\
&= d_0(0) + \int_0^t (-\partial_\eta^4 d(\tau, 0) + 3(\partial_\eta^2 d(\tau, 0))^3) d\tau \\
&= d_0(0) - \sigma t + \int_0^t \int_0^\tau (-\partial_s \partial_\eta^4 d(s, 0) + 3\partial_s(\partial_\eta^2 d(s, 0))^3) ds d\tau \\
&\leq \varepsilon - \sigma t + K(N) t^2,
\end{aligned}$$

which and the second equality of (15) imply

$$d(t, 0) = d(t, L) < 0 \quad \text{for } t_0 < t < \min(t_1, T_1(N)),$$

where

$$t_1 \equiv \frac{\sigma + \sqrt{\sigma^2 - 4\varepsilon K(N)}}{2K(N)}.$$

This shows that Γ_t ceases to be embedded for $t_0 < t < \min(t_1, T_1(N))$. This completes the proof. $\square$

3 Outline of the proof of the local existence theorem

In this section, we show an outline of the proof of Theorem 1 briefly. The proof of Corollary 2 is similar, so is omitted. Throughout this section, unless otherwise claimed, we denote by C and C_j $(j = 1, 2, \cdots)$ universal positive constants whose numerical values may be different in each occasion.

Outline of the proof of Theorem 1. Step 1. We begin by solving a linear equation for unknown function $w(t,x)$:

$$\begin{cases} w_t + a(x, u_0, u_{0x})w_{xxxx} = Av_{xxxx} + f, & (t,x) \in (0,T) \times \mathbf{T}, \\ w(0,x) = u_0(x), & x \in \mathbf{T}, \end{cases} \tag{16}$$

where $v(t,x)$ is any given function with

$$v \in L^2(0,T; H^6(\mathbf{T})), \quad v_t \in L^2(0,T; H^2(\mathbf{T}))$$

and A and f are defined by

$$A \equiv a(x, u_0, u_{0x}) - a(x, v, v_x),$$

$$f \equiv -b(x, v, v_x, v_{xx})v_{xxx} - c(x, v, v_x, v_{xx}).$$

We shall construct a mapping $\Phi : v \mapsto w$ by solving the equation (16).

Lemma 5 *(Unique existence for (16)). For any $u_0 \in H^4(\mathbf{T})$, there exists a unique solution $w(x,t)$ of (16) with*

$$w \in L^2(0,T; H^6(\mathbf{T})), \quad w_t \in L^2(0,T; H^2(\mathbf{T})).$$

Proof. We apply the classical Lax-Milgram type abstract existence theorem due to J. L. Lions [5] (see e.g. [6], Theorem 10.3 in [9]) to (16). To do this, it suffices to check the coercivity: Let $M > 0$. For $\varphi \in H^4(\mathbf{T})$ with $\|\varphi\|_{H^4(\mathbf{T})} \le M$, there are positive constants $c_0(M)$ and $\lambda_0(M)$ such that

$$(a(\cdot, \varphi, \varphi_x)u_{xxxx}, u)_{H^4(\mathbf{T})} \ge c_0(M)\|u_{xx}\|^2_{H^4(\mathbf{T})} - \lambda_0(M)\|u\|^2_{H^4(\mathbf{T})} \tag{17}$$

for $u \in H^6(\mathbf{T})$.

We prove (17). Take $\varphi \in H^4(\mathbf{T})$ and estimate $(\tilde{a}u_{xxxx}, u)_{H^4(\mathbf{T})}$, where we set $\tilde{a}(x) = a(x, \varphi(x), \varphi_x(x))$. We only estimate its leading term since the estimate for lower order terms is easy. It follows from (a) that there is a constant $a_1 > 0$ depending on $\|\varphi\|_{W^{1,\infty}(\mathbf{T})}$ such that $\tilde{a}(x) \ge a_1$ for $x \in \mathbf{T}$. Then,

$$\begin{aligned} &\int_{\mathbf{T}} \partial_x^4(\tilde{a}u_{xxxx})\partial_x^4 u \, dx = \int_{\mathbf{T}} \partial_x^2(\tilde{a}u_{xxxx})\partial_x^6 u \, dx \\ &= \int_{\mathbf{T}} \tilde{a}\partial_x^6 u \partial_x^6 u \, dx + \int_{\mathbf{T}} \partial_x^2\tilde{a}\partial_x^4 u \partial_x^6 u \, dx \\ &+2 \int_{\mathbf{T}} \partial_x\tilde{a}\partial_x^5 u \partial_x^6 u \, dx \\ &\ge a_1\|\partial_x^6 u\|^2_{L^2(\mathbf{T})} - \varepsilon a_1\|\partial_x^6 u\|^2_{L^2(\mathbf{T})} - C_\varepsilon(\|\varphi\|_{H^4(\mathbf{T})})\|u\|^2_{H^4(\mathbf{T})}, \end{aligned}$$

where we have used the Young inequality:

$$\forall\varepsilon > 0, \exists C_\varepsilon > 0; |ab| \le \varepsilon|a|^2 + C_\varepsilon|b|^2$$

and the interpolation inequality:

$$\|\partial_x^5 u\|_{L^2(\mathbf{T})} \le C\|\partial_x^6 u\|_{L^2(\mathbf{T})}^{1/2}\|\partial_x^4 u\|_{L^2(\mathbf{T})}^{1/2}.$$

After taking ε sufficiently small, we get (17). $\square$

Step 2. For $T > 0$ and $R > 0$, we define two sets of functions on $(0,T)\times\mathbf{T}$:

$$Z_T^4 = \{u \in L^2(0,T;H^6(\mathbf{T})); u_t \in L^2(0,T;H^2(\mathbf{T})), \quad u(0,x) = u_0(x)\},$$

$$B(T,R) = \{u \in Z_T^4; \|u\|_{Z_T^4} \le R\},$$

where

$$\|u\|_{Z_T^4} = \|u\|_{L^2(0,T;H^6(\mathbf{T}))} + \|u_t\|_{L^2(0,T;H^2(\mathbf{T}))}.$$

By *Step 1*, the mapping

$$\begin{array}{rccc} \Phi: & Z_T^4 & \to & Z_T^4 \\ & v & \mapsto & w = \Phi(v) \end{array}$$

is well-defined. We shall show:

Claim: Let $M > 0$. For any $u_0 \in H^4(\mathbf{T})$ with $\|u_0\|_{H^4(\mathbf{T})} \le M$, there are constants $T_0(M)$, $R_0(M) > 0$ and $\theta(M) \in (0,1)$ such that

$$v \in B_M \Rightarrow w = \Phi(v) \in B_M, \tag{18}$$

$$\|w_1 - w_2\|_{Z^4_{T_0(M)}} \le \theta(M)\|v_1 - v_2\|_{Z^4_{T_0(M)}} \quad \text{for } v_i \in B_M, \tag{19}$$

where $B_M = B(T_0(M), R_0(M))$ and $w_i = \Phi(v_i)$, $i = 1,2$.

If (18) and (19) are verfied, then the Banach fixed point theorem implies that there exists a unique $u \in B_M$ such that u is the fixed point of Φ restricted on B_M. This shows that u is the desired solution of (3) on $[0, T_0(M)]$.

It remains to prove the *Claim*. Here, we only verify (18), since (19) can be shown in a similar way. Let $T > 0$ and $R > 0$ be parameters determined later. Let $v \in B(T,R)$. Taking the $H^4(\mathbf{T})$-inner product of (16) and w and integrating it from 0 to T, we have

$$\begin{aligned} &\frac{1}{2}\|w(T)\|_{H^4(\mathbf{T})}^2 - \frac{1}{2}\|u_0\|_{H^4(\mathbf{T})}^2 + \int_0^T (a(\cdot,u_0,u_{0x})\partial_x^4 w, w)_{H^4(\mathbf{T})}dt \\ &= \int_0^T (A\partial_x^4 v, w)_{H^4(\mathbf{T})}dt + \int_0^T (f,w)_{H^4(\mathbf{T})}dt \\ &\equiv I + II. \end{aligned}$$

From the coercivity (17),

$$\begin{aligned} &c_0(M)\|w_{xx}\|_{L^2(0,T;H^4(\mathbf{T}))}^2 - \lambda_0(M)\int_0^T \|w\|_{H^4(\mathbf{T})}(t)^2 dt \\ &\le \int_0^T (a(\cdot,u_0,u_{0x})\partial_x^4 w, w)_{H^4(\mathbf{T})}dt. \end{aligned}$$

Estimate of I. We only estimate the leading term of I:

$$
\begin{aligned}
&\int_0^T \int_{\mathbf{T}} \partial_x^4(A\partial_x^4 v)\cdot \partial_x^4 w\,dxdt \\
&= \int_0^T \int_{\mathbf{T}} \partial_x^2(A\partial_x^4 v)\cdot \partial_x^6 w\,dxdt \\
&\leq \|\partial_x^2(A\partial_x^4 v)\|_{L^2(0,T;L^2(\mathbf{T}))}\|\partial_x^6 w\|_{L^2(0,T;L^2(\mathbf{T}))} \equiv I_1\|\partial_x^6 w\|_{L^2(0,T;L^2(\mathbf{T}))} \\
&\leq C(\delta, \|u_0\|_{H^4(\mathbf{T})})I_1^2 + \delta c_0(\|u_0\|_{H^4(\mathbf{T})})\|\partial_x^6 w\|^2_{L^2(0,T;L^2(\mathbf{T}))},
\end{aligned}
$$

where $\delta > 0$ is a sufficiently small parameter determined later. We only estimate of the leading term of I_1:

$$
\begin{aligned}
&\|A\partial_x^6 v\|_{L^2(0,T;L^2(\mathbf{T}))} \leq \|A\|_{L^\infty(0,T;L^\infty(\mathbf{T}))}\|\partial_x^6 v\|_{L^2(0,T;L^2(\mathbf{T}))} \\
&\leq C\|A\|_{L^\infty(0,T;H^1(\mathbf{T}))}\|\partial_x^6 v\|_{L^2(0,T;L^2(\mathbf{T}))},
\end{aligned}
$$

$$
\|A\|_{L^\infty(0,T;H^1(\mathbf{T}))} \leq C(\|u_0\|_{W^{1,\infty}(\mathbf{T})}, \|v\|_{L^\infty(0,T;W^{1,\infty}(\mathbf{T}))})\|u_0 - v\|_{L^\infty(0,T;H^2(\mathbf{T}))}.
$$

Observe that

$$
\begin{aligned}
&\|(u_0 - v)_{xx}\|_{L^\infty(0,T;L^2(\mathbf{T}))} = \sup_{t\in[0,T]} \|\int_0^t \partial_\tau v_{xx}(\tau,\cdot)d\tau\|_{L^2(\mathbf{T})} \\
&\leq \int_0^T \|\partial_\tau v_{xx}(\tau,\cdot)\|_{L^2(\mathbf{T})}d\tau \leq CT^{1/2}\|v_{txx}\|_{L^2(0,T;L^2(\mathbf{T}))}.
\end{aligned}
$$

Since the other terms are controlled more easily, from the above observations, we arrive at the following conclusion: there is a $C_1 > 0$; for δ, T, M, $R > 0$ and $i = 0, 1$, there are $C_I^{(i)}(\delta, T, M, R) > 0$ such that

$$
\begin{aligned}
I \ \leq\ & C_I^{(0)}(\delta, T, M, R) \\
& + C_I^{(1)}(\delta, T, M, R)\int_0^T \|w\|^2_{H^4(\mathbf{T})}(t)dt + C_1\delta c_0(M)\|w_{xx}\|^2_{L^2(0,T;H^4(\mathbf{T}))},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
& C_I^{(i)}(\delta, T, M, R) \to 0 \quad \text{as } T \to 0 \ (\delta,\ M,\ R\text{: fixed}), \\
& C_I^{(i)} \text{ is nondecreasing in } T.
\end{aligned}
\tag{21}
$$

Estimate of II. The following fact is fundamental.

Lemma 6 *Put*

$$
F(v) = f(x, v, v_x, v_{xx}, v_{xxx}).
$$

Then, F is a mapping

$$
F : Z_T^4 \to L^2(0, T; H^2(\mathbf{T}))
$$

satisfying:

(F1) for any $T > 0$ and $R > 0$, there is a $C(T, R) > 0$ such that

$$
\|F(v)\|_{L^2(0,T;H^2(\mathbf{T}))} \leq C(T, \|v\|_{Z_T^4}) \quad \textit{for } v \in Z_T^4,
$$

and $C(T, R) \to 0$ as $T \to 0$ for fixed R, *(F2) there holds*

$$\|F(v_2) - F(v_1)\|_{L^2(0,T;H^2(\mathbf{T}))} \leq C(T, \|v_1\|_{Z_T^4} + \|v_2\|_{Z_T^4})\|v_2 - v_1\|_{Z_T^4}$$

for v_1, $v_2 \in Z_T^4$.

Proof. We verify (F1). Here, we only estimate the term having the highest derivative:

$$II_1 \equiv \|b(\cdot, v, v_x, v_{xx})\partial_x^5 v\|_{L^2(0,T;L^2(\mathbf{T}))}.$$

Observe that

$$\begin{aligned}\|b(\cdot, v, v_x, v_{xx})\|_{L^\infty(0,T;L^\infty(\mathbf{T}))} &\leq \sup\{|b(x, \alpha)|; |x| \leq 2L, |\alpha| \leq \|v\|_{L^\infty(0,T;W^{2,\infty}(\mathbf{T}))}\} \\ &\leq \sup\{|b(x, \alpha)|; |x| \leq 2L, |\alpha| \leq C\|v\|_{L^\infty(0,T;H^4(\mathbf{T}))}\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\partial_x^5 v\|_{L^2(0,T;L^2(\mathbf{T}))} &\leq C\|\|\partial_x^4 v\|_{L^2(\mathbf{T})}^{1/2}\|\partial_x^6 v\|_{L^2(\mathbf{T})}^{1/2}\|_{L^2(0,T)} \\ &\leq C\|\partial_x^4 v\|_{L^\infty(0,T;L^2(\mathbf{T}))}^{1/2} T^{3/4}\|\partial_x^6 v\|_{L^2(0,T;L^2(\mathbf{T}))}^{1/2}.\end{aligned}$$

We now arrive at

$$II_1 \leq CT^{3/4}\sup\{|b(x, \alpha)|; |x| \leq 2L, |\alpha| \leq C\|v\|_{Z_T^4}\} \cdot \|v\|_{Z_T^4}.$$

Estimating the lower derivative terms, we thus prove (F1). The proof of (F2) can be done in a similar way. □

We turn to the estimate of the leading term of II. We set

$$g(t, x) = f(x, v(t, x), \cdots, v_{xxx}(t, x)).$$

From Lemma 6 (F1), it follows that

$$\begin{aligned}&\int_0^T \int_{\mathbf{T}} \partial_x^4 g \partial_x^4 w \, dx dt = \int_0^T \int_{\mathbf{T}} \partial_x^2 g \partial_x^6 w \, dx dt \\ &\leq \|\partial_x^2 g\|_{L^2(0,T;L^2(\mathbf{T}))}\|\partial_x^6 w\|_{L^2(0,T;L^2(\mathbf{T}))} \\ &\leq C(T, \|v\|_{Z_T^4})\|\partial_x^6 w\|_{L^2(0,T;L^2(\mathbf{T}))} \\ &\leq C(\delta, T, M, R) + \delta c_0(M)\|\partial_x^6 w\|_{L^2(0,T;L^2(\mathbf{T}))}^2\end{aligned}$$

with small parameter $\delta > 0$. Thus, estimating the lower derivative terms, we arrive at the following conclusion: there is a $C_2 > 0$; for δ, T, M, $R > 0$ and $i = 0, 1$, there are $C_{II}^{(i)}(\delta, T, M, R) > 0$ such that

$$\begin{aligned}II \leq\ & C_{II}^{(0)}(\delta, T, M, R) \\ &+ C_{II}^{(1)}(\delta, T, M, R)\int_0^T \|w\|_{H^4(\mathbf{T})}^2(t)dt + C_2\delta c_0(M)\|w_{xx}\|_{L^2(0,T;H^4(\mathbf{T}))}^2,\end{aligned} \tag{22}$$

$$\begin{aligned}&C_{II}^{(i)}(\delta, T, M, R) \to 0 \quad \text{as } T \to 0 \ (\delta,\ M,\ R\text{:fixed}), \\ &C_{II}^{(i)} \text{ is nondecreasing in } T.\end{aligned} \tag{23}$$

Put

$$C_{III}^{(i)}(\delta, T, M, R) = C_I^{(i)}(\delta, T, M, R) + C_{II}^{(i)}(\delta, T, M, R), \quad i = 0, 1.$$

From (20) and (22), we find

$$\begin{aligned}
&\|w(T_1)\|^2_{H^4(\mathbf{T})} + 2c_0(M)\|w_{xx}\|^2_{L^2(0,T_1;H^4(\mathbf{T}))} \\
&\leq M^2 + 2C_{III}^{(0)}(\delta, T, M, R) + 2(\lambda_0(M) + C_{III}^{(1)}(\delta, T, M, R)) \int_0^{T_1} \|w(t)\|^2_{H^4(\mathbf{T})} dt \\
&+2C_3\delta c_0(M)\|w_{xx}\|^2_{L^2(0,T_1;H^4(\mathbf{T}))}
\end{aligned}$$

for $0 \leq T_1 \leq T$, where $C_3 = C_1 + C_2$. Taking δ as $2C_3\delta = 1$ and applying the Gronwall lemma, we get

$$\|w(T_1)\|^2_{H^4(\mathbf{T})} \leq (M^2 + 2C_{III}^{(0)}(T, M, R))e^{2(\lambda_0(M) + C_{III}^{(1)}(T,M,R))T_1}$$

for $0 \leq T_1 \leq T$. Consequently,

$$\begin{aligned}
&\|w(T)\|^2_{H^4(\mathbf{T})} + c_0(M)\|w_{xx}\|^2_{L^2(0,T;H^4(\mathbf{T}))} \\
&\leq M^2 + 2C_{III}^{(0)}(T, M, R) \\
&+2(\lambda_0(M) + C_{III}^{(1)}(T, M, R))T(M^2 + 2C_{III}^{(0)}(T, M, R))e^{2(\lambda_0(M) + C_{III}^{(1)}(T,M,R))T}. \quad (24)
\end{aligned}$$

Then, utilyzing (21) and (23), one can take $R = R_0(M) > 0$ and $T_0(M) > 0$ sufficiently small such that

$$\text{the R.H.S. of (24)} \leq \min(4M^2, \frac{1}{16}c_0(M)R_0(M)^2), \quad (25)$$

and also

$$\|w\|_{L^2(0,T;H^1(\mathbf{T}))} \leq \frac{1}{4}R_0(M) \quad (26)$$

for $0 \leq T \leq T_0(M)$. Similarly, using (16), we may conclude

$$\|w_t\|_{L^2(0,T;H^2(\mathbf{T}))} \leq \frac{1}{2}R_0(M) \quad \text{for } 0 \leq T \leq T_0(M). \quad (27)$$

Combining (25)-(27), we have proved (18). (In the proof of (19), we use (F2) instead of (F1).) The proof of Theorem 1 is complete. □

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