Title	The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order
Author(s)	Nakamura, M.; Ozawa, T.
Citation	Hokkaido University Preprint Series in Mathematics, 383, 1-24
Issue Date	1997-6-1
DOI	10.14943/83529
Doc URL	http://hdl.handle.net/2115/69133
Type	bulletin (article)
File Information	pre383.pdf

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Series #383. June 1997

HOKKAIDO UNIVERSITY
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The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order

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Abstract

We show the local in time solvability of the Cauchy problem for nonlinear wave equations in the Sobolev space of critical order with nonlinear term of exponential type.

1 Introduction

We consider the Cauchy problem for the nonlinear wave equations in the Sobolev space of critical order with nonlinear term of exponential type:

$$\begin{aligned}
&\partial_t^2 u(t,x) = \Delta u(t,x) + f(u(t,x)),\\
&u(0,\cdot) = \phi \in H^{n/2}(\mathbf{R}^n),\ \partial_t u(0,\cdot) = \psi \in H^{n/2-1}(\mathbf{R}^n),
\end{aligned}$$

where u is a complex-valued function, ∂_t^2 is the second derivative of u with respect to time, Δ denotes the Laplacian of spatial variables, and f is a complex-valued function with two variables z and $\bar{z}$. There is a large literature on the Cauchy problem for the nonlinear wave equations(see the reference below and their references).

As usual, we regard the above equation as the following integral equation

$$(NLW) \qquad \begin{aligned}
&u(t) = V(t)\phi + U(t)\psi + \Gamma f(u)(t)\\
&u(0,\cdot) = \phi \in H^{n/2}(\mathbf{R}^n),\ \partial_t u(0,\cdot) = \psi \in H^{n/2-1}(\mathbf{R}^n).
\end{aligned}$$

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Here V, U, and Γ are given by

$$V(t)\phi \equiv F^{-1} \cos t|\xi| F\phi,$$
$$U(t)\psi \equiv F^{-1}(\sin t|\xi|)/|\xi| F\psi,$$
$$\Gamma h(t) \equiv \int_0^t F^{-1}(\sin(t-\tau)|\xi|)/|\xi| F h(\tau) d\tau,$$

where F and F^{-1} denote Fourier transform and its inverse, respectively.

Our purpose is to show the local solvability of (NLW) with the nonlinear term f of exponential type, for example $f(u) = u^2 \exp(\lambda|u|^2)$, $\lambda > 0$. The local solvability of (NLW) with data in the Sobolev space $H^s \times H^{s-1}$ is already shown by L.Kapitanski($0 \le s \le 1$)[5] and H.Pecher($1/2 \le s < n/2$)[16] if the nonlinear term is of a power type, such as $|u|^{p-1}u$. As for the nonlinearity of power type it seems natural that (NLW) has a solution with data in $H^s \times H^{s-1}$ in the case $0 \le s < n/2$ and $0 \le p-1 \le 2(n/2-s)^{-1}$ in view of the scaling argument on (NLW) (for details, see L.Kapitanski[5]). Indeed, the above two authors have shown the solvability of (NLW) essentially in this case. On the other hand, on the cases $s = n/2$ and $s > n/2$, it is believed that there is a local in time solution of (NLW) for any large p and for any function which satisfies $f \in C^{[s]}(\mathbf{C}, \mathbf{C})$ and $f(0) = 0$ respectively. The difference of the conditions for the solvability among $0 \le s < n/2$, $s = n/2$, and $s > n/2$ is basically from the embeddings $H^s \subset L^p$ if $2 \le p \le n(n/2-s)^{-1}$, $H^s \subset L^p$ if $2 \le p < \infty$ and $H^s \subset L^p$ if $2 \le p \le \infty$, respectively. Especially in the case $s > n/2$, the reason why it suffices only to assume that f satisfies $f \in C^{[s]}(\mathbf{C}, \mathbf{C})$ and $f(0) = 0$ is from the fact

$$\|f^{(k)}(u); L^\infty\| \le \sup_{|z| \le \|u; L^\infty\|} |f^{(k)}(z)| \le \sup_{|z| \le C\|u; H^s\|} |f^{(k)}(z)|, \qquad 0 \le k \le [s],$$

where C is a constant which is from the embedding $H^s \subset L^\infty$, so that we need not consider the behavior of f at infinity. In the second appendix in this paper, we show the local solvability of (NLW) with $s > n/2$.

In the case $s = n/2$, the usual Sobolev embedding $H^s \subset L^\infty$ breaks down and we cannot control the nonlinearity in L^∞. Accordingly, we use the inequality[12, 13]

$$\max(\|u; L^q(\mathbf{R}^n)\|, \|u; \dot{B}^0_q(\mathbf{R}^n)\|) \le C q^{1/2} \|u; H^{n/2}(\mathbf{R}^n)\|,$$

where $2 \le q < \infty$ and C is independent of q and u. Using this inequality, we show the solvability of (NLW) with the nonlinear term of exponential type. Specifically, we use the above inequality to ensure the convergence of Taylor's expansion of f, and at this stage we need to assume the smallness of initial data. If we suppose slower rate of growth of the nonlinearity, the smallness assumption on the data is removed as is shown in the first appendix below.

Our method in this paper requires only easy part of more sophisticated Strichartz inequality. We would emphasize here that as for the linear estimates the L^2 framework could essentially suffice in our problem, namely, the local solvability for the Sobolev space $H^s \times H^{s-1}$ with $s \geq n/2$.

Finally, we mention notations in this paper.

C denotes a positive constant independent of time and of the length of the time interval in question. For $T > 0$, $[0,T]$ denotes the time interval. For $s \in \mathbf{R}$, $[s]$ is the largest integer less than or equal to s. For $\xi \in \mathbf{R}^n$, $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. We follow the notation of [1, Chapter 6] on the (homogeneous) Sobolev and (homogeneous) Besov spaces. We make abbreviation such as $B^s_q = B^s_{q,2}(\mathbf{R}^n)$, $\dot{B}^s_q = \dot{B}^s_{q,2}(\mathbf{R}^n)$, $H^{s,q} = H^{s,q}(\mathbf{R}^n)$, $\dot{H}^{s,q} = \dot{H}^{s,q}(\mathbf{R}^n)$.

We frequently use the following facts

$$\begin{aligned}
&H^{s,q} \subset \dot{H}^{s,q}, \quad \text{if } s > 0, 1 < q < \infty,\\
&H^{s,q} = (H^{s_0,q_0}, H^{s_1,q_1})_{\theta,q},\\
&\quad \text{if } s = (1-\theta)s_0 + \theta s_1, 1/q = (1-\theta)/q_0 + \theta/q_1 \text{ with } 0 < \theta < 1,
\end{aligned}$$

where H may be replaced by B.

2 Linear estimate

Proposition 2.1 (Linear estimate) *For any $\hat{q}$ and μ with $1 < \hat{q} \leq 2$ and $\mu = n(1/\hat{q} - 1/2 - 1/n)$, Γ satisfies*

$$\|\Gamma h; L^\infty(0,T;L^2(\mathbf{R}^n))\| \leq C(1+T)\|h; L^1(0,T;H^{\mu,\hat{q}}(\mathbf{R}^n))\|.$$

Let $\hat{q}$ satisfy $1/2 \leq 1/\hat{q} < 1$ for $n = 1,2$, and $1/2 \leq 1/\hat{q} \leq 1/2 + 1/n$ for $n \geq 3$. Then Γ satisfies

$$\|\Gamma h; L^\infty(0,T;L^2(\mathbf{R}^n))\| \leq C(1+T)\|h; L^1(0,T;L^{\hat{q}}(\mathbf{R}^n))\|.$$

Moreover Γ is an operator from $L^1(0,T;H^{-1})$ to $C([0,T];L^2)$.

Proof) By the inequality

$$\left|((\sin t|\xi|)\langle\xi\rangle)/|\xi|\right| \leq \min(\langle\xi\rangle/|\xi|, t\langle\xi\rangle) \leq C(1+t), \tag{2.1}$$

we have

$$\begin{aligned}
&\|\Gamma h; L^\infty(0,T;L^2(\mathbf{R}^n))\|\\
= \ &\left\|\int_0^t F^{-1}\frac{\sin(t-\tau)|\xi|}{|\xi|}\langle\xi\rangle\langle\xi\rangle^{-1}Fh(\tau)d\tau; L^\infty(0,T;L^2(\mathbf{R}^n))\right\|\\
\leq \ &C(1+T)\|h; L^1(0,T;H^{-1}(\mathbf{R}^n))\|.
\end{aligned}$$

The two estimates in the proposition then follow by the embeddings $H^{\mu,\hat{q}} \subset H^{-1}$ and $L^{\hat{q}} \subset H^{-1}$.

For the last part of the proposition, it suffices to show, with $t \in [0,T]$ fixed,

$$\|\Gamma h(t+\epsilon) - \Gamma h(t); L^2\| \longrightarrow 0 \text{ as } \epsilon \to 0,$$

for any $h \in L^1(0,T;H^{-1})$. We have

$$\begin{aligned}
&\|\Gamma h(t+\epsilon) - \Gamma h(t); L^2\| \\
= \;& \|\int_0^{t+\epsilon} U(t+\epsilon-\tau)h(\tau)d\tau - \int_0^t U(t-\tau)h(\tau)d\tau; L^2\| \\
\leq \;& \|\int_0^t \{U(t+\epsilon-\tau) - U(t-\tau)\}h(\tau)d\tau; L^2\| + \|\int_t^{t+\epsilon} U(t+\epsilon-\tau)h(\tau)d\tau; L^2\|.
\end{aligned} \tag{2.2}$$

The first term on the right hand side of the inequality in (2.2) is rewritten as

$$\int_0^t \|\left(\frac{\sin(t+\epsilon-\tau)|\xi| - \sin(t-\tau)|\xi|}{|\xi|}\langle\xi\rangle\right)\langle\xi\rangle^{-1}Fh(\tau); L^2\|d\tau.$$

Using the pointwise convergence as $\epsilon \to 0$ of the last integrand and (2.1), we conclude by the Lebesgue convergence theorem that the first term on the right hand side of the inequality in (2.2) converges to 0 as $\epsilon \to 0$.

For the second term, we similarly have

$$\begin{aligned}
&\|\int_t^{t+\epsilon} U(t+\epsilon-\tau)h(\tau)d\tau; L^2\| \\
\leq \;& \int_t^{t+\epsilon} \|\Big(\left(\sin(t+\epsilon-\tau)|\xi|/|\xi|\right)\langle\xi\rangle\Big)\langle\xi\rangle^{-1}Fh(\tau); L^2\|d\tau \\
\leq \;& C(1+T)\int_t^{t+\epsilon} \|h(\tau); H^{-1}\|d\tau \\
\longrightarrow \;& 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

□

Remark Using the interpolation, we may replace $L^2(\mathbf{R}^n), H^{\mu,\hat{q}}(\mathbf{R}^n)$, and $L^{\hat{q}}(\mathbf{R}^n)$ with $B_2^s(\mathbf{R}^n), B_{\hat{q}}^{s+\mu}(\mathbf{R}^n)$, and $B_{\hat{q}}^s(\mathbf{R}^n)$ respectively in the above proposition, where s is any real number and B_p^s denotes the Besov space.

Proposition 2.2 *For any $s \in \mathbf{R}$, V and U satisfies*

$$\begin{aligned}
&\|V\phi; L^\infty(0,T;H^s)\| \leq \|\phi; H^s\|, \\
&\|U\psi; L^\infty(0,T;H^s)\| \leq C(1+T)\|\psi; H^{s-1}\|.
\end{aligned}$$

Moreover V and U are operators from H^s and H^{s-1} to $C([0,T];H^s)$, respectively.

Proof) The first inequality follows by the boundedness of $\cos t|\xi|$ and the second by (2.1).

For the continuity of $V\phi$ and $U\psi$, we only prove that of the latter since the other case follows similarly. It suffices to show that with $t \in [0, T]$ fixed

$$\|U(t+\epsilon)\phi - U(t)\phi; H^s\| \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0. \tag{2.3}$$

As before, passing to the Fourier representation, we have the pointwise convergence with respect to the Fourier transformed variables and therefore (2.3) follows by the Lebesgue dominated convergence theorem with (2.1). □

3 Estimate on nonlinear term

In this section, we consider estimates of the nonlinear term with growth rate of exponential type.

To clarify what nonlinear term we consider, we define a class of functions.

Definition Let $n \geq 1$ be an integer and let $\lambda > 0$. We define G_λ as

$$\begin{aligned} G_\lambda \;=\; & \{f \in C^1(\mathbf{C}, \mathbf{C}) \cap C^{[n/2]}(\mathbf{C}, \mathbf{C}) \mid f(0) = 0,\ |f'(z)| \leq C|z|e^{\lambda|z|^2}, \\ & |f^{(k)}(z)| \leq Ce^{\lambda|z|^2} \text{ for all } k \text{ with } 2 \leq k \leq [n/2]\}, \end{aligned}$$

where the last condition is disregarded if $[n/2] < 2$.

Below we show some examples of the functions in G_λ

1. $f(z) = z^2 e^{\lambda|z|^2}$,
2. $f(z) = |z|^{p-1} z e^{\lambda|z|^2}$, $|z|^p e^{\lambda|z|^2}$ with $p \in [2, \infty) \cap (n/2, \infty)$,
3. $f(z) = e^{\lambda|z|^2} - 1$,
4. $f(z) = P(z)e^{\lambda|z|^2}$,

and any linear combination of the above functions, where $P(z)$ is a polynomial of z and $\bar{z}$, having no constant and linear terms and λ is modified to be large if necessary.

Lemma 3.1 *If f is in G_λ, then*

1. $|f(z)| \leq C|z|^2 e^{\lambda|z|^2}$,
2. $|f(z_1) - f(z_2)| \leq C \sum_{i,j=1}^{2} |z_i| e^{\lambda|z_j|^2} |z_1 - z_2|$,
3. $|f^{k-1}(z_1) - f^{k-1}(z_2)| \leq C(e^{\lambda|z_1|^2} + e^{\lambda|z_2|^2})|z_1 - z_2|$, *for any k with $2 \leq k \leq [n/2]$* .

Proof) The lemma follows by direct calculations. For instance,

$$\begin{aligned}
& |f(z_1) - f(z_2)| \\
\le\ & \int_0^1 |f'(tz_1 + (1-t)z_2)| dt |z_1 - z_2| \\
\le\ & C \max(|z_1|, |z_2|) \exp(\lambda \max(|z_1|^2, |z_2|^2)) |z_1 - z_2| \\
\le\ & C(|z_1| + |z_2|)(\exp(\lambda |z_1|^2) + \exp(\lambda |z_2|^2)) |z_1 - z_2|.
\end{aligned}$$

□

In the following, to show the norm convergence of the power series expansion of functions of exponential type we frequently use the inequality

$$\max(\|u; L^q\|, \|u; \dot{B}^0_q\|) \le C q^{1/2} \|u; H^{n/2}\|, \tag{3.4}$$

for $2 \le q < \infty$, where C is independent of q.

Proposition 3.1 *Let $n \ge 1$, let $f \in G_\lambda$, and let $1 \le \hat{q} < 2$, then*

$$\|f(u); L^\infty(0,T; L^{\hat{q}})\| \le C(\lambda, \hat{q}, \|u; L^\infty(0,T; H^{n/2})\|) \|u; L^\infty(0,T; H^{n/2})\|^2,$$

$$\begin{aligned}
& \|f(u) - f(v); L^\infty(0,T; L^{\hat{q}})\| \\
\le\ & C(\lambda, \hat{q}, \|u; L^\infty(0,T; H^{n/2})\| + \|v; L^\infty(0,T; H^{n/2})\|) \\
& \cdot (\|u; L^\infty(0,T; H^{n/2})\| + \|v; L^\infty(0,T; H^{n/2})\|) \|u - v; L^\infty(0,T; L^2)\|,
\end{aligned}$$

for any u, v with

$$\max\{\|u; L^\infty(0,T; H^{n/2})\|, \|v; L^\infty(0,T; H^{n/2})\|\} < (2^5 \lambda e)^{-1/2} (1/\hat{q} - 1/2)^{1/2},$$

where the above constant $C(\lambda, \hat{q}, x)$ decreases to a constant as x tends to zero.

Proof) We use d'Alembert's criterion to verify the convergence of the Taylor expansion of the functions of exponential type. By (3.4), we have

$$\begin{aligned}
\|f(u); L^{\hat{q}}\| &\le C \| |u|^2 e^{\lambda |u|^2}; L^{\hat{q}} \| \\
&\le C \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \| |u|^{2+2\ell}; L^{\hat{q}} \| \\
&\le C \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \|u; L^{(2+2\ell)\hat{q}}\|^{2+2\ell} \\
&\le C \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \{(2+2\ell)\hat{q}\}^{1+\ell} \|u; L^\infty(0,T; H^{n/2})\|^{2+2\ell}.
\end{aligned}$$

Here we show the relation between the convergence of the last series and the smallness of u in $L^\infty(0,T;H^{n/2})$. If we set a_ℓ as

$$a_\ell \equiv \lambda^\ell \{(2+2\ell)\hat{q}\}^{1+\ell}\|u; L^\infty(0,T;H^{n/2})\|^{2\ell}/\ell!,$$

then to show the convergence of the series $\sum_{\ell=0}^\infty a_\ell$ it suffices to require that the limit of the ratio

$$\begin{aligned}\lim_{\ell\to\infty} a_{\ell+1}/a_\ell &= \lim_{\ell\to\infty} 2\lambda(1+1/(\ell+1))^{\ell+1}\hat{q}\|u; L^\infty(0,T;H^{n/2})\|^2 \\ &= 2\lambda e\hat{q}\|u; L^\infty(0,T;H^{n/2})\|^2\end{aligned}$$

should be less than one, from which the first part of the proposition follows.

In the following, let $q^*(\ell) = (1+2\ell)(1/\hat{q}-1/2)^{-1}$, so that $2 \le q^*(\ell) < \infty$ and $1/\hat{q} = (1+2\ell)/q^*(\ell) + 1/2$. We use the inequality (3.4) to estimate the $L^{q^*(\ell)}$-norm of u, v.

In the same way as above, with $(u_1,u_2) = (u,v)$ we have

$$\begin{aligned}&\|f(u)-f(v); L^{\hat{q}}\| \\ \le\ & C\sum_{i,j=1}^{2} \|u_i e^{\lambda|u_j|^2}|u-v|; L^{\hat{q}}\| \\ \le\ & C\sum_{i,j=1}^{2}\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u_i|u_j|^{2\ell}|u-v|; L^{\hat{q}}\| \\ \le\ & C\sum_{i,j=1}^{2}\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u_i; L^{q^*(\ell)}\|\|u_j; L^{q^*(\ell)}\|^{2\ell}\|u-v; L^\infty(0,T;L^2)\| \\ \le\ & C\sum_{i,j=1}^{2}\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(1+2\ell)/2}\|u_i; H^{n/2}\|\|u_j; H^{n/2}\|^{2\ell}\|u-v; L^\infty(0,T;L^2)\| \\ \le\ & C\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(1+2\ell)/2}(\|u; L^\infty(0,T;H^{n/2})\| + \|v; L^\infty(0,T;H^{n/2})\|)^{1+2\ell} \\ & \cdot\|u-v; L^\infty(0,T;L^2)\|,\end{aligned}$$

where the convergence of the last series is guaranteed by the smallness of u and v in $L^\infty(0,T;H^{n/2})$. □

Proposition 3.2 *Let n be even, let $f \in G_\lambda$, and let $1 \le \hat{q} < 2$. Then*

$$\|f(u); L^\infty(0,T;\dot{H}^{n/2,\hat{q}})\| \le C(\lambda,\hat{q},\|u; L^\infty(0,T;H^{n/2})\|)\|u; L^\infty(0,T;H^{n/2})\|^2,$$

for any u with

$$\|u; L^\infty(0,T;H^{n/2})\| < (2\lambda e)^{-1/2}(1/\hat{q}-1/2)^{1/2},$$

where the above constant $C(\lambda,\hat{q},x)$ decreases to a constant as x tends to zero.

Proof) For simplicity we assume that f is a function of u. General case follows by a similar argument on functions of two variables u and $\bar{u}$. By differentiating the composite function $f \circ u$, we have

$$\|f(u); \dot{H}^{n/2,\hat{q}}\| \leq C \sum_{|\alpha|=n/2} \sum_{k=1}^{n/2} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_j|\geq 1}} \|f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u; L^{\hat{q}}\|,$$

We separately consider the terms on the right hand side of the last inequality, namely we first consider the term

$$I \equiv \|f'(u)\partial^{n/2}u; L^{\hat{q}}\|,$$

and then the terms of the form

$$II \equiv \|f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u; L^{\hat{q}}\|,$$

with $\beta_1 + \cdots + \beta_k = \alpha$ with $|\alpha| = n/2$, $|\beta_1|, \cdots, |\beta_k| \geq 1$, and $2 \leq k \leq n/2$.

To estimate I, we set $q^*(\ell) = (1+2\ell)(1/\hat{q} - 1/2)^{-1}$, so that $2 \leq q^*(\ell) < \infty$, $1/\hat{q} = (1+2\ell)/q^*(\ell) + 1/2$. By the Hölder inequality and (3.4), we have

$$\begin{aligned} I &\leq \|ue^{\lambda|u|^2}\partial^{n/2}u; L^{\hat{q}}\| \\ &\leq \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \||u|^{1+2\ell}\partial^{n/2}u; L^{\hat{q}}\| \\ &\leq \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \|u; L^{q^*(\ell)}\|^{1+2\ell} \|u; L^\infty(0,T; H^{n/2})\| \\ &\leq \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} q^*(\ell)^{(1+2\ell)/2} \|u; L^\infty(0,T; H^{n/2})\|^{2+2\ell}, \end{aligned}$$

where the last series converges by the same argument as in the proof of Proposition 3.1.

To estimate II, we set $q^*(\ell) = (k-1+2\ell)(1/\hat{q} - 1/2)^{-1}$ and $1/q_j = (1 - 2\beta_j/n)/q^*(\ell) + |\beta_j|/n$, $j = 1, \cdots, k$, so that $2 \leq q^*(\ell) < \infty$, $2 \leq q_j < n$ and $1/\hat{q} = 2\ell/q^*(\ell) + \sum_{j=1}^{k} 1/q_j$. We use the interpolation inequality

$$\|\partial^{\beta_j} u; L^{q_j}\| \leq C\|u; \dot{B}^0_{q^*(\ell)}\|^{1-2|\beta_j|/n} \|u; H^{n/2}\|^{2|\beta_j|/n}, \tag{3.5}$$

where the constant C is independent of ℓ because q_j, $j = 1, \cdots, k$ are all in a compact set $[2, n]$ (see [1, Th 6.3.1, Th 6.4.4] and [4, Lemma A.1]). We estimate II as

$$II \equiv \|f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u; L^{\hat{q}}\|$$

$$\leq C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\||u|^{2\ell}\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat{q}}\|$$

$$\leq C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u; L^{q^*(\ell)}\|^{2\ell}\|u; \dot{B}^0_{q^*(\ell)}\|^{k-1}\|u; H^{n/2}\|$$

$$\leq C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(k-1+2\ell)/2}\|u; L^\infty(0,T; H^{n/2})\|^{2\ell}\|u; L^\infty(0,T; H^{n/2})\|^k,$$

where the last series converges by the same argument as above. Combining these estimates yields the propposition. □

Proposition 3.3 *Let n be odd, let $f \in G_\lambda$, and let $\hat{q}$ and μ satisfy $1/2 < 1/\hat{q} < 1/2 + 1/2n$ and $\mu = n(1/\hat{q} - 1/2 - 1/n)$. Then*

$$\begin{aligned} &\|f(u); L^\infty(0,T; \dot{B}^{n/2}_{\hat{q}})\| \\ \leq\ & C(\lambda, \hat{q}, \|u; L^\infty(0,T; H^{n/2})\|)\|u; L^\infty(0,T; H^{n/2})\|^2 \quad \text{if } n=1, \\ &\|f(u); L^\infty(0,T; \dot{B}^{n/2+\mu}_{\hat{q}})\| \\ \leq\ & C(\lambda, \hat{q}, \|u; L^\infty(0,T; H^{n/2})\|)\|u; L^\infty(0,T; H^{n/2})\|^2 \quad \text{if } n\geq 3, \end{aligned}$$

for any u with $\|u; L^\infty(H^{n/2})\| < (2^3\lambda e)^{-1/2}(1/\hat{q} - 1/2)^{1/2}$, where the constant $C(\lambda, \hat{q}, x)$ decrease to a constant as x tends to zero.

Proof) First we note that the following equivalent norm on the homogeneous Besov norm on $\dot{B}^\nu_r$ with $\nu \in \mathbf{R}_+\backslash\mathbf{Z}$ and $1 \leq r \leq \infty$

$$\|h; \dot{B}^\nu_r\| \simeq \{\int_0^\infty (t^{[\nu]-\nu}\sup_{|y|<t}\|\partial^{[\nu]}h - \partial^{[\nu]}\tau_y h; L^r\|)^2\frac{dt}{t}\}^{1/2}, \tag{3.6}$$

where τ_y is the shift operator by $y \in \mathbf{R}^n$.

We consider separately the cases $n = 1$ and $n \geq 3$.

• **Case n=1**

In this case we set $q^*(\ell) = (1+2\ell)(1/\hat{q} - 1/2)^{-1}$, so that $2 \leq q^*(\ell) < \infty$, $1/\hat{q} = (1+2\ell)/q^*(\ell) + 1/2$. With $(u_1, u_2) = (u, \tau_y u)$, we estimate the norm in the ingrand in (3.6) as

$$\begin{aligned} &\|f(u) - f(\tau_y u); L^{\hat{q}}\| \\ \leq\ & C\sum_{i,j=1}^{2}\|u_i e^{\lambda|u_j|^2}|u - \tau_y u|; L^{\hat{q}}\| \\ \leq\ & C\sum_{i,j=1}^{2}\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u_i|u_j|^{2\ell}|u - \tau_y u|; L^{\hat{q}}\| \end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u;L^{q^*(\ell)}\|^{1+2\ell}\|u-\tau_y u;L^2\| \\
&\leq C\sum_{l=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(1+2\ell)/2}\|u;L^\infty(0,T;H^{n/2})\|^{1+2\ell}\|u-\tau_y u;L^2\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|f(u);\dot{B}_{\hat{q}}^{1/2}\| \\
&\leq C(\lambda,\hat{q},\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\| \\
&\quad\cdot\{\int_0^\infty(t^{-1/2}\sup_{|y|<t}\|u-\tau_y u;L^2\|)^2\frac{dt}{t}\}^{1/2} \\
&\leq C(\lambda,\hat{q},\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|\|u;\dot{B}_2^{n/2}\| \\
&\leq C(\lambda,\hat{q},\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|^2, \qquad (3.7)
\end{aligned}
$$

where we have used the embedding $H^s\subset\dot{B}_2^s$ with $s>0$.

- **Case** $n\geq 3$

Let $N=[n/2+\mu]$. Note that $N=(n-3)/2$.

In the case $n=3$, we have $N=0$, so that an argument analogous to that in the preceding case works to yield

$$
\begin{aligned}
&\|f(u);\dot{B}_{\hat{q}}^{n/2+\mu}\| \\
&\leq C(\lambda,\hat{q},\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|\|u;\dot{B}_2^{n/2+\mu}\| \\
&\leq C(\lambda,\hat{q},\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|^2,
\end{aligned}
$$

where we have used the embedding $B_p^{s_1}\subset B_p^{s_2}$ with $s_1>s_2, 1\leq p\leq\infty$.

In the following, we assume $n\geq 5$, *i.e* $N\geq 1$. Let α satisfy $|\alpha|=N$.

To estimate an equivalent norm on $\dot{B}_{\hat{q}}^{n/2+\mu}$, we rewrite the difference by shifted functions as

$$
\begin{aligned}
&\partial^\alpha(f(u))-\partial^\alpha(f(\tau_y u)) \\
&=\sum_{k=1}^{N}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\geq 1}}C(\alpha,k,\beta)\{f^{(k)}(u)\prod_{j=1}^{k}\partial^{\beta_j}u-f^{(k)}(\tau_y u)\prod_{j=1}^{k}\partial^{\beta_j}\tau_y u\} \\
&=\sum_{k=1}^{N}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\geq 1}}C(\alpha,k,\beta)\{(f^{(k)}(u)-f^{(k)}(\tau_y u))\prod_{j=1}^{k}\partial^{\beta_j}u \\
&\quad+f^{(k)}(\tau_y u)\sum_{j=1}^{k}(\partial^{\beta_j}u-\partial^{\beta_j}\tau_y u)\prod_{\ell=1}^{j-1}\partial^{\beta_\ell}u\prod_{\ell=j+1}^{k}\partial^{\beta_\ell}\tau_y u\}. \qquad (3.8)
\end{aligned}
$$

We separately consider the terms of the form $(f^{(k)}(u)-f^{(k)}(\tau_y u))\prod_{j=1}^k \partial^{\beta_j}u$ and of the form $f^{(k)}(\tau_y u)(\partial^{\beta_j}u-\partial^{\beta_j}\tau_y u)\prod_{\ell=1}^{j-1}\partial^{\beta_\ell}u\prod_{\ell=j+1}^{k}\partial^{\beta_\ell}\tau_y u$.

We estimate the $L^{\hat{q}}$-norm of the terms in (3.8) of the first type as

$$\begin{aligned}
&\|(f^{(k)}(u)-f^{(k)}(\tau_y u))\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat{q}}\| \\
\leq\ & C\|e^{\lambda|u|^2}|u-\tau_y u|\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat{q}}\| + C\|e^{\lambda|\tau_y u|^2}|u-\tau_y u|\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat{q}}\|,
\end{aligned} \tag{3.9}$$

so that we have only to consider the first term on the right hand side of (3.9), since the second term is treated analogously.

We set $q^*(\ell)=(k-(n-3)/n+2\ell)/(1/\hat{q}-1/2)$ and $1/q_j=(1-2|\beta_j|/n)/q^*(\ell)+|\beta_j|/n$. Then we have

$$2\leq q^*(\ell)<\infty,\quad 2\leq q_j<n,\ j=1,\cdots,k,$$

$$1/\hat{q}=2\ell/q^*(\ell)+3/(2n)+\sum_{j=1}^{k}1/q_j,$$

$$\|\partial^{\beta_j}u; L^{q_j}\|\leq C\|u;\dot{B}^0_{q^*(\ell)}\|^{1-2|\beta_j|/n}\|u;H^{n/2}\|^{2|\beta_j|/n}, \tag{3.10}$$

where the constant C is independent of ℓ by the same reason for (3.5). Using these properties and the embedding $\dot{H}^{(n-3)/2}\subset L^{2n/3}$, we estimate the first term on the right hand side of (3.9) as

$$\begin{aligned}
&\|e^{\lambda|u|^2}|u-\tau_y u|\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat{q}}\| \\
\leq\ & C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\||u|^{2\ell}|u-\tau_y u|\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat{q}}\| \\
\leq\ & C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u;L^{q^*(\ell)}\|^{2\ell}\|u;\dot{B}^0_{q^*(\ell)}\|^{k-(n-3)/n} \\
&\cdot\|u;H^{n/2}\|^{(n-3)/n}\|u-\tau_y u;\dot{H}^{(n-3)/2}\| \\
\leq\ & C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(k-(n-3)/n+2\ell)/2}\|u;H^{n/2}\|^{2\ell+k}\|\partial^N u-\partial^N\tau_y u; L^2\| \\
\leq\ & C(\lambda,\hat{q},k,\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|^k\|\partial^N u-\partial^N\tau_y u;L^2\|.
\end{aligned}$$

Substituting these estimates into (3.9) and then resulting inequality into the honogeneous Besov norm, we obtain

$$
\begin{aligned}
&\{\int_0^\infty (t^{N-n/2-\mu} \sup_{|y|<t} \|(f^{(k)}(u)-f^{(k)}(\tau_y u))\prod_{j=1}^{k}\partial^{\beta_j}u; L^{\hat q}\|)^2\frac{dt}{t}\}^{1/2}\\
\le\ & C(\lambda,\hat q,k,\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|^k\|u;\dot B_2^{n/2+\mu}\|\\
\le\ & C(\lambda,\hat q,k,\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|^{k+1}, \qquad (3.11)
\end{aligned}
$$

where we have used the embedding $H^{n/2}\subset \dot B_2^{n/2+\mu}$.

We proceed to estimate the $L^{\hat q}$-norm of the terms in (3.8) of the second type. We consider only the term

$$
f^{(k)}(\tau_y u)(\partial^{\beta_1}u-\partial^{\beta_1}\tau_y u)\prod_{j=2}^{k}\partial^{\beta_j}\tau_y u,
$$

for simplicity, since the others are treated analogously.

Because of the property of f, we need to consider the norm in the integral of Besov norm separetely according to the case $k=1$ and the case $k\ge 2$.

First we consider the case $k=1$.

In this case, we set $1/q^*(\ell)=(1+2\ell)^{-1}(1/\hat q-1/2)$. Then we have

$$
\begin{aligned}
&\|f^{(k)}(\tau_y u)(\partial^{\beta_1}u-\partial^{\beta_1}\tau_y u)\prod_{j=2}^{k}\partial^{\beta_j}\tau_y u; L^{\hat q}\|\\
\le\ & \sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|(\tau_y u)^{1+2\ell}(\partial^N u-\partial^N\tau_y u); L^{\hat q}\|\\
\le\ & \sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u;L^{q^*(\ell)}\|^{1+2\ell}\|\partial^N u-\partial^N\tau_y u;L^2\|\\
\le\ & \sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(1+2\ell)/2}\|u;H^{n/2}\|^{2\ell+1}\|\partial^N u-\partial^N\tau_y u;L^2\|\\
\le\ & C(\lambda,\hat q,k,\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|\|\partial^N u-\partial^N\tau_y u;L^2\|.
\end{aligned}
$$

The corresponding Besov norm is estimated by

$$
\begin{aligned}
&\{\int_0^\infty (t^{N-n/2-\mu}\sup_{|y|<t}\|f^{(k)}(\tau_y u)(\partial^{\beta_1}u-\partial^{\beta_1}\tau_y u)\prod_{j=2}^{k}\partial^{\beta_j}\tau_y u; L^{\hat q}\|)^2\frac{dt}{t}\}^{1/2}\\
\le\ & C(\lambda,\hat q,k,\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|\|u;\dot B_2^{n/2+\mu}\|\\
\le\ & C(\lambda,\hat q,k,\|u;L^\infty(0,T;H^{n/2})\|)\|u;L^\infty(0,T;H^{n/2})\|^2. \qquad (3.12)
\end{aligned}
$$

Secondly we consider the case $k \geq 2$.

In this case, we set $q^*(\ell) = (k-1+2\ell-2(N-\beta_1)/n)/(1/\hat{q}-1/2)$, $1/q_1 = 1/2-(N-\beta_1)/n$, and $1/q_j = (1-2\beta_j/n)/q^*(\ell)+\beta_j/n$ for $j \geq 2$. Under these setting, we note that

$$
\begin{aligned}
&2 \leq q^*(\ell) < \infty, \quad 2 \leq q_1 \leq 2n/5, \quad 2 \leq q_j < n, \ 2 \leq j \leq k, \\
&1/\hat{q} = 2\ell/q^*(\ell) + \sum_{i=1}^{k} 1/q_i, \\
&\|\partial^{\beta_1} u - \partial^{\beta_1}\tau_y u; L^{q_1}\| \leq C\|\partial^N u - \partial^N \tau_y u; L^2\|, \\
&\|\partial^{\beta_j} u; L^{q_j}\| \leq C\|u; \dot{B}^0_{q^*(\ell)}\|^{1-2|\beta_j|/n}\|u; H^{n/2}\|^{2|\beta_j|/n}, \ j \geq 2,
\end{aligned}
$$

where the constant C is independent of ℓ by the same reason for (3.5). Using these properties, we have

$$
\begin{aligned}
&\|f^{(k)}(\tau_y u)(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u)\prod_{j=2}^{k}\partial^{\beta_j}\tau_y u; L^{\hat{q}}\| \\
\leq\ & \sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\||\tau_y u|^{2\ell}(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u)\prod_{j=2}^{k}\partial^{\beta_j}\tau_y u; L^{\hat{q}}\| \\
\leq\ & C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u; L^{q^*(\ell)}\|^{2\ell}\|u; \dot{B}^0_{q^*(\ell)}\|^{k-1-2(N-|\beta_1|)/n} \\
&\cdot\|u; H^{n/2}\|^{2(N-|\beta_1|)/n}\|\partial^N u - \partial^N\tau_y u; L^2\|.
\end{aligned}
$$

The corresponding Besov norm is estimated by

$$
\begin{aligned}
&\{\int_0^\infty (t^{N-n/2-\mu}\sup_{|y|<t}\|f^{(k)}(\tau_y u)(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u)\prod_{j=2}^{k}\partial^{\beta_j}\tau_y u; L^{\hat{q}}\|)^2\frac{dt}{t}\}^{1/2} \\
\leq\ & C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\|u; L^{q^*(\ell)}\|^{2\ell}\|u; \dot{B}^0_{q^*(\ell)}\|^{k-1-2(N-|\beta_1|)/n} \\
&\cdot\|u; L^\infty(0,T; H^{n/2})\|^{2(N-|\beta_1|)/n}\|u; \dot{B}_2^{n/2+\mu}\| \\
\leq\ & C\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}q^*(\ell)^{(2\ell+k-1-2(N-|\beta_1|)/n)/2}\|u; L^\infty(0,T; H^{n/2})\|^{2\ell+k} \\
\leq\ & C(\lambda, \hat{q}, k, \|u; L^\infty(0,T; H^{n/2})\|)\|u; L^\infty(0,T; H^{n/2})\|^k. \qquad (3.13)
\end{aligned}
$$

In the case $n \geq 3$, collecting (3.8), (3.11), (3.12), and (3.13), we obtain

$$
\|f(u); \dot{B}_{\hat{q}}^{n/2+\mu}\| \leq C(\lambda, \hat{q}, \|u; L^\infty(0,T; H^{n/2})\|)\|u; L^\infty(0,T; H^{n/2})\|^2. \qquad (3.14)
$$

By (3.7) and (3.14), we have proved the required inequality in the proposition. □

By the embedding $H^{s,p} \subset B^s_p$ for $1 < p \leq 2$ and Propositions 3.1, 3.2, and 3.3, we have the following proposition.

Proposition 3.4 *Let $n \geq 1$, let $\lambda > 0$, and let $f \in G_\lambda$. Let $\hat{q}$ and μ satisfy $1/2 < \hat{q} < 1/2 + 1/2n$ and $\mu = n(1/\hat{q} - 1/2 - 1/n)$. Then there exists a positive constant C_1 independent on λ such that for any u and v with*

$$\max\{\|u; L^\infty(0,T; H^{n/2})\|, \|v; L^\infty(0,T; H^{n/2})\|\} < C_1 \lambda^{-1/2},$$

the following inequalities hold

$$\begin{aligned}
\|f(u); B^{n/2}_{\hat{q}}\| &\leq C_2 \|u; L^\infty(0,T; H^{n/2})\|^2 \quad \text{for } n \text{ even or } n = 1\ , \\
\|f(u); B^{n/2+\mu}_{\hat{q}}\| &\leq C_2 \|u; L^\infty(0,T; H^{n/2})\|^2 \quad \text{for } n \geq 3 \text{ odd}\ ,
\end{aligned}$$

$$\begin{aligned}
\|f(u) - f(v); L^{\hat{q}}\| \leq\ & C_2(\|u; L^\infty(0,T; H^{n/2})\| + \|v; L^\infty(0,T; H^{n/2})\|) \\
& \cdot \|u - v; L^\infty(0,T; L^2)\|,
\end{aligned}$$

where C_2 is a constant depending on λ and C_1.

4 Theorem and its proof

Notation For $T, R > 0$, let $X(T,R)$ and d be defined by

$$\begin{aligned}
& X(T,R) \equiv \{u \in L^\infty(0,T; H^{n/2}(\mathbf{R}^n)) \mid \|u; L^\infty(0,T; H^{n/2}(\mathbf{R}^n))\| \leq R\}, \\
& d(u,v) \equiv \|u - v; L^\infty(0,T; L^2(\mathbf{R}^n))\|, \quad \text{for } u, v \in X(T,R)\ .
\end{aligned}$$

Proposition 4.1 *$(X(T,R), d)$ is a complete metric space.*

Theorem 4.1 *Let $\lambda > 0$ and let $f \in G_\lambda$. Then there exists a positive constant C_0 independent of λ such that for any initial data $(\phi, \psi) \in H^{n/2}(\mathbf{R}^n) \times H^{n/2-1}(\mathbf{R}^n)$ with*

$$\|\phi; H^{n/2}(\mathbf{R}^n)\| + \|\psi; H^{n/2-1}(\mathbf{R}^n)\| < C_0 \lambda^{-1/2},$$

(NLW) has a unique solution $u \in X(T,R)$ for some $T, R > 0$.

Moreover, $u \in C([0,T]; H^{n/2})$. Let $\{(\phi_m, \psi_m)\}_{m=1}^\infty$ be a sequence in $H^{n/2} \times H^{n/2-1}$ which converges to (ϕ, ψ) in $H^{n/2} \times H^{n/2-1}$. Then for m sufficiently large, the corresponding solutions u_m belongs to $X(T,R)$ and converges to u in $X(T,R)$.

Proof) Let $\hat{q}, \mu, C_1$, and C_2 be as in Proposition 3.4.
By Propositions 2.1, 2.2, and Remark in between, for $\phi \in H^{n/2}$, $\psi \in H^{n/2-1}$,

and $u \in X(T,R)$, we have

$$\begin{aligned}
& \|\Phi(u); L^\infty(0,T;H^{n/2})\| \\
\le\ & \|V\phi; L^\infty(0,T;H^{n/2})\| + \|U\psi; L^\infty(0,T;H^{n/2})\| \\
& +\|\Gamma f(u); L^\infty(0,T;H^{n/2})\| \\
\le\ & (1+C)(\|\phi; H^{n/2}\| + \|\psi; H^{n/2-1}\|) + CT\|\psi; H^{n/2-1}\| \\
& + \begin{cases} C(1+T)T\|f(u); L^\infty(0,T;B_{\hat{q}}^{n/2})\| & \text{for } n \text{ even or } n=1\ , \\ C(1+T)T\|f(u); L^\infty(0,T;B_{\hat{q}}^{n/2+\mu})\| & \text{for } n \ge 3 \text{ odd}\ . \end{cases}
\end{aligned}$$

We now set $C_0 = C_1/(1+C)$ and let (ϕ,ψ) satisfy

$$\|\phi; H^{n/2}\| + \|\psi; H^{n/2-1}\| < C_0(\lambda e)^{-1/2}.$$

Then we choose R as

$$(1+C)(\|\phi; H^{n/2}\| + \|\psi; H^{n/2-1}\|) < R < C_1(\lambda e)^{-1/2}. \tag{4.15}$$

By Proposition 3.4, we obtain

$$\|\Phi(u); L^\infty(0,T;H^{n/2})\| \ < \ R + CT\|\psi; H^{n/2-1}\| + C(1+T)TC_2R^2,$$

and therefore, for sufficiently small T

$$\|\Phi(u); L^\infty(0,T;H^{n/2})\| \le R,$$

which ensures that Φ maps $X(T,R)$ into itself. In the same way as above, we have

$$\begin{aligned}
& d(\Phi(u), \Phi(v)) \\
=\ & \|\Gamma(f(u)-f(v)); L^\infty(0,T;L^2)\| \\
\le\ & C(1+T)T\|f(u)-f(v); L^\infty(0,T;L^{\hat{q}})\| \\
\le\ & C(1+T)TC_2(\|u; L^\infty(0,T;H^{n/2})\| + \|v; L^\infty(0,T;H^{n/2})\|) \\
& \cdot\|u-v; L^\infty(0,T;L^2)\| \\
\le\ & C(1+T)TC_2Rd(u,v),
\end{aligned}$$

and therefore, for sufficiently small T, Φ becomes a contraction in the metric d. We have thus proved the existence and uniqueness of fixed points of Φ in $X(T,R)$.

The continuity in time of the solution is derived from the Propositions 2.1 and 2.2.

We proceed to the continuous dependence on the initial data. In view of (4.15), R and T are determined only by the size of the norm of initial data, so that T and R are independent of approximating sequences in the sense that

$u_m \in X(T,R)$ if m is sufficiently large.
With m sufficiently large, we have

$$\begin{aligned} d(u,u_m) &\leq \|V(\phi-\phi_m);L^\infty(0,T;L^2)\| + \|U(\psi-\psi_m);L^\infty(0,T;L^2)\| \\ &\quad + \|\Gamma(f(u)-f(u_m));L^\infty(0,T;L^2)\| \\ &\leq \|\phi-\phi_m;L^\infty(0,T;L^2)\| + C(1+T)\|\psi-\psi_m;L^\infty(0,T;H^{-1})\| \\ &\quad + C(1+T)TC_2Rd(u,u_m), \end{aligned}$$

where T is chosen so small that $C(1+T)TC_2R < 1$. Using the embeddings $H^{n/2} \subset L^2$ and $H^{n/2-1} \subset H^{-1}$, we have

$$\begin{aligned} &\{1-C(1+T)TC_2R\}d(u,u_m) \\ \leq\ &\|\phi-\phi_m;L^\infty(0,T;H^{n/2})\| + C(1+T)\|\psi-\psi_m;L^\infty(0,T;H^{n/2-1})\|, \end{aligned}$$

so that we conclude that $d(u,u_m) \longrightarrow 0$ as $m \to \infty$. □

5 Appendix 1; Nonlinearity with lower growth rate

In this section, we show that the smallness assumption on the initial data (ϕ,ψ) is removed in the local existence theorem at the cost of lower growth rate of the nonlinearity.

To be specific we introduce:

Definition For $\lambda > 0$,

$$\begin{aligned} G'_\lambda = \{f \in C^1(\mathbf{C},\mathbf{C}) \cap C^{[n/2]}(\mathbf{C},\mathbf{C}) \mid f(0)=0,\ |f'(z)| \leq C|z|e^{\lambda|z|}, \\ |f^{(k)}(z)| \leq Ce^{\lambda|z|} \text{ for any } k \text{ with } 2 \leq k \leq [n/2]\}, \end{aligned}$$

where the last condition is disregarded if $[n/2] < 2$.

The following functon belongs to G'_λ:

$$f(z) = z^m e^{\mu|z|},$$

where $m \geq \max(2,[n/2])$ and $\mu < \lambda$.

Concerning Propositions 3.1, 3.2 and 3.3, the assumption on the size of the norm of u and v is not necessary if G_λ is replaced by G'_λ throughout the statement of the propositions. Indeed we show the convergence of the series of norms arising from the power series expansion of $e^{\lambda|u|}$ or $e^{\lambda|v|}$ without the smallness of the norms of u and v. For instance, for the first part of Proposition

3.1, by (3.4) we estimate

$$\begin{aligned}
\|f(u); L^{\hat{q}}\| &\le C\||u|^2 e^{\lambda|u|}; L^{\hat{q}}\| \\
&\le C\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}\||u|^{2+\ell}; L^{\hat{q}}\| \\
&\le C\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}\|u; L^{(2+\ell)\hat{q}}\|^{2+\ell} \\
&\le C\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}\{(2+\ell)\hat{q}\}^{(2+\ell)/2}\|u; L^\infty(0,T; H^{n/2})\|^{2+\ell}.
\end{aligned}$$

Here the last series converges since $\lim_{\ell\to\infty} a_{\ell+1}/a_\ell = 0$, where

$$a_\ell \equiv \lambda^\ell\{(2+\ell)\hat{q}\}^{(2+\ell)/2}\|u; L^\infty(0,T; H^{n/2})\|^\ell/\ell!.$$

Using the fact above, we now have the following theorem.

Theorem 5.1 *Let $\lambda > 0$ and let $f \in G'_\lambda$. Then for any initial data $(\phi,\psi) \in H^{n/2}(\mathbf{R}^n) \times H^{n/2-1}(\mathbf{R}^n)$, (NLW) has a unique solution $u \in X(T,R)$ for some $T, R > 0$.*

Moreover, $u \in C([0,T]; H^{n/2})$. Let $\{(\phi_m,\psi_m)\}_{m=1}^\infty$ be a sequence in $H^{n/2} \times H^{n/2-1}$. Then for m sufficiently large, the corresponding solution u_m belongs to $X(T,R)$ and converges to u in $X(T,R)$.

Proof) Since $f \in G'_\lambda$, the conclusion of Proposition 3.4 holds without the smallness of the norms of u and v, so that the restriction $R < C_1(\lambda e)^{-1/2}$ in the proof of Theorem 4.1 is not necessary in the sequel.

Let $T_0, R > 0$ and let $0 < T \le T_0$. In the same way as in the proof of Theorem 4.1, for any $\phi \in H^{n/2}$, $\psi \in H^{n/2-1}$, and $u, v \in X(T,R)$,

$$\begin{aligned}
&\|\Phi(u); L^\infty(0,T; H^{n/2})\| \\
\le\ &\|\phi; H^{n/2}\| + C(1+T)\|\psi; H^{n/2-1}\| + C(1+T)TC_2R^2,
\end{aligned}$$

$$d(\Phi(u),\Phi(v)) \le C(1+T)TC_2Rd(u,v).$$

Therefore, if we choose R and T such that

$$\|\phi; H^{n/2}\| + C(1+T_0)\|\psi; H^{n/2-1}\| < R,$$

$$\begin{aligned}
&\|\phi; H^{n/2}\| + C(1+T)\|\psi; H^{n/2-1}\| + C(1+T)TC_2R^2 \le R, \\
&C(1+T)TC_2R < 1,
\end{aligned}$$

then Φ is a contraction map on $(X(T,R), d)$.

The rest of the proof proceeds in the same way as in the proof of Theorem 4.1.

□

6 Appendix 2; Local Cauchy problem in the Sobolev space of supercritical order

In this section, we show the existence of local solutions in the Sobolev space H^s with $s > n/2$.

In this case we make extensive use of the embedding

$$H^s \subset L^2 \cap L^\infty.$$

Definition For $s \geq 0$,

$$G(s) \equiv \{f \in C^1(\mathbf{C}, \mathbf{C}) \cap C^{[s]}(\mathbf{C}, \mathbf{C}) | f(0) = 0\}.$$

Let $s > n/2$, let $f \in G(s)$, let j be an integer with $0 \leq j \leq [s]$, and let M be a nonnegative, monotone increasing function on the interval $(n/2, \infty)$. Then $\omega(k, M(s))$ be defined by

$$\omega(k, M(s)) \equiv \sup_{|z| \leq C(s)M(s)} |f'(z)| + \sum_{k=0}^{[s]} \sup_{|z| \leq C(s)M(s)} |z^{k-1} f^{(k)}(z)|,$$

where $C(s)$ is defined by

$$C(s) = \inf\{C; \|u; L^\infty\| \leq C\|u; H^s\| \text{ for all } u \in H^s\}.$$

Proposition 6.1 *Let $s > n/2$ and let $f \in C^1(\mathbf{C}, \mathbf{C})$. Then*

$$\begin{aligned} \|f(u); L^2\| &\leq \omega(0, \|u; L^\infty(0,T; H^s)\|)\|u; L^\infty(0,T; L^2)\|, \\ \|f(u) - f(v); L^2\| &\leq \omega(0, \max(\|u; L^\infty(0,T; H^s)\|, \|v; L^\infty(0,T; H^s)\|)) \\ &\quad \cdot \|u - v; L^\infty(0,T; L^2)\|, \end{aligned}$$

for any $u, v \in L^\infty(0,T; H^s)$.

Proof) We only prove the first inequality, since the proof of the last is analogous. We have

$$\begin{aligned} \|f(u); L^2\| &= \|\int_0^1 f'(\theta u) u d\theta; L^2\| \\ &\leq \omega(0, \|u; L^\infty(0,T; H^s)\|)\|u; L^2\|. \qquad \square \end{aligned}$$

Proposition 6.2 *Let $s > n/2$ be an integer and let $f \in G(s)$. Then*

$$\|f(u); \dot{H}^s\| \leq C\omega(s, \|u; L^\infty(0,T; H^s)\|)\|u; L^\infty(0,T; \dot{H}^s)\|,$$

for any $u \in L^\infty(0,T; H^s)$.

Proof) We estimate

$$\begin{aligned}
&\|f(u);\dot H^s\| \\
\le\ & C\sum_{|\alpha|=s}\sum_{k=1}^{s}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\ge 1}}\|f^{(k)}(u)\prod_{j=1}^{k}\partial^{\beta_j}u;L^2\| \\
\le\ & C\sum_{|\alpha|=s}\sum_{k=1}^{s}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\ge 1}}\|f^{(k)}(u);L^\infty\|\prod_{j=1}^{k}\|\partial^{\beta_j}u;L^{2s/|\beta_j|}\|,
\end{aligned}$$

where we have used the Hölder inequality with $1/2=\sum_{j=1}^k|\beta_j|/2s$.

By the Gagliardo-Nirenberg inequality, we have

$$\|\partial^{\beta_j}u;L^{2s/|\beta_j|}\|\le C\|u;\dot H^s\|^{|\beta_j|/s}\|u;L^\infty\|^{1-|\beta_j|/s},$$

so that

$$\begin{aligned}
\|f(u);\dot H^s\| &\le C\sum_{k=1}^{s}\|u;L^\infty\|^{k-1}\|f^{(k)}(u);L^\infty\|\|u;\dot H^s\| \\
&\le C\omega(s,\|u;L^\infty(0,T;H^s)\|)\|u;L^\infty(0,T;\dot H^s)\|. \qquad \square
\end{aligned}$$

Proposition 6.3 *Let s satisfy $s>n/2$ and $s\notin \mathbf{Z}$ and let $f\in G(s)$. Then*

$$\begin{aligned}
\|f(u);\dot H_2^s\| &\le C\omega(0,\|u;L^\infty(0,T;H^s)\|)\|u;L^\infty(0,T;\dot H^s)\| \ \text{if } s<1, \\
\|f(u);\dot H_2^{s-1}\| &\le C\omega([s],\|u;L^\infty(0,T;H^s)\|)\|u;L^\infty(0,T;\dot H^{s-1})\| \ \text{if } s>1,
\end{aligned}$$

for any $u\in L^\infty(0,T;H^s)$.

Proof) For definiteness we consider the sase $s>1$ only. The case $0<s<1$ is treated in the same way.

We use the equivalent norm (3.6) on the homogeneous Besov space.

First we consider the case $[s-1]=0$.

For the modulus of continuity in (3.6) for the Besov norm on $\dot B_2^{s-1}=\dot H_2^{s-1}$ with $1<s<2$, we have

$$\begin{aligned}
\|f(u)-f(\tau_y u);L^2\| &= \|\int_0^1 f'(\tau_y u+\theta(u-\tau_y u))(u-\tau_y u)d\theta;L^2\| \\
&\le C\|\sup_{|z|\le\max(|u|,|\tau_y u|)}|f'(z)|(u-\tau_y u);L^2\| \\
&\le C\omega(0,\|u;L^\infty(0,T;H^s)\|)\|u-\tau_y u;L^2\|, \qquad (6.16)
\end{aligned}$$

and therefore

$$\|f(u);\dot H_2^{s-1}\|\le C\omega(0,\|u;L^\infty(0,T;H^s)\|)\|u;L^\infty(0,T;\dot H^{s-1})\|.$$

Second we consider the case $[s-1] \geq 1$. For the modulus of continuity in (3.6) for the Besov norm on $\dot{B}_2^{s-1} = \dot{H}_2^{s-1}$ with $s > 2$, we have

$$\begin{aligned}
&\|\partial^{[s-1]}(f(u)) - \partial^{[s-1]}(f(\tau_y u)); L^2\| \\
\leq\ & C \sum_{|\alpha|=[s-1]} \sum_{k=1}^{[s-1]} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_j|\geq 1}} \{\|(f^{(k)}(u) - f^{(k)}(\tau_y u)) \prod_{j=1}^{k} \partial^{\beta_j} u; L^2\| \\
& + C \sum_{j=1}^{k} \|f^{(k)}(\tau_y u)(\partial^{\beta_j} u - \partial^{\beta_j} \tau_y u) \prod_{i<j} \partial^{\beta_i} u \prod_{i>j} \partial^{\beta_i} \tau_y u; L^2\|\}. \quad (6.17)
\end{aligned}$$

We consider two types of the terms on the right hand side of (6.17) individually.

To estimate the terms of the first type on the right hand side of (6.17), we set $1/q^* = (1/2)(1-\theta)$, $\theta = [s-1]/(s-1)$, $\mu = (n/2)\theta$. Then we have

$$1/2 = 1/q^* + \sum_{j=1}^{k} |\beta_j|/(2s-2),$$

and therefore by the Hölder and Gagliardo-Nirenberg-Sobolev inequalities,

$$\begin{aligned}
&\|(f^{(k)}(u) - f^{(k)}(\tau_y u)) \prod_{j=1}^{k} \partial^{\beta_j} u; L^2\| \\
\leq\ & \|f^{(k+1)}(u); L^\infty\| \|u - \tau_y u; L^{q^*}\| \prod_{j=1}^{k} \|\partial^{\beta_j} u; L^{2(s-1)/|\beta_j|}\| \\
\leq\ & C \sup_{|z|\leq\|u;L^\infty\|} |f^{(k+1)}(z)| \|u - \tau_y u; \dot{H}^{\mu}\| \|u; L^\infty\|^{k-\theta} \|u; \dot{H}^{s-1}\|^{\theta}.
\end{aligned}$$

Returning back to the Besov norm, we obtain

$$\begin{aligned}
&\{\int_0^\infty (t^{[s]-s} \sup_{|y|<t} \|(f^{(k)}(u) - f^{(k)}(\tau_y u)) \prod_{j=1}^{k} \partial^{\beta_j} u; L^2\|)^2 \frac{dt}{t}\}^{1/2} \\
\leq\ & C \sup_{|z|\leq\|u;L^\infty\|} |f^{(k+1)}(z)| \|u; \dot{H}^{\mu+s-[s]}\| \|u; L^\infty\|^{k-\theta} \|u; \dot{H}^{s-1}\|^{\theta} \\
\leq\ & C \sup_{|z|\leq C(s)\|u;H^s\|} |z^k f^{(k+1)}(z)| \|u; \dot{H}^{s-1}\|, \quad (6.18)
\end{aligned}$$

where we have used the convexity property of the embedding

$$\|u; \dot{H}^{\mu+s-[s]}\| \leq C\|u; \dot{H}^{s-1}\|^{1-\theta} \|u; \dot{H}^{n/2}\|^{\theta}$$

since $\mu + s - [s] = (s-1)(1-\theta) + (n/2)\theta$.

To estimate the terms of the second type on the right hand side of (6.17), we set $1/q_j = (1/2)(1-\theta)+|\beta_j|/(2s-2)$, $\theta = [s-1]/(s-1)$, $\mu_j = |\beta_j|+n\theta/2-n|\beta_j|/(2s-2)$, and $1/q_i = |\beta_i|/(2s-2)$ if $i \neq j$. Then we have

$$1/2 = \sum_{\ell=1}^{k} 1/q_\ell$$

and therefore by the Hölder and Gagliardo-Nirenberg-Sobolev inequalities,

$$\begin{aligned} &\|f^{(k)}(\tau_y u)(\partial^{\beta_j}u - \partial^{\beta_j}\tau_y u)\prod_{i<j}\partial^{\beta_i}u\prod_{i>j}\partial^{\beta_i}\tau_y u; L^2\| \\ \leq\ & \|f^{(k)}(\tau_y u); L^\infty\|\|\partial^{\beta_j}u - \partial^{\beta_j}\tau_y u; L^{q_j}\|\prod_{i\neq j}\|\partial^{\beta_i}u; L^{q_i}\| \\ \leq\ & C\sup_{|z|\leq\|u;L^\infty\|}|f^{(k)}(z)|\|u-\tau_y u; \dot{H}^{\mu_j}\| \\ & \cdot\|u; L^\infty\|^{k-1-([s-1]-|\beta_j|)/(s-1)}\|u; \dot{H}^{s-1}\|^{([s-1]-|\beta_j|)/(s-1)}. \end{aligned}$$

Returning back to the Besov norm, we obtain

$$\begin{aligned} &\{\int_0^\infty (t^{[s]-s}\sup_{|y|<t}\|f^{(k)}(\tau_y u)(\partial^{\beta_j}u - \partial^{\beta_j}\tau_y u)\prod_{i<j}\partial^{\beta_i}u\prod_{i>j}\partial^{\beta_i}\tau_y u; L^2\|)^2\frac{dt}{t}\}^{1/2} \\ \leq\ & C\sup_{|z|\leq\|u;L^\infty\|}|f^{(k)}(z)|\|u; \dot{H}^{\mu_j+s-[s]}\| \\ & \cdot\|u; L^\infty\|^{k-1-([s-1]-|\beta_j|)/(s-1)}\|u; \dot{H}^{s-1}\|^{([s-1]-|\beta_j|)/(s-1)} \\ \leq\ & C\sup_{|z|\leq C(s)\|u;H^s\|}|z^{k-1}f^{(k)}(z)|\|u; \dot{H}^{s-1}\|, \end{aligned} \tag{6.19}$$

where we have used the convexity property of the embedding

$$\|u; \dot{H}^{\mu_j+s-[s]}\| \leq C\|u; \dot{H}^{s-1}\|^{1-\theta+|\beta_j|/(s-1)}\|u; \dot{H}^{n/2}\|^{\theta-|\beta_j|/(s-1)}$$

since $\mu_j + s - [s] = (s-1)(1-\theta+|\beta_j|/(s-1)) + (n/2)(\theta-|\beta_j|/(s-1))$.

By (6.17), (6.18) and (6.19), we obtain the required inequality for $[s-1] \geq 1$. □

Remark In the second inequality in Proposition 6.3, we may replace $\dot{H}_2^{s-1}$ by $\dot{H}_2^s$ if we impose additional differentiability by one on f. The proof is almost the same as above.

We introduce the function space as follows.

Notation Let $s > 0$. For $T, R > 0$, let $Y(T,R)$ and d be defined by

$$\begin{aligned} &Y(T,R) \equiv \{u \in L^\infty(0,T; H^s(\mathbf{R}^n)) \mid \|u; L^\infty(0,T; H^s(\mathbf{R}^n))\| \leq R\}, \\ &d(u,v) \equiv \|u-v; L^\infty(0,T; L^2(\mathbf{R}^n))\|, \text{ for } u,v \in Y(T,R)\ . \end{aligned}$$

Proposition 6.4 *$(Y(T,R),d)$ is a complete metric space.*

Theorem 6.1 *Let $s > n/2$ and let $f \in G(s)$. Then for any initial data $(\phi,\psi) \in H^s(\mathbf{R}^n) \times H^{s-1}(\mathbf{R}^n)$, (NLW) has a unique solution $u \in (Y(T,R),d)$ for some $T, R > 0$.*

Moreover $u \in C([0,T]:H^s)$. Let $\{(\phi_m,\psi_m)\}_{m=1}^{\infty}$ be a sequence in $H^s \times H^{s-1}$ which converges to (ϕ,ψ) in $H^s \times H^{s-1}$. Then for m sufficiently large, the corresponding solution u_m belongs to $Y(T,R)$ and converges to u in $Y(T,R)$.

Proof) By the Proposition 2.1, 2.2, 6.1, 6.2 and 6.3, for any $u \in Y(T,R)$ we have

$$
\begin{aligned}
& \|\Phi(u);Y(T,R)\| \\
\le\ & \|\phi;H^s\| + C(1+T)\|\psi;H^{s-1}\| \\
& + \begin{cases} C(1+T)T\|f(u);L^\infty(0,T;H^s)\| & \text{if } \ 0 \le s < 1\,, \\ C(1+T)T\|f(u);L^\infty(0,T;H^{s-1})\| & \text{if } \ 1 \le s\,, \end{cases} \\
\le\ & \|\phi;H^s\| + C(1+T)\|\psi;H^{s-1}\| \\
& + C(1+T)T\omega([s],\|u;L^\infty(0,T;H^s)\|)\|u;L^\infty(0,T;H^s)\| \\
\le\ & \|\phi;H^s\| + C(1+T)\|\psi;H^{s-1}\| \\
& + C(1+T)T\omega([s],R)R.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
d(\Phi(u),\Phi(v)) &\le C(1+T)T\|f(u)-f(v);L^\infty(0,T;L^2)\| \\
&\le C(1+T)T\omega([s],\max(\|u;L^\infty(0,T;H^s)\|,\|v;L^\infty(0,T;H^s)\|)) \\
&\qquad \cdot\|u-v;L^\infty(0,T;L^2)\| \\
&\le C(1+T)T\omega([s],R)d(u,v).
\end{aligned}
$$

We choose R sufficiently large and T sufficiently small so that

$$
\begin{aligned}
& \|\phi;H^s\| + C(1+T)\|\psi;H^{s-1}\| + C(1+T)T\omega([s],R)R \le R, \\
& C(1+T)T\omega([s],R) < 1.
\end{aligned}
$$

The rest of the proof proceeds in the same way as that of Theorem 4.1. □

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