Asymptotic Stability of Planar Rarefaction Wave for Scalar Viscous Conservation Law

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Asymptotic Stability of Planar Rarefaction Wave for Scalar Viscous Conservation Law

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Abstract

We study asymptotic stability of the planar rarefaction wave in one or two space dimensional scalar viscous conservation law for nonsmooth initial data.

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Keywords: planar rarefaction wave, asymptotic stability, viscous conservation law.

1 Introduction

We study the initial value problem of the scalar viscous conservation law in one or two space dimensions:

$$\begin{cases} u_t + \sum_{i=1}^{N} (f_i(u))_{x_i} = \sum_{i=1}^{N} u_{x_i x_i}, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

Here $N = 1$ or 2 and $u = u(t,x)$ is unknown real-valued function of time $t \geq 0$ and position $x = x_1 \in R$ or $x = (x_1, x_2) \in R^2$. We assume that each f_i is smooth and f_1 is convex, i.e., there is an $\alpha > 0$ such that $f_1''(\lambda) \geq \alpha$ for $\lambda \in R$. We also assume that initial data $u_0(x)$ satisfy

$$u_0(x) \to u_\pm \quad \text{as } x_1 \to \infty \text{ for any } x_2 \in R,$$

where u_- and u_+ are given constants with $u_- < u_+$. Our purpose in this paper is to study asymptotic behavior of solutions to (1) as $t \to \infty$ when initial data $u_0(x)$ are not necessarily smooth.

Asymptotic behavior of solutions to (1) as $t \to \infty$ is related to the planar rarefation wave $r(t, x_1)$, which is defined by the unique weak solution of the Riemann problem of the form

$$\begin{cases} r_t + (f(r))_{x_1} = 0, & t > 0,\ x_1 \in R, \\ r(0, x_1) = r_0(x_1), & x_1 \in R, \end{cases} \tag{2}$$

where

$$r_0(x_1) = \begin{cases} u_-, & x_1 < 0, \\ u_+, & x_1 > 0. \end{cases} \tag{3}$$

In fact there are several works on asymptotic stability of the planar rarefaction wave in (1), see [4, 1, 8, 5, 7] and references cited there. Especially Xin [8] first studied the case $N = 2$ and showed that the planar rarefaction wave $r(t, x_1)$ was stable in $L^\infty(R^2)$ when $u_0(x)$ were close to smooth functions $u_0^r(x_1)$ in $H^2(R^2)$; here $u_0^r(x_1)$ have the properties such as

$$\lim_{x_1 \to \pm} u_0^r(x_1) = u_\pm, \quad \frac{du_0^r}{dx_1}(x_1) > 0, \quad \left| \frac{d^2 u_0^r}{dx_1^2}(x_1) \right| \leq k_0 \frac{du_0^r}{dx_1}(x_1),$$

where k_0 is a positive constant. Under the same assumption as [8], the author [5] gave the convergence rate of solutions to (1) toward $r(t,x_1)$ when $|u_+ - u_-|$ was small and $u_0 - u_0^r \in L^1(R^2)$. Recently Nishikawa and Nishihara [7] obtained the same result as [5] without smallness conditions.

Our aim in this paper is to give the same result as [7] for a class containing nonsmooth initial data, which properly includes the class treated in [7]. In the next section we give our main result. After stating some preliminaries in Section 3, we prove main result in the last section. The ordering principle plays an essential role in the proof.

2 Main result

To begin with, we define smooth rarefaction wave $w(t,x_1)$, which is first introduced by Nishikawa and Nishihara [7], by the unique solution of the following initial value problem:

$$\begin{cases} w_t + (f_1(w))_{x_1} = w_{x_1x_1} + \dfrac{f'''(w)}{f''(w)} w_{x_1}^2, & t > -1,\ x_1 \in R, \\ w(-1,x_1) = r_0(x_1), & x_1 \in R. \end{cases} \tag{4}$$

Put $w_0(x_1) \equiv w(0,x_1)$. The smooth rarefaction wave $w(t,x_1)$ has the following properties.

Lemma 1 [2], [6] *The smooth rarefaction wave $w(t,x_1)$ satisfies the following properties on $[0,\infty) \times R$: for any $p \in [1,\infty]$, there are constants $c_d > 0$, $C_d > 0$ (depending on $d = |u_+ - u_-|$), and $C > 0$ such that*

$$u_- < w(t,x_1) < u_+, \quad |w(t,x_1) - u_\pm| \le C_d e^{-c_d x_1^2},$$

$$w_{x_1}(t,x_1) > 0,$$

$$\|w_{x_1}(t)\|_{L^p(R)} \le C d^{1/p} (1+t)^{-(1-1/p)},$$

$$\|(w-r)(t)\|_{L^p(R)} \le C \begin{cases} (1+t)^{-(1-1/p)/2}, & p \in (1,\infty], \\ \log(2 + dt^{1/2}), & p = 1. \end{cases}$$

Let us introduce the viscous rarefaction wave $U(t,x_1)$ by the unique solution of the following initial value problem:

$$\begin{cases} U_t + (f_1(U))_{x_1} = U_{x_1x_1}, & t > 0,\ x_1 \in R, \\ U(0,x_1) = w_0(x_1), & x_1 \in R. \end{cases} \tag{5}$$

$U(t,x_1)$ has the following properties.

Lemma 2 [8], [5] *$U(t,x_1)$ is smooth and $U_{x_1}(t,x_1) > 0$ in $[0,\infty) \times R$. Furthermore,*

$$U - w \in C([0,\infty); H^2(R)),$$

$$\|U(t) - r(t)\|_{L^2(R)} = O(t^{-1/4} \log t) \quad \textit{as } t \to \infty.$$

We define a series of sets of functions:

$$\begin{aligned} E_0(R^N) &= \{v_0 \in (L^2 \cap L^1)(R^N); \exists \varphi_0, \exists \psi_0 \in (H^2 \cap L^1)(R^N) \text{ s.t.} \\ &\qquad \varphi_0(x) \le v_0(x) \le \psi_0(x) \text{ a.e.}\}, \\[1em] Y^0(R^N) &= \{v \in C([0,\infty); (L^2 \cap L^1)(R^N)) \cap L^\infty((0,\infty) \times R^N); \\ &\qquad Dv \in L^2(0,T; L^2(R^N))\ (\forall T > 0)\}, \\[1em] Y^2(R^N) &= \{v \in C([0,\infty); (H^2 \cap L^1)(R^N)) \cap L^\infty((0,\infty) \times R^N); \\ &\qquad Dv \in L^2(0,T; H^2(R^N)), \\ &\qquad t^{1/2} Dv \in L^\infty(0,T; L^1(R^N))\ (\forall T > 0)\} \end{aligned}$$

where $D = \partial_{x_1}$ for $N = 1$ and $D = (\partial_{x_1}, \partial_{x_2})$ for $N = 2$,

$$D_0(R^N) = \{w_0\} + E_0(R^N),$$

$$X(R) = \{w\} + Y^0(R), \quad X(R^2) = \{U\} + Y^0(R^2).$$

Definition Let $u_0 \in D_0(R^N)$. We call $u(t,x)$ a global *mild solution* of (1), if $u(t,x)$ satisfies the following conditions: $u \in X(R^N)$ and

$$u(t) = S(t)u_0 - \sum_{i=1}^{N} \int_0^t \partial_{x_i} S(t-\tau) f_i(u)(\tau) d\tau \quad \text{for } t \geq 0.$$

Here

$$(S(t)u_0)(x) = \int_{R^N} G(t, x - x') u_0(x') dx'$$

and $G(t,x) = (4\pi t)^{-N/2} \exp(-|x|^2/(4t))$ is the Gauss kernel.

Then our main result is stated as follows.

Theorem 3 *(Asymptotic stability of rarefaction wave). Let $N = 1$ or 2. For any $u_0 \in D_0(R^N)$, there exists a unique global mild solution $u \in X(R^N)$ of (1) satisfying*

$$\sup_{x_2 \in R} \|u(t,\cdot,x_2) - r(t,\cdot)\|_{L^2(R)} = O(t^{-1/4} \log t) \tag{6}$$

as $t \to \infty$.

Remark. When $N = 1$, it follows from Lemma 1 that

$$D_0(R) = \{r_0\} + E_0(R).$$

This shows that $r(t,x_1)$ is asymptotically stable for $E_0(R)$-perturbations to $r_0(x_1)$.

3 Preliminaries

In this section we state some preliminaries needed in the proof of Theorem 3.

First of all, with the aid of Theorems 4, 5, 6 and 7 in [7], we define the mapping

$$\begin{array}{rccl} F: & (H^2 \cap L^1)(R^N) & \to & Y^2(R^N), \\ & \varphi_0 & \mapsto & \varphi = F(\varphi_0) \end{array}$$

by utilyzing the unique solution of the following initial value problem:

$$\begin{cases} \varphi_t + (f_1(\varphi + w) - f_1(w))_{x_1} = \varphi_{x_1 x_1} - \dfrac{f'''(w)}{f''(w)} w_{x_1}^2, & t > 0, \ x_1 \in R, \\ \varphi(0, x_1) = \varphi_0(x_1), \quad x_1 \in R \end{cases}$$

for $N = 1$ and

$$\begin{cases} \varphi_t + (f_1(\varphi + U) - f_1(U))_{x_1} + (f_2(\varphi + U))_{x_2} = \displaystyle\sum_{i=1}^{2} \varphi_{x_i x_i}, & t > 0, \ x \in R^2, \\ \varphi(0,x) = \varphi_0(x), \quad x \in R^2 \end{cases}$$

for $N = 2$.

From now on we will often use the following notations: Let $j(x_1)$, $x_1 \in R$, be a nonnegative smooth function with $\int_R j(x_1) dx_1 = 1$, whose support is in $[-1, 1]$. For $k = 1, 2, \cdots$, set $j_k(x_1) = kj(kx_1)$ and

$$h^k(x_1) = \int_R j_k(x_1 - x_1') h(x_1') dx_1'$$

for functions $h(x_1)$. This is called the smoothing of h. We denote by C universal positive constants, whose values are different in each occasion.

The mapping $\varphi = F(\varphi_0)$ has the following properties.

Theorem 4 [7] *(i) Let $N = 1$. Then,*

$$\|\varphi(t)\|^2_{H^m(R)} + \int_0^t \int_R w_{x_1} \varphi^2 dx_1 d\tau$$
$$+ \int_0^t \|\varphi_{x_1}(\tau)\|^2_{H^m(R)} d\tau \leq C(\|\varphi_0\|^2_{H^m(R)} + 1), \quad t \geq 0,$$

for $m = 0, 1, 2$, and

$$\|\varphi(t)\|_{L^2(R)} = O(t^{-1/4} \log t) \quad \text{as } t \to \infty.$$

(ii) Let $N = 2$. Then,

$$\|\varphi(t)\|^2_{H^m(R^2)} + \int_0^t \int_{R^2} U_{x_1} \varphi^2 dx d\tau$$
$$+ \int_0^t \|D\varphi(\tau)\|^2_{H^m(R^2)} d\tau \leq C\|\varphi_0\|^2_{H^m(R^2)}, \quad t \geq 0,$$

for $m = 0, 1, 2$, and

$$\sup_{x_2 \in R} \|\varphi(t, \cdot, x_2)\|_{L^2(R)} = O(t^{-1/4} \log t) \quad \text{as } t \to \infty.$$

Lemma 5 *Let $N = 1$ or 2.*

(i) (L^1-contraction). Let v_0, $V_0 \in (H^2 \cap L^1)(R^N)$. Then,

$$\|F(v_0)(t) - F(V_0)(t)\|_{L^1(R^N)} \leq \|v_0 - V_0\|_{L^1(R^N)}, \quad t \geq 0. \tag{7}$$

(ii) (Ordering principle). In (i), if $v_0(x) \leq V_0(x)$ a.e., then

$$F(v_0)(t,x) \leq F(V_0)(t,x) \quad a.e.. \tag{8}$$

Proof. We only show the proof for the case $N = 1$, since the case $N = 2$ can be treated in a similar way.

(i): For $v = F(v_0)$ and $V = F(V_0)$, set $z = v - V$. Then z satisfies

$$\begin{cases} z_t + (f_1(v+w) - f_1(V+w))_{x_1} = z_{x_1 x_1}, \\ z(0, x_1) = z_0(x_1) \equiv (v_0 - V_0)(x_1). \end{cases} \tag{9}$$

Let

$$\chi(x_1) = \begin{cases} -1, & x_1 < 0, \\ 0, & x_1 = 0, \\ 1, & x_1 > 0. \end{cases}$$

Multiplying (9) by $\chi^k(z)$ (χ^k is the smoothing of χ) and integrating it over $(0,t) \times R$, we have

$$\int_R \int_0^{z(t,x_1)} \chi^k(\lambda) d\lambda dx_1 - \int_R \int_0^{z_0(x_1)} \chi^k(\lambda) d\lambda dx_1$$
$$+ \int_0^t \int_R \chi^k(z)(f_1(v+w) - f_1(V+w))_{x_1} dx_1 d\tau$$
$$= -\int_0^t \int_R (\chi^k)'(z)(z_{x_1})^2 dx_1 d\tau \leq 0. \tag{10}$$

The third term in the left hand side (denoting it by $I(t)$) is estimated as follows by using the boundedness of v, V and w:

$$\begin{aligned} I(t) &= -\int_0^t \int_R (\chi^k)'(z) z_{x_1} z \frac{f_1(v+w) - f_1(V+w)}{z} dx_1 d\tau \\ &= -\int_0^t \int_R \partial_{x_1} \int_0^z (\chi^k)'(\lambda)\lambda d\lambda \int_0^1 f_1'(V + w + \theta z) d\theta dx_1 d\tau \\ &= \int_0^t \int_R \int_0^z (\chi^k)'(\lambda)\lambda d\lambda \int_0^1 f_1''(V + w + \theta z)(V + w + \theta z)_{x_1} d\theta dx_1 d\tau \\ &\leq \frac{C}{k} \int_0^t (\|v_{x_1}\|_{L^1(R)} + \|V_{x_1}\|_{L^1(R)} + \|w_{x_1}\|_{L^1(R)})(\tau) d\tau. \end{aligned}$$

Noting v, $V \in Y^2(R)$ and using Lemma 1, we find that the last term is integrable for any $t \geq 0$. Letting $k \to \infty$ in (10), we get (7).

(ii): It is easy to see

$$\int_R z(t, x_1) dx_1 = \int_R z_0(x_1) dx_1.$$

Adding the above and (7), we have

$$\int_R (|z| + z)(t, x_1) dx_1 \leq \int_R (|z_0| + z_0)(x_1) dx_1 = 0,$$

since $z_0(x_1) \leq 0$ a.e.. Nonneativeness of $(|z| + z)(t, x_1)$ implies $(|z| + z)(t, x_1) = 0$ a.e., which gives $z(t, x_1) \leq 0$ a.e.. This completes the proof. □

4 Proof of the main theorem

Proof of Theorem 3. We only show the proof for $N = 2$, since the case $N = 1$ can be proved in a similar way.

Let $u_0(x) = w_0(x) + v_0(x)$, where $v_0 \in E_0(R^2)$. Put $v^k \equiv F(v_0^k)$, where v_0^k is the smoothing of v_0. We begin by showing the following lemma.

Lemma 6 *There are a subsequence of $\{v^k\}_{k=1}^\infty$ (denoting it by $\{y^k\}_{k=1}^\infty$) and a $v \in Y^0(R^2)$ such that the following convergences hold:*

$$\begin{aligned} y^k &\to v && \text{in } C([0,T]; L^1(R^2)), \\ y^k &\to v && \text{in } L^\infty(0,T; L^2(R^2)) \text{ weakly*}, \\ Dy^k &\to Dv && \text{in } L^2(0,T; L^2(R^2)) \text{ weakly}, \\ y^k &\to v && \text{a.e. } (0,T) \times R^2 \end{aligned}$$

as $k \to \infty$ for any $T > 0$.

Proof of Lemma 6. Throughout this proof, we denote $Q_n = C([0,n]; L^1(R^2))$ for $n = 1, 2, \cdots$. It follows from Lemma 5 (i) that

$$\|v^k(t) - v^l(t)\|_{L^1(R^2)} \leq \|v_0^k - v_0^l\|_{L^1(R^2)}, \quad t \geq 0, \tag{11}$$

for $k, l = 1, 2, \cdots$, which shows that in particular $\{v^k\}_{k=1}^\infty$ is a Cauchy sequence in Q_1. Then, there are a subsequence $\{v^{k,1}\}_{k=1}^\infty$ of $\{v^k\}_{k=1}^\infty$ and a $q^1 \in Q_1$ such that

$$v^{k,1} \to q^1 \quad \text{in } Q_1 \text{ as } k \to \infty.$$

Inductively, it follows from (11) that $\{v^{k,n}\}_{k=1}^\infty$ is a Cauchy sequence in Q_{n+1}. Then, there are a subsequence $\{v^{k,n+1}\}_{k=1}^\infty$ of $\{v^{k,n}\}_{k=1}^\infty$ and a $q^{n+1} \in Q_{n+1}$ such that

$$v^{k,n+1} \to q^{n+1} \quad \text{in } Q_{n+1} \text{ as } k \to \infty$$

and

$$q^{n+1} = q^n \quad \text{a.e. } [0,n] \times R^2.$$

Define

$$\begin{aligned} &y^k = v^{k,k}, \\ &v(t,x) = q^n(t,x) \quad \text{if } (t,x) \in [0,n] \times R^2. \end{aligned}$$

Then, we obtain

$$y^k \to v \quad \text{in } C([0,T]; L^1(R^2)) \text{ as } k \to \infty$$

for any $T > 0$. We can also choose a subsequence of $\{y^k\}_{k=1}^\infty$ (denoting it again by $\{y^k\}_{k=1}^\infty$) such as

$$y^k \to v \quad \text{as } k \to \infty \text{ a.e.} (0,\infty) \times R^2.$$

On the other hand, it folows from Theorem 4 that

$$\|y^k(t)\|^2_{L^2(R^2)} + \int_0^t \int_{R^2} U_{x_1}(y^k)^2 dx d\tau$$
$$+ \int_0^t \|Dy^k(\tau)\|^2_{L^2(R^2)} d\tau \le C\|y^k(0)\|^2_{L^2(R^2)} \le \|v_0\|^2_{L^2(R^2)}, \tag{12}$$

which shows the boundedenesses of $\{y^k\}_{k=1}^{\infty}$ in $L^\infty(0,T;L^2(R^2))$ and of $\{Dy^k\}_{k=1}^{\infty}$ in $L^2(0,T;L^2(R^2))$ for any $T>0$. Then, there is a subsequence of $\{y^k\}_{k=1}^{\infty}$ (denoting it again by $\{y^k\}_{k=1}^{\infty}$) such that

$$\begin{aligned} y^k &\to v && \text{in } L^\infty(0,T;L^2(R^2)) \text{ weakly*}, \\ Dy^k &\to Dv && \text{in } L^2(0,T;L^2(R^2)) \text{ weakly} \end{aligned}$$

as $k \to \infty$ for any $T > 0$.

We show $v \in L^\infty((0,\infty) \times R^2)$. Note

$$\varphi_0^k(x) \le y^k(0,x) \le \psi_0^k(x) \quad \text{a.e.,}$$

since $v_0 \in E_0(R^2)$. It then follows from Lemma 5 (ii) that

$$\varphi^k(t,x) \le y^k(t,x) \le \psi^k(t,x) \quad \text{a.e.,} \tag{13}$$

where $\varphi^k = F(\varphi_0^k)$ and $\psi^k = F(\psi_0^k)$. Furthermore, we can show by using the above procedure that

$$\varphi^k \to \varphi, \quad \psi^k \to \psi \quad \text{as } k \to \infty \text{ a.e.}(0,\infty) \times R^2,$$

where $\varphi = F(\varphi_0)$ and $\psi = F(\psi_0)$. With this fact in mind, letting $k \to \infty$ in (13), we obtain

$$\varphi(t,x) \le v(t,x) \le \psi(t,x) \quad \text{a.e..} \tag{14}$$

Since φ and $\psi \in L^\infty((0,\infty) \times R^2)$, we thus prove $v \in L^\infty((0,\infty) \times R^2)$.

It remains to prove $v \in C([0,\infty); L^2(R^2))$. For this purpose, we first check that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \|(f_i(y^k+U) - f_i(v+U))(t)\|_{L^1(R^2)} = 0 \tag{15}$$

for any $T > 0$ and $i = 1, 2$. In fact,

$$\begin{aligned} &\sup_{t\in[0,T]} \|(f_i(y^k+U) - f_i(v+U))(t)\|_{L^1(R^2)} \\ &= \sup_{t\in[0,T]} \| \int_0^1 f_i'(v+U+\theta(y^k-v))d\theta(y^k-v)\|_{L^1(R^2)}(t) \\ &\le C \sup_{t\in[0,T]} \|(y^k-v)(t)\|_{L^1(R^2)} \to 0 \quad \text{as } k \to \infty, \end{aligned}$$

where we have used Lemma 2, (13), Theorem 4 and (14).

Note y^k satisfies

$$\begin{aligned} y^k(t) &= S(t)y(0)^k - \int_0^t \partial_{x_1} S(t-\tau)(f_1(y^k+U) - f_1(U))(\tau)d\tau \\ &\quad - \int_0^t \partial_{x_2} S(t-\tau) f_2(y^k+U)(\tau)d\tau. \end{aligned}$$

Letting $k \to \infty$ in $C([0,T]; L^1(R))$ for any fixed $T > 0$ with the aid of (15), we obtain

$$\begin{aligned} v(t) &= S(t)v_0 - \int_0^t \partial_{x_1} S(t-\tau)(f_1(v+U) - f_1(U))(\tau)d\tau \\ &\quad - \int_0^t \partial_{x_2} S(t-\tau) f_2(v+U)(\tau)d\tau. \end{aligned} \tag{16}$$

Let $t_0 \geq 0$ be any fixed time. We show that the last term in the right hand side of (16) is continuous at t_0 in $L^2(R^2)$:

$$\begin{aligned}
&\| \int_0^t \partial_{x_2} S(t-\tau) f_2(v+U)(\tau) d\tau - \int_0^{t_0} \partial_{x_2} S(t_0-\tau) f_2(v+U)(\tau) d\tau \|_{L^2(R^2)} \\
&\leq \| \int_{t_0}^t \partial_{x_2} S(t-\tau) f_2(v+U)(\tau) d\tau \|_{L^2(R^2)} \\
&+ \| \int_0^{t_0} \partial_{x_2} (S(t-\tau) - S(t_0-\tau)) f_2(v+U)(\tau) d\tau \|_{L^2(R^2)} \\
&\leq C | \int_{t_0}^t \|v_{x_2}(\tau)\|_{L^2(R^2)} d\tau | + \int_0^{t_0} \|(S(t-\tau) - S(t_0-\tau))(f_2'(v+U) v_{x_2})(\tau)\|_{L^2(R^2)} d\tau.
\end{aligned}$$

Then the Lebesgue dominated convergence theorem implies that the last integral tends to 0 as $t \to t_0$. The continuity of the second term in the right hand side of (16) in $L^2(R^2)$ can be shown in a similar way. Thus it is proved that $v \in C([0,\infty); L^2(R^2))$, and hence $v \in Y^0(R^2)$.

This completes the proof of Lemma 6. □

We show $u(t,x) = U(t,x_1) + v(t,x)$ is the desired mild solution of (1). Adding (16) and

$$U(t) = S(t) w_0 - \int_0^t \partial_{x_1} S(t-\tau) f_1(U)(\tau) d\tau,$$

we get

$$u(t) = S(t) u_0 - \sum_{i=1}^2 \int_0^t \partial_{x_i} S(t-\tau) f_i(u)(\tau) d\tau,$$

which shows that $u(t,x)$ is the desired global mild solution of (1).

We show (6). Utilyzing (14), Theorem 4 and Lemma 2, we have

$$\begin{aligned}
&\sup_{x_2 \in R} \|u(t,\cdot,x_2) - r(t,\cdot)\|_{L^2(R)} \\
&\leq \sup_{x_2 \in R} \|u(t,\cdot,x_2) - U(t,\cdot)\|_{L^2(R)} + \|U(t) - r(t)\|_{L^2(R)} \\
&\leq \sup_{x_2 \in R} \max(\|\varphi(t,\cdot,x_2)\|_{L^2(R)}, \|\psi(t,\cdot,x_2)\|_{L^2(R)}) + \|U(t) - r(t)\|_{L^2(R)} \\
&\leq O(t^{-1/4} \log t) \quad \text{as } t \to \infty,
\end{aligned}$$

which shows (6).

It remains to prove uniqueness of mild solutions of (1). Let u and $\bar{u}$ be two mild solutions of (1) with initial data $u(0) = \bar{u}(0) = u_0$. Then,

$$(u - \bar{u})(t) = -\sum_{i=1}^N \int_0^t \partial_{x_i} S(t-\tau)(f_i(u) - f_i(\bar{u}))(\tau) d\tau.$$

It follows from the boundednesses of u and $\bar{u}$ that

$$\begin{aligned}
\|(u-\bar{u})(t)\|_{L^1(R^N)} &\leq C \sum_{i=1}^N \int_0^t \partial_{x_i} S(t-\tau) \|(f_i(u) - f_i(\bar{u}))(\tau)\|_{L^1(R^2)} d\tau \\
&\leq C \int_0^t (t-\tau)^{-1/2} \|(u-\bar{u})(\tau)\|_{L^1(R^2)} d\tau.
\end{aligned}$$

Applying Lemma 7.1.1 of [3], we get $\|(u-\bar{u})(t)\|_{L^1(R^2)} = 0$, $t \geq 0$.

The proof of Theorem 3 is complete. □

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