NORMS OF SOME SINGULAR INTEGRAL OPERATORS AND THEIR INVERSE OPERATORS

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NORMS OF SOME SINGULAR INTEGRAL OPERATORS AND THEIR INVERSE OPERATORS

by

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Abstract. Let α and β be bounded measurable functions on the unit circle T. Then the singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f, (f \in L^2(T))$ where P_+ is an analytic projection and P_- is a co-analytic projection. In this paper, the norms of $S_{\alpha,\beta}$ and its inverse operator on the Hilbert space $L^2(T)$ are calculated in general, using α, β and $\alpha\bar{\beta} + H^\infty$. Moreover, the relations between these and the norms of Hankel operators are established. As an application, in some special case in which α and β are nonconstant functions, the norm of $S_{\alpha,\beta}$ is calculated in a completely explicit form. If α and β are constant functions, then it is well known that the norm of $S_{\alpha,\beta}$ on $L^2(T)$ is equal to $\max\{|\alpha|, |\beta|\}$. If α and β are nonzero constant functions, then it is also known that $S_{\alpha,\beta}$ on $L^2(T)$ has an inverse operator $S_{\alpha^{-1},\beta^{-1}}$ whose norm is equal to $\max\left\{|\alpha|^{-1}, |\beta|^{-1}\right\}$.

Keywords: Singular integral operators, Hardy spaces, Hankel operators, Toeplitz operators.

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1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $T = \{\zeta; |\zeta| = 1\}$. That is, $dm(\zeta) = d\theta/2\pi$ for $\zeta = e^{i\theta}$. The inner product in the Hilbert space $L^2 = L^2(T)$ is given by

$$(f, g) = \int_T f(\zeta)\overline{g(\zeta)}dm(\zeta).$$

The norm in L^2 is given by $\|f\|_2 = \sqrt{(f, f)}$. By $L^\infty = L^\infty(T)$ we denote the space of essentially bounded Lebesgue measurable functions. The norm in L^∞ is given by $\|f\|_\infty = \operatorname{ess\,sup}_T |f|$. Let H^2 (resp. H^∞) be the Hardy space of functions $f \in L^2$ (resp. $f \in L^\infty$) whose negative Fourier coefficients are zero. Let $\overline{H_0^2}$ be the space of functions $f \in L^2$ whose nonnegative Fourier coefficients are zero. Then $L^2 = H^2 \oplus \overline{H_0^2}$. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi i}\int_T \frac{f(\eta)}{\eta - \zeta}d\eta \qquad (a.e. \zeta \in T),$$

where the integral is understood in the sence of Cauchy's principal value (cf. [4, p.12]). If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ on T, and Sf becomes a Lebesgue measurable function on T. Let an analytic projection and a co-analytic projection be

$$P_+ = (I + S)/2, \quad \text{and} \quad P_- = (I - S)/2,$$

where I denotes the identity operator. If $\alpha, \beta \in L^\infty$, then the singular integral operator $S_{\alpha,\beta}$ on L^2 is defined by

$$S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f, \quad (f \in L^2).$$

Then $S_{1,1} = I$, $S_{1,-1} = S$, $S_{1,0} = P_+$ and $S_{0,1} = P_-$. The norm of $S_{\alpha,\beta}$ is defined by

$$\|S_{\alpha,\beta}\| = \sup_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta}f\|_2.$$

Since $\|P_+\| = \|P_-\| = 1$, we have

$$\|S_{\alpha,\beta}\| \le \|\alpha\|_\infty + \|\beta\|_\infty < \infty.$$

Hence, $S_{\alpha,\beta}$ is a bounded operator on L^2. Furthermore, it is well known that

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|S_{\alpha,\beta}\| \leq \left\|\sqrt{|\alpha|^2+|\beta|^2}\right\|_\infty .$$

If α and β are constant functions, then it is well known and not difficult to establish that

$$\|S_{\alpha,\beta}\| = \max\{|\alpha|, |\beta|\}$$

(cf. [3]). If α and β are nonconstant functions, then we will show in Section 2 that the formula of $\|S_{\alpha,\beta}\|$ is more complicated.

If $\phi \in L^\infty$, then the Toeplitz operator T_ϕ is defined by $T_\phi f = P_+(\phi f)$ for $f \in H^2$. This operator is bounded, and it's norm is equal to $\|\phi\|_\infty$ (cf. [2, p.179]). The Hankel operator H_ϕ is defined by $H_\phi f = P_-(\phi f)$ for $f \in H^2$. This operator is bounded, and by the Nehari theorem [8] (cf. [9, p.181]), it's norm is equal to $\inf\{\|\phi - k\|_\infty; k \in H^\infty\}$. Hence,

$$\|H_\phi\| \leq \|T_\phi\| \leq \|S_{\phi,1}\|.$$

Though the norm of T_ϕ or H_ϕ is known, the norm of $S_{\phi,1}$ is not known.

In this paper, we consider the operator $S_{\alpha,\beta}$ on L^2 for functions $\alpha, \beta \in L^\infty$. In Section 2, we give the formula of the norm of $S_{\alpha,\beta}$ on L^2 for $\alpha, \beta \in L^\infty$, which involve lower bounds over the algebra H^∞. It is a little surprising that the norm of the singular integral operator $S_{\alpha,\beta}$ is related to the norm of the Hankel operator $H_{\alpha\bar{\beta}}$ for some special α and β. In Section 3, we also give the formula of the norm of $S_{\alpha,\beta}^{-1}$ on L^2 for $\alpha, \beta \in L^\infty$, which involve upper bounds over the algebra H^∞. If $S_{\alpha,\beta}$ is invertible, then $\operatorname{ess\,inf}_T(\min\{|\alpha|, |\beta|\}) > 0$. When $\phi = \alpha/\beta$ and $\operatorname{ess\,inf}_T(\min\{|\alpha|, |\beta|\}) > 0$,

$$S_{\alpha,\beta} = \beta S_{\phi,1} = \beta(P_+\phi P_+ + P_-)(I + P_-\phi P_+).$$

Then $I+P_-\phi P_+$ is invertible and $(I+P_-\phi P_+)^{-1} = I - P_-\phi P_+$ (cf. [9, p.393]). Hence, $S_{\alpha,\beta}$ is invertible if and only if $\min\{\operatorname{ess\,inf}_T|\alpha|, \operatorname{ess\,inf}_T|\beta|\} > 0$ and $T_{\alpha/\beta}$ is invertible.

The first author (cf. [7]) calculated essentially the norm of the operator inverse to T_ϕ. We will show in Section 3 that the formula of the norm of the operator inverse to T_ϕ is similar to the fomula of the norm of the operator inverse to $S_{\alpha,\beta}$. The second author (cf. [11]) considered the norm of $S_{\alpha,\beta}$ on the weighted space $L^2(T, W)$ with a weight function W on T, and proved the Feldman-Krupnik-Markus theorem [3] using the Cotlar-Sadosky lifting theorem [1] when α and β are constant functions.

2. NORM OF THE OPERATOR $S_{\alpha,\beta}$

If $\alpha, \beta \in L^\infty$, then the following inequality is well known and not difficult to establish.

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \le \|S_{\alpha,\beta}\| \le \left\|\sqrt{|\alpha|^2+|\beta|^2}\right\|_\infty .$$

We should mention that

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} = \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{0+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty^{1/2},$$

and

$$\left\|\sqrt{|\alpha|^2+|\beta|^2}\right\|_\infty = \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-0|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty^{1/2}.$$

It is not difficult to establish that

$$\|S_{\alpha,\beta}\|^2 \le \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty,$$

for any $k \in H^\infty$. The formula of the norm of $S_{\alpha,\beta}$ is as follows.

Theorem 1. Let $\alpha, \beta \in L^\infty$. Then

$$\|S_{\alpha,\beta}\|^2 = \inf_{k\in H^\infty} \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty.$$

The infimum is attained.

Proof. For any $k \in H^\infty$, we define the quantity M_k according to

$$M_k = \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty.$$

We prove that $\|S_{\alpha,\beta}\|^2 \geq \inf\{M_k; k \in H^\infty\}$. Let $\gamma = \|S_{\alpha,\beta}\|$. Then

$$\|S_{\alpha,\beta}f\|_2 \leq \gamma\|f\|_2, \quad (f \in L^2).$$

Let $W_1 = \gamma^2 - |\alpha|^2, W_2 = \gamma^2 - |\beta|^2$ and $W_3 = \gamma^2 - \alpha\bar{\beta}$. Then

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re\,(W_3 f_1, f_2) \geq 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem [1], $W_1 \geq 0, W_2 \geq 0$, and there exists a $g \in H^\infty$ such that

$$|W_3 - g|^2 \leq W_1 W_2.$$

Hence, $\gamma \geq \max\{|\alpha|, |\beta|\}$ and

$$|\gamma^2 - \alpha\bar{\beta} - g|^2 \leq (\gamma^2 - |\alpha|^2)(\gamma^2 - |\beta|^2).$$

Let $k_0 = \gamma^2 - g$. Then $k_0 \in H^\infty$ and $|\gamma^2 - \alpha\bar{\beta} - g| = |\alpha\bar{\beta} - k_0|$. Hence,

$$\begin{aligned} 0 &\leq (\gamma^2 - |\alpha|^2)(\gamma^2 - |\beta|^2) - |\alpha\bar{\beta} - k_0|^2 \\ &= \gamma^4 - (|\alpha|^2 + |\beta|^2)\gamma^2 + |\alpha\beta|^2 - |\alpha\bar{\beta} - k_0|^2. \end{aligned}$$

Suppose

$$\gamma^2 \leq \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on some measurable subset E of T. Since

$$\gamma^2 \geq \max\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} + \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right|$$

on T, we have

$$\left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \leq 0$$

on E. This implies $|\alpha| - |\beta| = |\alpha\bar{\beta} - k_0| = 0$ on E. Hence,

$$\gamma^2 \geq \max\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on E. Therefore,

$$\gamma^2 \geq \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}$$

on T. Hence $M_{k_0} \leq \gamma^2$. Since $\gamma = \|S_{\alpha,\beta}\|$, we have

$$\inf_{k\in H^\infty} M_k \leq M_{k_0} \leq \|S_{\alpha,\beta}\|^2.$$

We prove that $\|S_{\alpha,\beta}\|^2 \leq \inf\{M_k; k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, we have

$$(kf_1, f_2) = 0, \quad (f_1 \in H^2, f_2 \in \overline{H_0^2}).$$

Since

$$\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \leq M_k,$$

we have

$$\begin{aligned}
&(M_k - |\alpha|^2)(M_k - |\beta|^2) \\
\geq &\left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} - \frac{|\alpha|^2-|\beta|^2}{2}\right) \\
\times &\left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} + \frac{|\alpha|^2-|\beta|^2}{2}\right) \\
&= |\alpha\bar{\beta} - k|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&M_k\|f_1+f_2\|_2^2 - \|\alpha f_1 + \beta f_2\|_2^2 \\
&= \left\|\sqrt{M_k - |\alpha|^2} f_1\right\|_2^2 + \left\|\sqrt{M_k - |\beta|^2} f_2\right\|_2^2 - 2Re(\alpha\bar{\beta} f_1, f_2) \\
&\geq 2\left\|\sqrt{M_k - |\alpha|^2} f_1\right\|_2 \left\|\sqrt{M_k - |\beta|^2} f_2\right\|_2 - 2Re((\alpha\bar{\beta} - k) f_1, f_2) \\
&\geq 2\int_T \left(\sqrt{M_k - |\alpha|^2}\sqrt{M_k - |\beta|^2} - |\alpha\bar{\beta} - k|\right)|f_1 f_2| dm \geq 0.
\end{aligned}$$

Hence, for any $k \in H^\infty$,

$$\|S_{\alpha,\beta} f\|_2^2 \le M_k \|f\|_2^2, \quad (f \in L^2).$$

Therefore,

$$\inf_{k \in H^\infty} M_k \le M_{k_0} \le \|S_{\alpha,\beta}\|^2 \le \inf_{k \in H^\infty} M_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof.

Remark 1. Let $\alpha, \beta \in L^\infty$, let $\phi = \alpha\bar{\beta}$, and let $\psi = (|\alpha|^2 - |\beta|^2)/2$. Then

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \left\| \sqrt{|\phi|^2 + \psi^2} + \sqrt{|\phi - k|^2 + \psi^2} \right\|_\infty.$$

The infimum is attained. If $|\alpha| = |\beta|$, then $\psi = 0$. Hence

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \||\phi| + |\phi - k|\|_\infty.$$

Corollary 1. If $|\alpha|$ and $|\beta|$ are constant functions, then

$$\|S_{\alpha,\beta}\|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left\|H_{\alpha\bar{\beta}}\right\|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

Proof. It follows from Theorem 1 that

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty$$

$$= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left(\inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty\right)^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

By the Nehari theorem [8], this proves the corollary.

Corollary 2. Let $\alpha, \beta \in L^\infty$. Then

$$\|S_{\alpha,\beta}\|^2 \le \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \|H_{\alpha\bar{\beta}}\|.$$

Proof. It follows from the easy direction of Theorem 1 that

$$\|S_{\alpha,\beta}\|^2 \le \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right\|_\infty$$

$$\le \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2+|\beta|^2}{2} + |\alpha\bar{\beta}-k| + \left|\frac{|\alpha|^2-|\beta|^2}{2}\right| \right\|_\infty$$

$$= \inf_{k\in H^\infty} \left\| \max\left\{|\alpha|^2, |\beta|^2\right\} + |\alpha\bar{\beta}-k| \right\|_\infty$$

$$\le \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \inf_{k\in H^\infty} \|\alpha\bar{\beta}-k\|_\infty.$$

By the Nehari theorem [8], this proves the corollary.

Remark 2. If $\alpha\bar{\beta} \in H^\infty$, then the infimum in Theorem 1 is attained by $k = \alpha\bar{\beta}$. In this case,

$$\|S_{\alpha,\beta}\| = \|\max\{|\alpha|, |\beta|\}\|_\infty = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Hence the equality holds in Corollary 2 because $\|H_{\alpha\bar{\beta}}\| = 0$. By Corollary 1, if $|\alpha| = |\beta| = constant$, then the equality holds in Corollary 2.

By Corollary 2, if $\phi \in L^\infty$ and $\|\phi\|_\infty \le 1$, then $\|S_{\phi,1}\|^2 \le 1 + \|H_\phi\|$. By Corollary 1, if $|\phi| = 1$, then $\|S_{\phi,1}\|^2 = 1 + \|H_\phi\|$. When ϕ is a real function whose range consists of only two points, the norm of $S_{\phi,1}$ will be calculated in a completely explicit form in Corollary 3. For example, by Corollary 3, if E is a measurable subset of the unit circle T satisfying $0 < m(E) < 1$, then $\|S_{\chi_E,1}\| = 2/\sqrt{3}$. It is well known that $\|H_{\chi_E}\| = 1/2$. In this case, $\|S_{\chi_E,1}\|^2 < 1 + \|H_{\chi_E}\|$. Hence the equality does not hold in Corollary 2. We need Lemma 1 to prove Corollary 3.

Lemma 1. Let a and b be real numbers satisfying $a \neq b$. Then the equality

$$\frac{a^2+1}{2} + \sqrt{(a-x)^2 + \left(\frac{a^2-1}{2}\right)^2} = \frac{b^2+1}{2} + \sqrt{(b-x)^2 + \left(\frac{b^2-1}{2}\right)^2}$$

holds for some real number x if and only if $|a+b| < 2$. Then the x is unique and it is given by

$$x = x_0 = \frac{2(a+b)(1-ab)}{4-(a+b)^2}.$$

Proof. Suppose the equality holds for $x = 0$. Then $a^2 = b^2$. Since $a \neq b$, we have $a = -b$. Hence $|a+b| = 0 < 2$. Suppose the equality holds for some real number x satisfying $x \neq 0$. Then

$$\left\{\frac{b^2-a^2}{2} + \sqrt{(b-x)^2 + \left(\frac{b^2-1}{2}\right)^2}\right\}^2 = (a-x)^2 + \left(\frac{a^2-1}{2}\right)^2.$$

It follows by direct computation that

$$2(b^2-a^2)\sqrt{(x-b)^2 + \left(\frac{b^2-1}{2}\right)^2} = (b-a)\left\{4x - (a+b)(b^2+1)\right\}.$$

Since $a \neq b$, we have

$$2(a+b)\sqrt{(x-b)^2 + \left(\frac{b^2-1}{2}\right)^2} = 4x - (a+b)(b^2+1).$$

Hence,

$$\left\{4-(a+b)^2\right\}x^2 = 2(a+b)(1-ab)x.$$

Since $x \neq 0$, we have

$$\left\{4-(a+b)^2\right\}x = 2(a+b)(1-ab).$$

If $(a+b)^2 = 4$, then $ab = 1$. Hence $a^2 = b^2 = 1$. Since $a \neq b$, we have $a+b = 0$. This contradiction implies $(a+b)^2 \neq 4$. Hence,

$$x = \frac{2(a+b)(1-ab)}{4-(a+b)^2}.$$

It follows by direct computation that

$$4x-(a+b)(b^2+1)=\frac{(a+b)\left\{(a-b)^2+(b^2+ab-2)^2\right\}}{4-(a+b)^2}.$$

If $a\neq b$, then

$$2\sqrt{(x-b)^2+\left(\frac{b^2-1}{2}\right)^2}=\frac{(a-b)^2+(b^2+ab-2)^2}{4-(a+b)^2}>0$$

because $(a-b)^2>0$. Hence, $|a+b|<2$. This proof is reversible.

Corollary 3. Let E be a measurable subset of T satisfying $0<m(E)<1$, let a and b be real numbers, and let $\phi=a\chi_E+b\chi_{E^c}$. Then

(1) If $\max\{a^2,b^2\}\geq 2-ab$, then

$$\|S_{\phi,1}\|=\max\{|a|,|b|\}.$$

(2) If $\max\{a^2,b^2\}<2-ab$, then

$$\|S_{\phi,1}\|^2=\frac{4(1-ab)}{4-(a+b)^2}.$$

Proof. We define the functions $f(x)$ and $g(x)$ according to

$$f(x)=\frac{a^2+1}{2}+\sqrt{(a-x)^2+\left(\frac{a^2-1}{2}\right)^2},$$

and

$$g(x)=\frac{b^2+1}{2}+\sqrt{(b-x)^2+\left(\frac{b^2-1}{2}\right)^2}.$$

For the real number x_0 defined in Lemma 1, it follows that

$$f(x_0)=g(x_0)$$

$$=\frac{a^2+1}{2}+\sqrt{\left(\frac{(a-b)(a^2+ab-2)}{4-(a+b)^2}\right)^2+\left(\frac{a^2-1}{2}\right)^2}$$

$$= \frac{b^2+1}{2} + \sqrt{\left(\frac{(a-b)(b^2+ab-2)}{4-(a+b)^2}\right)^2 + \left(\frac{b^2-1}{2}\right)^2}$$
$$= \frac{4(1-ab)}{4-(a+b)^2}.$$

We prove (1). Since $\max\{a^2, b^2\} \geq ab$ and $\max\{a^2, b^2\} \geq 2 - ab$, we have $\max\{a^2, b^2\} \geq 1$. Hence, $\max\{a^2, b^2, 1\} = \max\{a^2, b^2\}$. Since $\|\phi\|_\infty = \max\{|a|, |b|\}$ and $\max\{\|\phi\|_\infty, 1\} \leq \|S_{\phi,1}\|$, we have

$$\max\left\{a^2, b^2\right\} = \max\left\{a^2, b^2, 1\right\} \leq \|S_{\phi,1}\|^2.$$

Suppose $a = b$. Then ϕ becomes a constant function. Hence, by Remark 2,

$$\|S_{\phi,1}\| = \max\{|\phi|, 1\} = \max\{|a|, |b|, 1\} = \max\{|a|, |b|\}.$$

Suppose $a \neq b$. Since $0 < m(E) < 1$, we have

$$\|S_{\phi,1}\|^2 \leq \inf_{x\in R} \left\| \frac{|\phi|^2+1}{2} + \sqrt{(\phi - x)^2 + \left(\frac{\phi^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{x\in R} \|f(x)\chi_E + g(x)\chi_{E^c}\|_\infty = \inf_{x\in R} \left(\max\{f(x), g(x)\}\right).$$

By Lemma 1, if $|a + b| \geq 2$, then the equality $f(x) = g(x)$ does not hold for any real number x. Hence, $f(x) < g(x)$ or $f(x) > g(x)$. Hence,

$$\inf_{x\in R} \left(\max\{f(x), g(x)\}\right) = \max\{f(a), g(b)\}.$$

Since $f(a) = \max\{a^2, 1\}$ and $g(b) = \max\{b^2, 1\}$, we have

$$\max\left\{a^2, b^2\right\} \leq \|S_{\phi,1}\|^2 \leq \max\{f(a), g(b)\} = \max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}.$$

By Lemma 1, if $|a + b| < 2$, then the equation $f(x) = g(x)$ has a unique solution $x = x_0$ which is given in Lemma 1. Hence, $f(x_0) = g(x_0)$. Without loss of generality, we assume $a < b$. If $a \leq x_0 \leq b$, then $\max\{a^2, b^2\} \leq 2 - ab$. Since $\max\{a^2, b^2\} \geq 2 - ab$, we have $\max\{a^2, b^2\} = 2 - ab$. By this equality,

$$\max\left\{a^2, b^2\right\} \leq \|S_{\phi,1}\|^2 \leq \inf_{x\in R} \left(\max\{f(x), g(x)\}\right)$$

$$= f(x_0) = g(x_0) = \frac{4(1-ab)}{4-(a+b)^2} = \max\left\{a^2, b^2\right\}.$$

If $x_0 \leq a \leq b$ or $a \leq b \leq x_0$, then

$$\max\left\{a^2, b^2\right\} \leq \|S_{\phi,1}\|^2 \leq \inf_{x \in R} (\max\{f(x), g(x)\})$$

$$= \max\{f(a), g(b)\} = \max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}.$$

We prove (2). Suppose $a = b$. Then ϕ becomes a constant function and $\phi = a = b$. Since $\max\{a^2, b^2\} < 2 - ab$, we have $a^2 = b^2 < 1$. By Remark 2,

$$\|S_{\phi,1}\|^2 = \max\{\phi^2, 1\} = \max\{a^2, 1\} = \max\{b^2, 1\} = 1.$$

Suppose $a \neq b$. It is sufficient to prove that $\|S_{\phi,1}\|^2 = f(x_0) = g(x_0)$. Without loss of generality, we assume $a < b$. Since $\max\{a^2, b^2\} < 2 - ab$, we have $a^2 + b^2 < 2(2 - ab)$. Hence $|a + b| < 2$. Let $f(x)$ and $g(x)$ be functions defined in the proof of (1). By Lemma 1, the equation $f(x) = g(x)$ has a unique solution $x = x_0$ which is given in Lemma 1. Hence, $f(x_0) = g(x_0)$. It follows by direct calculation that

$$a - x_0 = \frac{(b-a)(a^2+ab-2)}{4-(a+b)^2} < 0,$$

$$b - x_0 = \frac{(a-b)(b^2+ab-2)}{4-(a+b)^2} > 0.$$

Hence, $a < x_0 < b$. By Theorem 1, there exists a $k_0 \in H^\infty$ such that

$$\|S_{\phi,1}\|^2 = \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k_0|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{k \in H^\infty} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty.$$

$$\leq \inf_{x \in R} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - x|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{x\in R} \|f(x)\chi_E + g(x)\chi_{E^c})\|_\infty \leq \inf_{x\in R} (\max\{f(x), g(x)\}).$$

Since $a < x_0 < b$, we have

$$\inf_{x\in R} (\max\{f(x), g(x)\}) = f(x_0) = g(x_0).$$

Hence,

$$\|S_{\phi,1}\|^2 \leq f(x_0) = g(x_0).$$

Then

$$f(x_0) = \frac{a^2+1}{2} + \sqrt{(a-x_0)^2 + \left(\frac{a^2-1}{2}\right)^2},$$

$$g(x_0) = \frac{b^2+1}{2} + \sqrt{(b-x_0)^2 + \left(\frac{b^2-1}{2}\right)^2}.$$

Suppose there exists an $\varepsilon > 0$ such that $\|S_{\phi,1}\|^2 \leq f(x_0) - \varepsilon$. Then

$$\left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k_0|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty \leq \frac{a^2+1}{2} + \sqrt{(a-x_0)^2 + \left(\frac{a^2-1}{2}\right)^2} - \varepsilon.$$

Since $f(x_0) = g(x_0)$, we have $\|S_{\phi,1}\|^2 \leq g(x_0) - \varepsilon$. Hence,

$$\left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k_0|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty \leq \frac{b^2+1}{2} + \sqrt{(b-x_0)^2 + \left(\frac{b^2-1}{2}\right)^2} - \varepsilon.$$

Hence, there exists an $\varepsilon' > 0$ such that

$$|a - k_0| \leq |a - x_0| - \varepsilon' \quad \text{on} \quad E,$$

$$|b - k_0| \leq |b - x_0| - \varepsilon' \quad \text{on} \quad E^c.$$

Since $a < x_0 < b$, we have $|a - x_0| + |b - x_0| = b - a$. If $|a - x_0| \geq |b - x_0|$, then

$$\begin{aligned} &|2|a - x_0|\chi_{E^c} + a - k_0| \\ &= |a\chi_E + (b + |a - x_0| - |b - x_0|)\chi_{E^c} - k_0| \\ &\leq |a - k_0|\chi_E + |b - k_0|\chi_{E^c} + (|a - x_0| - |b - x_0|)\chi_{E^c} \end{aligned}$$

$$\leq |a - x_0| - \varepsilon'.$$

Hence, $\inf\{\|2\chi_{E^c} - k\|_\infty; k \in H^\infty\} < 1$. This is a contradiction (cf. [5, p.198]). If $|a - x_0| \leq |b - x_0|$, then

$$\begin{aligned}
&| - 2|b - x_0|\chi_E + b - k_0| \\
&= |(a + |a - x_0| - |b - x_0|)\chi_E + b\chi_{E^c} - k_0| \\
&\leq |a - k_0|\chi_E + |b - k_0|\chi_{E^c} + (|b - x_0| - |a - x_0|)\chi_E \\
&\leq |b - x_0| - \varepsilon'.
\end{aligned}$$

Hence, $\inf\{\|2\chi_E - k\|_\infty; k \in H^\infty\} < 1$. This is a contradiction (cf. [5, p.198]). These two contradictions imply that ε' must be zero. This contradiction implies that ε must be zero. Hence,

$$\|S_{\phi,1}\|^2 = f(x_0) = g(x_0).$$

This completes the proof.

When a and b are complex numbers, we give Corollary 4. When $|\phi|$ is not constant, Corollary 4 does not contain the completely explicit form of the norm of $S_{\phi,1}$. We need Lemma 2 to prove Corollary 4.

Lemma 2. Let a and b be complex numbers, and let θ be a real number satisfying $Re\left(e^{i\theta}(a - b)\right) = 0$. If the equality

$$\frac{|a|^2 + 1}{2} + \sqrt{|e^{i\theta}a - x|^2 + \left(\frac{|a|^2 - 1}{2}\right)^2} = \frac{|b|^2 + 1}{2} + \sqrt{|e^{i\theta}b - x|^2 + \left(\frac{|b|^2 - 1}{2}\right)^2}$$

holds for some real number x, then $|a| = |b|$ and the equality holds for any real numbers x.

Proof. Suppose the equality holds for some real number x. Then

$$\left\{\frac{|a|^2 - |b|^2}{2} + \sqrt{|e^{i\theta}a - x|^2 + \left(\frac{|a|^2 - 1}{2}\right)^2}\right\}^2 = |e^{i\theta}b - x|^2 + \left(\frac{|b|^2 - 1}{2}\right)^2.$$

Since $Re\left(e^{i\theta}(a-b)\right)=0$, we have

$$(|a|^2-|b|^2)\sqrt{|e^{i\theta}a-x|^2+\left(\frac{|a|^2-1}{2}\right)^2}$$

$$=2Re\left(e^{i\theta}(a-b)\right)x+|b|^2-|a|^2+\frac{(|b|^2-|a|^2)(|a|^2-1)}{2}$$

$$=\frac{(|b|^2-|a|^2)(|a|^2+1)}{2}.$$

Hence, $|a|=|b|$ and the equality

$$|e^{i\theta}a-x|^2=|e^{i\theta}b-x|^2$$

holds for any x because $Re\left(e^{i\theta}(a-b)\right)=0$.

Corollary 4. Let E be a measurable subset of T satisfying $0<m(E)<1$, let a and b be complex numbers, and let $\phi=a\chi_E+b\chi_{E^c}$. Then

(1) If θ is a real number satisfying $Re\left(e^{i\theta}(a-b)\right)=0$, then

$$\|S_{\phi,1}\|^2\le\max_{z=a,b}\left\{\frac{|z|^2+1}{2}+\sqrt{(Im\,(e^{i\theta}z))^2+\left(\frac{|z|^2-1}{2}\right)^2}\right\}.$$

The equality does not hold in general.

(2) If $|a|=|b|$, then the equality holds for some θ in (1), and

$$\|S_{\phi,1}\|^2=\frac{|a|^2+1}{2}+\sqrt{\left(\frac{|a-b|}{2}\right)^2+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Proof. We prove (2). Suppose $a=b$. Then ϕ becomes a constant function. Hence, by Remark 2,

$$\|S_{\phi,1}\|^2=\max\left\{|\phi|^2,1\right\}=\max\left\{|a|^2,1\right\}=\frac{|a|^2+1}{2}+\sqrt{0+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Suppose $a \neq b$. Since $|\phi| = |a| = |b|$, it follows from Corollary 1 that

$$\|S_{\phi,1}\|^2 = \frac{|a|^2+1}{2} + \sqrt{\|H_\phi\|^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

Let

$$\psi = \frac{2\phi - (a+b)}{a-b}.$$

Then

$$\psi = \begin{cases} 1 & \text{on } E \\ -1 & \text{on } E^c \end{cases}$$

Hence, $\inf\{\|\psi - k\|_\infty; k \in H^\infty\} = 1$ (cf. [5,p.198]). By the Nehari theorem [8],

$$\|H_\phi\| = \inf_{k \in H^\infty} \|\phi - k\|_\infty = \frac{|a-b|}{2}.$$

Hence,

$$\|S_{\phi,1}\|^2 = \frac{|a|^2+1}{2} + \sqrt{\left(\frac{|a-b|}{2}\right)^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

Since $a \neq b, |a| = |b|$ and $Re\left(e^{i\theta}(a-b)\right) = 0$, it follows that there exist real numbers u and v such that

$$e^{i\theta}a = u + iv, \quad \text{and} \quad e^{i\theta}b = u - iv.$$

Since $|a| = |b|$, we have

$$\left(\frac{|a-b|}{2}\right)^2 = \frac{|a|^2 - Re(a\bar{b})}{2} = \frac{u^2+v^2-(u^2-v^2)}{2}$$

$$= v^2 = \left(Im\left(e^{i\theta}a\right)\right)^2 = \left(Im\left(e^{i\theta}b\right)\right)^2.$$

We prove (1). By (2), it is sufficient to prove (1) when $|a| \neq |b|$. We define the functions $f(x)$ and $g(x)$ according to

$$f(x) = \frac{|a|^2+1}{2} + \sqrt{|e^{i\theta}a - x|^2 + \left(\frac{|a|^2-1}{2}\right)^2},$$

and

$$g(x) = \frac{|b|^2+1}{2} + \sqrt{|e^{i\theta}b - x|^2 + \left(\frac{|b|^2-1}{2}\right)^2}.$$

Since $|a| \neq |b|$, it follows from Lemma 2 that the equality $f(x) = g(x)$ does not hold for any real number x. If $|a| < |b|$, then $f(x) < g(x)$ because $Re\left(e^{i\theta}(a-b)\right) = 0$. Hence, by Theorem 1,

$$\begin{aligned}
\|S_{\phi,1}\|^2 &= \inf_{k\in H^\infty} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty \\
&\leq \inf_{x\in R} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - e^{-i\theta}x|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty \\
&= \inf_{x\in R} \|f(x)\chi_E + g(x)\chi_{E^c})\|_\infty \leq \inf_{x\in R} g(x) = g\left(Re\left(e^{i\theta}b\right)\right) \\
&= \frac{|b|^2+1}{2} + \sqrt{(Im\,(e^{i\theta}b))^2 + \left(\frac{|b|^2-1}{2}\right)^2}.
\end{aligned}$$

Similarly, if $|a| > |b|$, then

$$\|S_{\phi,1}\|^2 \leq \frac{|a|^2+1}{2} + \sqrt{(Im\,(e^{i\theta}a))^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

This completes the proof.

3. NORM OF THE OPERATOR INVERSE TO $S_{\alpha,\beta}$

The first author essentially gave Theorem 2 (cf. [7, Corollary 3]). It is not difficult to establish that

$$\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 \geq \operatorname*{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right),$$

for any $k \in H^\infty$. The formula of the norm of T_ϕ is as follows.

Theorem 2. Let $\phi \in L^\infty$. Then

$$\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 = \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right)\right).$$

The supremum is attained. When T_ϕ is invertible, this is equal to $\|T_\phi^{-1}\|^{-2}$.

Proof. For any $k \in H^\infty$, we define the quantity J_k according to

$$J_k = \operatorname*{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right).$$

We prove that $\inf\{\|T_\phi f\|_2^2; f \in H^2, \|f\|_2 = 1\} \leq \sup\{J_k; k \in H^\infty\}$. Let $\varepsilon = \inf\{\|T_\phi f\|_2; f \in H^2, \|f\|_2 = 1\}$. Then

$$\varepsilon\|f_1\|_2 \leq \|T_\phi f_1\|_2, \quad (f_1 \in H^2).$$

Since $\|T_\phi f_1\|_2^2 + \|H_\phi f_1\|_2^2 = \|\phi f_1\|_2^2$, we have

$$\|H_\phi f_1\|_2^2 \leq \left(\left(|\phi|^2 - \varepsilon^2\right) f_1, f_1\right).$$

Let $W_1 = |\phi|^2 - \varepsilon^2, W_2 = 1, W_3 = \phi$. Since $(\phi f_1, f_2) = (H_\phi f_1, f_2)$, for any $f_1 \in H^2, f_2 \in \overline{H_0^2}$, we have

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re\,(W_3 f_1, f_2)$$

$$\geq \|H_\phi f_1\|_2^2 + \|f_2\|_2^2 + 2Re\,(H_\phi f_1, f_2) = \|H_\phi f_1 + f_2\|_2^2 \geq 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem [1], $W_1 \geq 0, W_2 \geq 0$, and there exists a $k_0 \in H^\infty$ such that

$$|W_3 - k_0|^2 \leq W_1 W_2.$$

Hence,

$$|\phi - k_0|^2 \leq |\phi|^2 - \varepsilon^2.$$

Since $\varepsilon = \inf\{\|T_\phi f\|_2; f \in H^2, \|f\|_2 = 1\}$, we have

$$\inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 \leq J_{k_0} \leq \sup_{k\in H^\infty} J_k.$$

We prove that $\inf\{\|T_\phi f\|_2^2; f \in H^2, \|f\|_2 = 1\} \geq \sup\{J_k; k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, $|\phi|^2 - J_k \geq |\phi - k|^2$ and $H_{\phi-k} = H_\phi$. Hence, for any $f_1 \in H^2$,

$$\|T_\phi f_1\|_2^2 - J_k\|f_1\|_2^2 = \|\phi f_1\|_2^2 - \|H_\phi f_1\|_2^2 - J_k\|\phi f_1\|_2^2$$

$$= \left((|\phi|^2 - J_k)f_1, f_1\right) - \|H_\phi f_1\|_2^2 \geq \left(|\phi - k|^2 f_1, f_1\right) - \|H_{\phi-k} f_1\|_2^2$$

$$= \|(\phi - k)f_1\|_2^2 - \|H_{\phi-k} f_1\|_2^2 = \|T_{\phi-k} f_1\|_2^2 \geq 0.$$

Hence, for any $k \in H^\infty$,

$$\|T_\phi f_1\|_2^2 \geq J_k\|f_1\|_2^2, \quad (f_1 \in H^2).$$

Therefore,

$$\sup_{k\in H^\infty} J_k \leq \inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 \leq J_{k_0} \leq \sup_{k\in H^\infty} J_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof.

Corollary 5. If $|\phi|$ is a constant function, then

$$\inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 = |\phi|^2 - \|H_\phi\|^2.$$

Proof. It follows from Theorem 2 that

$$\inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 = \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right)\right)$$

$$= |\phi|^2 + \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(-|\phi - k|^2\right)\right) = |\phi|^2 - \inf_{k\in H^\infty} \|\phi - k\|_\infty^2.$$

By the Nehari theorem [8], this proves the corollary.

Corollary 6. Let $\phi \in L^\infty$. Then

$$\inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 \geq \operatorname*{ess\,inf}_T |\phi|^2 - \|H_\phi\|^2.$$

Proof. It follows from the easy direction of Theorem 2 that

$$\inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 \geq \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right)\right)$$

$$\geq \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_T |\phi|^2 + \operatorname*{ess\,inf}_T \left(-|\phi-k|^2\right)\right)$$

$$= \operatorname*{ess\,inf}_T |\phi|^2 + \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_T \left(-|\phi-k|^2\right)\right)$$

$$= \operatorname*{ess\,inf}_T |\phi|^2 - \inf_{k\in H^\infty} \|\phi - k\|_\infty^2.$$

By the Nehari theorem [8], this proves the corollary.

If $\alpha, \beta \in L^\infty$, then the following inequality is well known and not difficult to establish.

$$\inf_{f\in L^2, \|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2 \leq \operatorname*{ess\,inf}_T \left(\min\left\{|\alpha|^2, |\beta|^2\right\}\right)$$

$$= \operatorname*{ess\,inf}_T \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{0 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}\right).$$

It is not difficult to establish that

$$\inf_{f\in L^2, \|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2$$

$$\geq \operatorname*{ess\,inf}_T \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}\right),$$

for any $k \in H^\infty$. The formula of the norm of $S_{\alpha,\beta}^{-1}$ is as follows. The proof of Theorem 3 is essentially the same as the proof of Theorem 1.

Theorem 3. Let $\alpha, \beta \in L^\infty$. Then

$$\inf_{f\in L^2, \|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2$$

$$= \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_{T} \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right) \right).$$

The supremum is attained. When $S_{\alpha,\beta}$ is invertible, the supremum is equal to $\|S_{\alpha,\beta}^{-1}\|^{-2}$.

Proof. For any $k \in H^\infty$, we define the quantity N_k according to

$$N_k = \operatorname*{ess\,inf}_{T} \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right).$$

We prove that $\inf\{\|S_{\alpha,\beta}f\|_2^2; f \in L^2, \|f\|_2 = 1\} \le \sup\{N_k; k \in H^\infty\}$. Let $\varepsilon = \inf\{\|S_{\alpha,\beta}f\|_2; f \in L^2, \|f\|_2 = 1\}$. Then

$$\varepsilon\|f\|_2 \le \|S_{\alpha,\beta}f\|_2, \quad (f \in L^2).$$

Let $W_1 = |\alpha|^2 - \varepsilon^2, W_2 = |\beta|^2 - \varepsilon^2$ and $W_3 = \alpha\bar{\beta} - \varepsilon^2$. Then

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re\,(W_3 f_1, f_2) \ge 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem [1], $W_1 \ge 0, W_2 \ge 0$, and there exists a $g \in H^\infty$ such that

$$|W_3 - g|^2 \le W_1 W_2.$$

Hence, $\varepsilon \le \min\{|\alpha|, |\beta|\}$ and there exists a $g \in H^\infty$ such that

$$|\alpha\bar{\beta} - \varepsilon^2 - g|^2 \le \left(|\alpha|^2 - \varepsilon^2\right)\left(|\beta|^2 - \varepsilon^2\right).$$

Let $k_0 = \varepsilon^2 + g$. Then $k_0 \in H^\infty$ and $|\alpha\bar{\beta} - \varepsilon^2 - g| = |\alpha\bar{\beta} - k_0|$. Hence,

$$0 \le \left(|\alpha|^2 - \varepsilon^2\right)\left(|\beta|^2 - \varepsilon^2\right) - |\alpha\bar{\beta} - k_0|^2$$

$$= \varepsilon^4 - \left(|\alpha|^2 + |\beta|^2\right)\varepsilon^2 + |\alpha\beta|^2 - |\alpha\bar{\beta} - k|^2.$$

Suppose

$$\varepsilon^2 \geq \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on some measurable subset E of T. Since

$$\varepsilon^2 \leq \min\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} - \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right|$$

on T, we have

$$\left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \leq 0$$

on E. This implies $|\alpha| - |\beta| = |\alpha\bar{\beta} - k_0| = 0$ on E. Hence,

$$\varepsilon^2 \leq \min\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on E. Therefore,

$$\varepsilon^2 \leq \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on T. Hence $\varepsilon^2 \leq N_{k_0}$. Since $\varepsilon = \inf\{\|S_{\alpha,\beta}f\|_2; f \in L^2, \|f\|_2 = 1\}$, we have

$$\inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta}f\|_2^2 \leq N_{k_0} \leq \sup_{k \in H^\infty} N_k.$$

We prove that $\inf\{\|S_{\alpha,\beta}f\|_2^2; f \in L^2, \|f\|_2 = 1\} \geq \sup\{N_k; k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, we have

$$(kf_1, f_2) = 0, \quad (f_1 \in H^2, f_2 \in \overline{H_0^2}).$$

Since

$$\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \geq N_k,$$

we have

$$(|\alpha|^2 - N_k)(|\beta|^2 - N_k)$$

$$\geq \left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} + \frac{|\alpha|^2 - |\beta|^2}{2}\right)$$
$$\times \left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} - \frac{|\alpha|^2 - |\beta|^2}{2}\right)$$
$$= |\alpha\bar{\beta} - k|^2.$$

Hence,

$$\begin{aligned}
&\|\alpha f_1 + \beta f_2\|_2^2 - N_k \|f_1 + f_2\|_2^2 \\
&= \left\|\sqrt{|\alpha|^2 - N_k} f_1\right\|_2^2 + \left\|\sqrt{|\beta|^2 - N_k} f_2\right\|_2^2 + 2Re(\alpha\bar{\beta} f_1, f_2) \\
&\geq 2\left\|\sqrt{|\alpha|^2 - N_k} f_1\right\|_2 \left\|\sqrt{|\beta|^2 - N_k} f_2\right\|_2 + 2Re((\alpha\bar{\beta} - k) f_1, f_2) \\
&\geq 2\int_T \left(\sqrt{|\alpha|^2 - N_k}\sqrt{|\beta|^2 - N_k} - |\alpha\bar{\beta} - k|\right) |f_1 f_2| dm \geq 0.
\end{aligned}$$

Hence, for any $k \in H^\infty$,

$$\|S_{\alpha,\beta} f\|_2^2 \geq N_k \|f\|_2^2, \quad (f \in L^2).$$

Therefore,

$$\sup_{k \in H^\infty} N_k \leq \inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 \leq N_{k_0} \leq \sup_{k \in H^\infty} N_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof.

Remark 3. Let $\alpha, \beta \in L^\infty$, let $\phi = \alpha\bar{\beta}$, and let $\psi = (|\alpha|^2 - |\beta|^2)/2$. Then

$$\inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname*{ess\,inf}_T \left(\sqrt{|\phi|^2 + \psi^2} - \sqrt{|\phi - k|^2 + \psi^2}\right)\right).$$

The supremum is attained. If $|\alpha| = |\beta|$, then $\psi = 0$. Hence

$$\inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname*{ess\,inf}_T (|\phi| - |\phi - k|)\right).$$

Corollary 7. If $|\alpha|$ and $|\beta|$ are constant functions, then

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 = \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{\|H_{\alpha\bar\beta}\|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}.$$

Proof. It follows from Theorem 3 that

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2$$

$$= \sup_{k\in H^\infty}\left(\operatorname{ess\,inf}_T \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)\right)$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} + \sup_{k\in H^\infty}\left(\operatorname{ess\,inf}_T \left(-\sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)\right)$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} + \sup_{k\in H^\infty}\left(-\left\|\sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty\right)$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} - \inf_{k\in H^\infty}\left\|\sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{\inf_{k\in H^\infty}\|\alpha\bar\beta - k\|_\infty^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}.$$

By the Nehari theorem [8], this proves the corollary.

Corollary 8. Let $\alpha,\beta \in L^\infty$. Then

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 \geq \operatorname{ess\,inf}_T \left(\min\{|\alpha|^2,|\beta|^2\}\right) - \|H_{\alpha\bar\beta}\|.$$

Proof. It follows from the easy direction of Theorem 3 that

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2$$

$$
\begin{aligned}
&\geq \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\frac{|\alpha|^2+|\beta|^2}{2}-\sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)\right)\\
&\geq \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\frac{|\alpha|^2+|\beta|^2}{2}-|\alpha\bar{\beta}-k|-\left|\frac{|\alpha|^2-|\beta|^2}{2}\right|\right)\right)\\
&= \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}-|\alpha\bar{\beta}-k|\right)\right)\\
&\geq \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\operatorname*{ess\,inf}_T\left(-|\alpha\bar{\beta}-k|\right)\right)\\
&= \operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(-|\alpha\bar{\beta}-k|\right)\right)\\
&= \operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\sup_{k\in H^\infty}\left(-\|\alpha\bar{\beta}-k\|_\infty\right)\\
&= \operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)-\inf_{k\in H^\infty}\|\alpha\bar{\beta}-k\|_\infty.
\end{aligned}
$$

By the Nehari theorem [8], this proves the corollary.

Remark 4. If $\alpha\bar{\beta}\in H^\infty$, then the supremum in Theorem 3 is attained by $k=\alpha\bar{\beta}$. In this case,

$$
\inf_{f\in L^2,\|f\|_2=1}\|S_{\alpha,\beta}f\|_2=\operatorname*{ess\,inf}_T\left(\min\{|\alpha|,|\beta|\}\right).
$$

For functions $\alpha,\beta\in L^\infty$, $S_{\alpha,\beta}$ is left invertible if and only if

$$
\operatorname*{ess\,inf}_T\left(\frac{|\alpha|^2+|\beta|^2}{2}-\sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)>0,
$$

for some $k\in H^\infty$. By Corollary 5, for a function $\phi\in L^\infty$ satisfying $|\phi|=1$, T_ϕ is left invertible if and only if $S_{\phi,1}$ is left invertible if and only if $\|H_\phi\|<1$ (cf. [9, p.203]).

Corollary 9. Let $\phi\in L^\infty, |\phi|=1$. Then

$$
\inf_{f\in H^2,\|f\|_2=1}\|T_\phi f\|_2=\|S_{\phi,1}\|\left(\inf_{f\in L^2,\|f\|_2=1}\|S_{\phi,1}f\|_2\right).
$$

Proof. It follows from Corollary 5 that

$$\inf_{f\in H^2, \|f\|_2=1} \|T_\phi f\|_2^2 = 1 - \|H_\phi\|^2.$$

It follows from Corollary 1 that

$$\|S_{\phi,1}\|^2 = 1 + \|H_\phi\|.$$

It follows from Corollary 7 that

$$\inf_{f\in L^2, \|f\|_2=1} \|S_{\phi,1} f\|_2^2 = 1 - \|H_\phi\|.$$

These equalities proves the corollary.

Corollary 10. If a and b are invertible functions in H^∞, then

$$\|S_{a,\bar{b}}L_{1/\bar{b}}\|^{-2} = \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{T}\left(\frac{|a|^2+|b|^2}{2|a|^2} - \sqrt{\left|\frac{\bar{b}}{a} - k\right|^2 + \left(\frac{|a|^2-|b|^2}{2|a|^2}\right)^2}\right)\right),$$

where $L_{1/\bar{b}}$ denotes the Laurent operator on L^2. The supremum is attained.

Proof. Since a and b are invertible functions H^∞, it follows that $S_{\bar{b}/a,1}$ is invertible, and

$$S_{\bar{b}/a,1}^{-1} = S_{a,\bar{b}}L_{1/\bar{b}},$$

(cf. [6, p.88], [10, Theorem 3]). Hence,

$$\|S_{a,\bar{b}}L_{1/\bar{b}}\|^{-2} = \|S_{\bar{b}/a,1}^{-1}\|^{-2} = \inf_{f\in L^2, \|f\|_2=1} \|S_{\bar{b}/a,1} f\|_2^2.$$

By Theorem 3, this proves the corollary.

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References

[1] M.Cotlar and C.Sadosky, On the Helson-Szegö theorem and a related class of modified Toeplitz kernels, Proc. Sym. Pure Math. 35 (1979), 383-407.

[2] R.G.Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.

[3] I.Feldman, N.Krupnik and A.Markus, On the norm of two adjoint projections, Integral Equations and Operator Theory 14 (1991), 69-90.

[4] I.Gohberg and N.Krupnik, One-Dimensional Linear Singular Integral Equations, Vols. I,II, Birkhäuser Verlag, Basel, 1992.

[5] P.Koosis, Introduction to H_p Spaces, Cambridge U. Press, London, 1980.

[6] G.S.Litvinchuk and I.M.Spitkovskii, Factorization of Measurable Matrix Functions, Birkhäuser Verlag, Basel, 1987.

[7] T.Nakazi, Absolute values of Toeplitz operators and Hankel operators, Canad. Math. Bull. 34 (1991), 249-253.

[8] Z.Nehari, On bounded bilinear forms, Ann. of Math. 65 (1957), 153-162.

[9] N.K.Nikol'skii, Treatise on the Shift Operator, Springer Verlag, Berlin, 1986.

[10] T.Yamamoto, Invertibility of some singular integral operators and a lifting theorem, Hokkaido Math. J. 22 (1993), 181-198.

[11] T.Yamamoto, Boundedness of some singular integral operators in weighted L^2 spaces, J. Operator Theory 32 (1994), 243-254.