Remarks on convergence of evolving graphs by nonlocal curvature

M.-H. Giga and Y. Giga

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MI-HO GIGA and YOSHIKAZU GIGA

Remarks on convergence of evolving graphs by nonlocal curvature

1. Introduction

This is a continuation of the work of the authors [GG1-5] on graph-like solutions for motion driven by nonlocal weighted curvature in the plane.

We consider a fully nonlinear evolution equation in one space dimension:

$$u_t + F(u_x, \Lambda_W(u)) = 0 \tag{1.1}$$

with $\Lambda_W(u) = (W'(u_x))_x$. Here W is a given convex function on $\mathbf{R}$ and its derivative W' may have jumps ; F is a given continuous function satisfying monotonicity condition:

$$F(p, X) \le F(p, Y) \quad \text{for} \quad X \ge Y \tag{1.2}$$

so that the equation (1.1) is at least degenerate parabolic. The subscripts t and x denote partial differentiation in time and space variables, respectively. If $W(p) = p^2/2$ and $F(p, X) = -X$, (1.1) is the heat equation. If $W(p) = (1+p^2)^{1/2}$, $\Lambda_W(u)$ is the curvature of the graph of $u(t,\cdot)$ so Λ_W is often called the weighted curvature. The value Λ_W is actually unchanged by adding an affine function to W but we still denote it by Λ_W rather than $\Lambda_{W''}$. If $W(p) = (1+p^2)^{1/2}$ and $F(p, X) = -(1+p^2)^{1/2}X$, then (1.1) is the curve shortening equation for the graph of $u(t,\cdot)$. We are interested in studying the case when W' has jump discontinuities. A typical example is $W(p) = |p|/2$. In this case the meaning of solutions of (1.1) is unclear even if u is smooth, since $\Lambda_W(u) = \delta(u_x)u_{xx}$ is not a well-defined distribution because of the presence of the Dirac δ measure.

To circumvent this inconvenience we introduced in [GG1, 3] a notion of generalized solution for (1.1) by extending the theory of viscosity solutions [CIL]. In [GG3] we established comparison principles and the unique global existence of generalized solutions for a given periodic continuous initial data, although $\Lambda_W(u)$ turns to be a nonlocal quantity. As anticipated, our solution is obtained by the limit of solutions of smoother strictly parabolic problem approximating the original problem [GG5].

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We shall first state it in a rigorous way. Let $\mathcal{E}$ be the subset of the space of convex functions of form

$$\begin{aligned}\mathcal{E} = \{W; W \text{ is convex in } \mathbf{R} \text{ and } W \text{ is } C^2 \text{ except some discrete set } P.\\ \text{Moreover, } \sup_{K\setminus P} W'' = c_K < \infty \text{ for every compact set } K \text{ in } \mathbf{R}\}.\end{aligned} \tag{1.3}$$

Of course, $\mathcal{E}$ includes all piecewise linear convex functions. The set P is called the *singularity* of W. Let $\mathcal{F} = \mathcal{F}_T$ be the set of all continuous function $F : [0,T) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying the monotonicity condition (1.2) for $F(t,p,X)$ with respect to the last variable X. Let $\mathcal{F}_\# = \mathcal{F}_{\#T}$ be the set of $F \in \mathcal{F}$ which is either time independent or uniformly continuous on $[0,T'] \times [-K_0, K_0] \times \mathbf{R}$ for each $T' < T$, $K_0 > 0$. The last assumption together with boundedness of c_K in (1.3) seems to be technical in the next convergence theorem but we do not attempt to remove it. We consider the initial value problem with periodic boundary condition to avoid extra technicality; see [GG3] for other boundary conditions. Let $C(\mathbf{T})$ denote the space of all continuous functions on $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$, i.e. the space of ω-periodic continuous functions.

Convergence Theorem 1.1. *([GG5]) For $F_\varepsilon \in \mathcal{F}_{\#T}, W_\varepsilon \in \mathcal{E}$ with $\varepsilon \geq 0$ assume that $F_\varepsilon \to F_0$, $W_\varepsilon \to W_0$ locally uniformly as $\varepsilon \to 0$ in $[0,T) \times \mathbf{R} \times \mathbf{R}$ and $\mathbf{R}$, respectively. For $\varepsilon > 0$ let $u^\varepsilon \in C([0,T) \times \mathbf{T})$ be the generalized solution of*

$$\begin{cases} u_t + F_\varepsilon(t, u_x, \Lambda_{W_\varepsilon}(u)) = 0 & \text{in} \quad (0,T) \times \mathbf{R}, \\ u(0,x) = u_0^\varepsilon(x), & x \in \mathbf{R} \end{cases} \tag{1.4}$$

with $u_0^\varepsilon \in C(\mathbf{T})$. If $u_0^\varepsilon \to u_0$ in $C(\mathbf{T})$, then u^ε converges to a function $u \in C([0,T) \times \mathbf{T})$ locally uniformly in $[0,T) \times \mathbf{T}$ and u is a generalized solution of

$$\begin{cases} u_t + F_0(t, u_x, \Lambda_{W_0}(u)) = 0 & \text{in} \quad (0,T) \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{1.5}$$

The constant T may be taken as $+\infty$.

Theorem 1.1 is quite general because it allows any degeneracy of W_0'', W_ε''. This justifies a way to construct a solution u of (1.5) with general $W_0 \in \mathcal{E}, F_0 \in \mathcal{F}_\#$ by approximating W_0 by $W_\varepsilon \in C^\infty \cap \mathcal{E}$ with $W_\varepsilon'' > 0$ and F_0 by $F_\varepsilon \in C^\infty \cap \mathcal{F}_\#$ for $\varepsilon > 0$, so that the problem (1.4) is strictly parabolic and it is solvable by the classical theory (cf. Proposition 2.3).

In this note as an application of Theorem 1.1 we study the convergence of derivatives u_x^ε. The key ingredient is the $L^1(\mathbf{T})$ estimate of second derivatives, or more generally

$$\|(\xi(u_x^\varepsilon))_x\|_{L^1(\mathbf{T})}(t) = \int_0^\omega |(\xi(u_x^\varepsilon))_x(t,x)|dx \leq \|\xi(u_{0x}^\varepsilon)_x\|_{L^1(T)} \qquad (1.6)$$

for any C^1 nondecreasing function ξ. It is not difficult to prove this estimate for smooth solution with smooth F_ε and W_ε but Theorem 1.1 passes such an estimate to a general problem. This shows that Theorem 1.1 is useful. As noted in [GG5] Theorem 1.1 is also essential to prove the convergence of crystalline algorithm (initiated by S. Angenent and M. Gurtin [AG] and J. Taylor [T] and analyzed by P. Girão and R. Kohn [GK1-2]) for a general equation (1.5). The convergence results by [GK1-2] as well as [FG] apply, for example, the heat equation and curve shortening equation. In [GK1] L^2 convergence of derivatives of solutions by crystalline algorithm is proved with rate of convergence for these examples with both the Neumannn and Dirichlet boundary conditions. Our convergence of derivatives is L^p convergence for any $p \geq 1$ for a general problem with no rate of convergence.

It turns out that Theorem 1.1 is powerful to get a priori estimates (1.6) with $\xi(\sigma) = \sigma$ of solutions by analyzing crystalline algorithm. In this paper we extend crystalline algorithm in [GK1] for a general problem (1.5). We moreover derive a priori estimate (1.6) with $\xi(\sigma) = \sigma$, which yields the convergence of derivative u_x^ε, without using any analytic theory of parabolic equations other than Theorem 1.1. This is one of main goals of this paper.

In crystalline algorithm one restricts a class of solutions to a special class of piecewise linear function to construct solutions [AG], [T]. The problem roughly corresponds to the case when W is piecewise linear. The main ansatz is that each piecewise linear part (called a facet) of the graph of solution $u(t,\cdot)$ stays as a facet and does not change its slope and that only its end points move. This ansatz is natural and it is regarded as a generalized solution in our sense [GG2]. However, if F depends on space variable, such an assumption that a facet stays as a facet of the same slope (unless it disappears) is turns to be unnatural. In this paper we give an example that the comparison principle may not hold if 'facet-stay-facet assumption' is imposed.

For physical background of problems the reader is referred to [GG3] and [GG5] and references cited there especially a book of M. Gurtin [Gu], reviews of J. Taylor,

J. Cahn and C. Handwerker [TCH] and of P. Girão and R. Kohn [GK2]. The bibliography of [GG3] and [GG5] includes many references related to recent work on the motion by crystalline energy and nonlocal curvature. We won't repeat to cite these articles unless necessary in this note. This note is written so that no explicit definition of generalized solutions is necessary. To read this note we only need to admit that classical solutions for strictly parabolic problems are generalized solutions.

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2. Convergence of derivatives

We state a priori estimate (1.6) in its general form. For a finite (signed) Radon measure μ we denote its total variation by $||\mu||_1$ since $||\mu||_1$ equals $L^1(\mathbf{T})$ norm if μ is absolutely continuous with respect to the Lebesgue measure. We shall simply write $L^p(\mathbf{T})$ norm of f by $||f||_p$. The partial derivative in x is taken in the distribution sense.

Theorem 2.1 (A priori estimate). *Let $u \in C([0,T) \times \mathbf{T})$ be a generalized solution (1.5) with $u_0 \in C(\mathbf{T})$, where $W_0 \in \mathcal{E}, F_0 \in \mathcal{F}_{\#T}$. Let ξ be a nondecreasing C^1 function on $\mathbf{R}$. Assume that u_{0xx} is a finite Radon measure (so that $u_{0x} \in L^\infty(\mathbf{T})$ i.e., u_0 is Lipschitz). Then*

$$||u_x||_\infty(t) \le ||u_{0x}||_\infty, \qquad 0 \le t < T, \tag{2.1}$$

$$||(\xi(u_x))_x||_1(t) \le ||(\xi(u_{0x}))_x||_\infty, \quad 0 \le t < T. \tag{2.2}$$

In particular,

$$||\Lambda_{W_0}(u)||_1(t) \le ||\Lambda_{W_0}(u_0)||_1, \quad 0 \le t < T$$

if $W_0 \in C^2$, where $\Lambda_{W_0}(u) = W_0'(u_x)_x$.

To show (2.1)-(2.2) we first prove them for smooth solutions with smooth W_0 and F_0.

Proposition 2.2. *In addition to the hypotheses of Theorem 2.1 assume that W_0 is smooth and that $F_0(t,\cdot,\cdot)$ is smooth in $\mathbf{R}\times\mathbf{R}$ for each t. Assume that u is smooth in $[0,T)\times\mathbf{T}$. Assume that $\xi'>0$. Then the estimates (2.1)-(2.2)* are *valid.*

We approximate the original problem (1.5) by smooth strictly parabolic problem so that there exists a unique smooth solution.

Proposition 2.3. *Assume that $W_0\in\mathcal{E}$ is smooth and $W_0''>0$ everywhere. Assume that $F_0=F_0(t,p,X)\in\mathcal{F}_{\#T}$ is smooth in each variables and that for each $T'<T$, $K_0>0$ there is $\lambda>0$ such that $\partial F_0/\partial X\le-\lambda$ on $[0,T']\times[-K_0,K_0]\times\mathbf{R}$. Assume that u_0 is smooth. Then there is a unique classical solution u of (1.5) which is smooth in $[0,T)\times\mathbf{T}$.*

Proof of Proposition 2.3. We differentiate (1.5) in x and set $v=u_x$ to get

$$\begin{cases} v_t+\partial_x(F_0(t,v,W_0'(v)_x))=0, \\ v(0,x)=v_0,\ v_0=u_{0x}. \end{cases} \tag{2.3}$$

By the maximum principle we have a priori bound

$$||v||_\infty(t)\le||v_0||_\infty,\ 0\le t<T. \tag{2.4}$$

By (2.4) and assumptions of F_0 and W_0 we now apply the theory of uniformly parabolic equations [LSU] to get a unique global smooth solution v of (2.3) for smooth v_0. Since the condition $\int_0^\omega v_0=0$ implies $\int_0^\omega v(t,\cdot)=0$, we get smooth periodic solution u of (1.5) by integrating v in the space variable $u(t,x)=\int_0^x v(t,y)dy+u(t,0)$. □

Lemma 2.4 (Approximation). *Assume that $u_0\in C(\mathbf{T})$ and that u_{0xx} is a finite Radon measure. Let ξ be a nondecreasing C^∞ function on $\mathbf{R}$ with $\xi'\ge\delta$ uniformly for some $\delta>0$. Then there is $u_0^\varepsilon\in C^\infty(\mathbf{T})$ $(\varepsilon>0)$ that satisfies*

$$\lim_{\varepsilon\to0}||(\xi(u_{0x}^\varepsilon))_x||_1=||(\xi(u_{0x}))_x||_1, \tag{2.5}$$

$$\lim_{\varepsilon\to0}||u_{0x}^\varepsilon-u_{0x}||_p=0,\quad 1\le p<\infty, \tag{2.6}$$

$$\lim_{\varepsilon\to0}||u_{0x}^\varepsilon||_\infty=||u_{0x}||_\infty. \tag{2.7}$$

Sketch of Proof of Lemma 2.4. We first note that $||u_{0xx}||_1<\infty$ with $u_0\in C(\mathbf{T})$ implies $u_{0x}\in L^\infty(\mathbf{T})$. Indeed, by a fundamental formula of calculus we have $u_{0x}(x)=\int_a^x u_{0xx}(y)dy$ for $u_0\in C^2(T)$ if $u_{0x}(a)=0$; such $a\in\mathbf{T}$ exists since

$\int_0^\omega u_{0x} = 0$. This yields $||u_{0x}||_\infty \leq ||u_{0xx}||_1$. Extension to general functions is standard by approximation argument as in [Giu].

Since $\xi(u_{0x}) \in L^\infty(\mathbf{T})$, we approximate $\xi(u_{0x})$ by $\theta_\varepsilon \in C^\infty(\mathbf{T})$ (see e.g. [Giu]) so that $\theta_\varepsilon \to \xi(u_{0x})$ in $L^p(\mathbf{T})(1 \leq p < \infty)$, $||\theta_{\varepsilon x}||_1 \to ||(\xi(u_{0x}))_x||_1$, $||\theta_\varepsilon||_\infty \to ||\xi(u_{0x})x||_\infty$ as $\varepsilon \to 0$. We take $w_\varepsilon = \xi^{-1}(\theta_\varepsilon)$ to get $w_\varepsilon \to u_{0x}$ in $L^p(\mathbf{T})(p \geq 1)$ since $\xi' \geq \delta$ implies $|\xi(v) - \xi(w)| \geq \delta|v - w|$. Unfortunately w_ε may not satisfy $\int_0^\omega w_\varepsilon = 0$. However, since $\int_0^\omega u_{0x} = 0$, the convergence $w_\varepsilon \to u_{0x}$ implies $\lim_{\varepsilon\to 0} \int_0^\omega w_\varepsilon = 0$. We get $v_\varepsilon = w_\varepsilon - c_\varepsilon$ with $c_\varepsilon = \omega^{-1}\int_0^\omega w_\varepsilon$ to get $\int_0^\omega v_\varepsilon = 0$ and $v_\varepsilon \to u_{0x}$ in $L^p(\mathbf{T})$ with $||v_\varepsilon||_\infty \to ||u_{0x}||_\infty$. Moreover, $||(\xi(v_\varepsilon))_x||_1 \to ||(\xi(u_{0x}))_x||_1$ as $\varepsilon \to 0$, since $||(\xi(w_\varepsilon))_x||_1 \to ||(\xi(u_{0x}))_x||_1$ and

$$||(\xi(w_\varepsilon))_x - (\xi(v_\varepsilon))_x||_1 \leq ||\xi'(v_\varepsilon + c_\varepsilon) - \xi'(v_\varepsilon)||_\infty \, ||v_{\varepsilon x}||_1 \to 0.$$

The last convergence follows since $||v_\varepsilon||_\infty$ and $||v_{\varepsilon x}||_1$ is bounded and $c_\varepsilon \to 0$. If we set $u_0^\varepsilon(x) = \int_0^x v_\varepsilon(y)dy + u_0(0)$, then $\int_0^\omega v_\varepsilon = 0$ implies $u_0^\varepsilon \in C^\infty(\mathbf{T})$. By constructions it is clear that u_0^ε satisfies (2.5)-(2.7). □

Proof of Theorem 2.1. We first note that $||u_{0x}||_\infty \leq ||u_{0xx}||_1$ so that $u_{0x} \in L^\infty$. Since $||u_{0xx}||_1 < \infty$ we may assume that $\xi' \geq \delta > 0$ uniformly for some $\delta > 0$. Indeed, assume that $||(\xi_\delta(u_x))_x||_1(t) \leq ||(\xi_\delta(u_{0x}))_x||_1$ for $\xi_\delta(\sigma) = \xi(\sigma) + \delta\sigma$, $\delta > 0$. The right hand side converges to $||(\xi(u_{0x}))_x||_1$ since $\delta||u_{0xx}||_1 \to 0$ as $\delta \to 0$. By lowersemicontinuity $||(\xi(u_x))_x||_1(t) \leq \varliminf_{\delta\to 0} ||(\xi_\delta(u_x))_x||_1$. Thus (2.2) follows for general $\xi \in C^1$ with $\xi' \geq 0$.

For given $W_0 \in \mathcal{E}, F_0 \in \mathcal{F}_{\#T}$ it is not difficult to construct a sequence $W_\varepsilon, F_\varepsilon$ converging to W_0, F_0 locally uniformly in $\mathbf{R}$ and $[0,T) \times \mathbf{R} \times \mathbf{R}$, respectively as $\varepsilon \to 0$, such that W_ε and F_ε satisfy all assumptions on W_0 and F_0 in Proposition 2.3. We approximate u_0 by u_0^ε by Lemma 2.4. By Proposition 2.2 we get (2.1) and (2.2) for u_0^ε and u^ε. By Theorem 1.1 and lowersemicontinuity of norms we get (2.1) and (2.2) by sending $\varepsilon \to 0$. □

Proof of Proposition 2.2. We consider (2.3) instead of (1.5). Estimate $||u_{xx}||_1(t) \leq ||u_{0xx}||_1$ is proved by proving L^1-contruction property of solutions which is standard; see references of [O] for example. We rather estimate directly, although the idea of approximating sign function by $\mathrm{sgn}_\delta(p)$ (which equals p/δ for $|p| \leq \delta$ and $\mathrm{sgn}(p)$ for $|p| \geq \delta$) with $\delta > 0$ is standard. We set $\psi_\delta(p) = \int_0^p \mathrm{sgn}_\delta(q)dq$. For

$w = \xi(v)_x$ we calculate, by integration by parts, to get

$$\frac{d}{dt}\int_0^\omega \psi_\delta(w)dx = \int_0^\omega \mathrm{sgn}_\delta(w)\xi(v)_{xt}dx = -\int_0^\omega \mathrm{sgn}'_\delta(w)w_x\ \xi(v)_t dx. \tag{2.8}$$

Using the equation (2.3) we get

$$\xi'(v)_t = -\xi'(v)\partial_x(F(t,v,z_x)) = -w\ F_p(t,v,z_x) - \xi'(v)F_X(t,v,z_x)z_{xx}$$

with $z = W'(v)$. Since $\xi' > 0$, we rewrite z_{xx} by using w_x and w to get

$$\begin{aligned}\xi(v)_t &= -w\ A - F_X(t,v,z_x)\ W''(v)w_x,\\ A &= F_p(t,v,z_x) + \xi'(v)F_X(t,v,z_x)(W''(v)/\xi'(v))_x.\end{aligned}$$

We use this formula in (2.8) to get

$$\begin{aligned}\frac{d}{dt}\int_0^\omega \psi_\delta(w)dx &= \int_0^\omega \mathrm{sgn}'_\delta(w)w_x\ w\ A\ dx\\ &\quad+\int_0^\omega \mathrm{sgn}'_\delta(w)w_x^2\ F_X(t,v,z_x)W''(v)dx\\ &\le \int_0^\omega \mathrm{sgn}'_\delta(w)w_x\ w\ A\ dx =: \sigma_\delta(t)\end{aligned} \tag{2.9}$$

since $W'' \ge 0, F_X \le 0$ and $\mathrm{sgn}'_\delta \ge 0$. By definition of sgn_δ we see $||\mathrm{sgn}'_\delta(w)w||_1(t) \to 0$ as $\delta \to 0$ uniformly in $[0,T)$. Since A and w_x is smooth in $[0,T)\times \mathbf{T}$, this now yields $\sigma_\delta(t) \to 0$ uniformly for $t \in [0,T']$ for each $T' < T$. Thus integrating (2.9) yields

$$\int_0^\omega |w|(t,x)dx - \int_0^\omega |w|(0,x)dx \le \lim_{\delta\downarrow 0}\int_0^t \sigma_\delta(s)ds = 0$$

for all $0 \le t < T$. This yields (2.2). The estimate (2.1) follows from the maximum principle (2.4). $\square$

Theorem 2.5 (Convergence of derivatives). *Assume the same hypotheses of Theorem 1.1. Assume moreover that $u^\varepsilon_{0xx}(\varepsilon > 0)$ is a finite Radon measure with $\overline{\lim}_{\varepsilon\to 0}||u^\varepsilon_{0xx}||_1 = \gamma < \infty$. Then $||u_{0xx}||_1 \le \gamma$, $||u^\varepsilon_x||_\infty(t) \le \gamma$, $||u_x||_\infty(t) \le \gamma$. Moreover, for every r, $1 \le r < \infty$ and $0 < T' < T$*

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le T'}||u^\varepsilon_x - u_x||_r(t) = 0. \tag{2.10}$$

Proof. Since $||u_{0x}^\varepsilon||_\infty \leq ||u_{0xx}^\varepsilon||_1$ the estimate $||u_x^\varepsilon||_\infty(t) \leq \gamma$ follows from (2.1). Since $u_0^\varepsilon \to u_0$ in $C(\mathbf{T})$ implies $||u_{0xx}||_1 \leq \underline{\lim}_{\varepsilon\to 0} ||u_{0xx}^\varepsilon||_1$ by duality representation of norms, we have $||u_{0xx}||_1 \leq \gamma$, so that $||u_{0x}||_\infty \leq \gamma$, which yields $||u_x||_\infty(t) \leq \gamma$ by (2.1).

If (2.10) were false, then for some $r, 1 \leq r < \infty$ there would exist sequences $t_j \in [0, T']$ $(j = 1, 2, \cdots)$, $\varepsilon_j \to 0 (\varepsilon_j > 0)$ and a constant $\delta > 0$ and $t_* \in [0, T']$ such that

$$||u_x^{\varepsilon_j}(t_j, \cdot) - u_x(t_*, \cdot)||_r \geq \delta \quad \text{and} \quad t_j \to t_* \qquad (\text{as } j \to \infty). \tag{2.11}$$

By (2.2) and BV version of Rellich's compactness (see [Giu]) $f_j := u_x^{\varepsilon_j}(t_j, \cdot)$ has a convergent subsequence (still denoted f_j) in $L^r(\mathbf{T})$. Since $u^{\varepsilon_j}(t_j, \cdot) \to u(t_*, \cdot)$ in $C(\mathbf{T})$ by Theorem 1.1, the limit of f_j in $L^r(\mathbf{T})$ should equal $u_x(t_*, \cdot)$. This contradicts (2.11). □

Remark 2.6 (Uniqueness of solution of (1.5)). There are a number of papers on (2.3) even if W' has jumps. The uniqueness of solutions as well as L^1 contraction property of solutions is well-studied. Although the boundary condition is different, the reader is referred to the article [O] of F. Otto and references cited there. However, if W' allows jumps together with degeneracy of W'', the uniqueness of our generalized solution of (1.5) does not follow from existing uniqueness theory (cf. [O] and references cited there) for (2.3) when $F_0(t, p, X)$ depends on X nonlinearily even if u_0 is smooth.

3. Method by crystalline algorithm

A typical example of an energy W in $\mathcal{E}$ is a piecewise linear convex function. Such an energy is called a *crystalline* energy (density). For simplicity we assume that the singularity P is a finite set, i.e., $P = \{p_1 < p_2 < \cdots < p_m\}$. To analyse (1.5) with crystalline $W = W_0$, one often considers solutions in a special class of piecewise linear functions. To explain the method we recall a special class of piecewise linear solutions as in [GK1] and [GG2].

For a piecewise linear continuous function f, a (bounded) closed (nontrivial) interval is called a *faceted region* of f if it is a maximal interval on which f is affine. We say that a piecewise linear continuous function is an *admissible crystal* if

(a) slope f_x belongs to P;

(b) slope f_x in adjacent faceted regions should be adjacent in P, i.e. if $f_x = p_i$ on a faceted region, then f_x on each adjacent faceted region equals either p_{i-1} or p_{i+1} with $i+1 \leq m, i-1 \geq 1$.

We say that $u \in C(J \times \mathbf{T})$ is an *admissible evolving crystal* (with respect to P) in a time interval J if $u(t,\cdot)$ is an admissible crystal and (the abscissa of) a jump of $u_x(t,\cdot)$ moves smoothly in time $t \in J$. We have implicitly assumed that each jump does not collide each other. We now recall a system of ordinary differential equations (ODEs) so that an admissible evolving crystal solves (1.5). By periodicity of $u(t,x)$ in x there are only finitely many points (for each $t \in J$) $\{x_1(t) < x_2(t) < \cdots < x_d(t)\}$ in $[0,\omega)$ for which any faceted region of $u(t,\cdot)$ is represented as $I_j(t) = [x_{j-1}(t), x_j(t)]$, $(j = 1, \cdots, d)$ with convention that $x_j + \omega = x_{j+d}$. On each $I_j(t)$, u_t is independent of x so its value is denoted $(u_t)_j(t)$ and u_x is independent of both x and t so its value is denoted $(u_x)_j$. We say that an admissible evolving crystal $u \in C(J \times \mathbf{T})$ is an *admissible solution* of $(1.5)_1$ (with $W = W_0$, $F = F_0$) if

$$(u_t)_j(t) + F(t, (u_x)_j, \chi_j\ \Delta((u_x)_j)/L_j(t)) = 0, \quad j = 1, \cdots, d, \quad t \in J \tag{3.1}$$

with

$$\begin{aligned} \Delta(p_i) &= W'(p_i + 0) - W'(p_i - 0), \qquad p_i \in P, \\ L_j(t) &= \text{ the length of faceted region } I_j(t), \end{aligned}$$

where $\chi_j = 1$ (resp. -1) if $u(t,\cdot)$ is convex (resp. concave) around I_j; otherwise $\chi_j = 0$. The quantity $\chi_j \Delta((u_x)_j)/L_j(t)$ corresponding to $\Lambda_W(u)$ is *nonlocal weighted curvature.* An elementary geometric consideration shows that

$$dL_j(t)/dt = \rho_j^0 (u_t)_j + \rho_j^{-1}(u_t)_{j-1} + \rho_j^1 (u_t)_{j+1}, \quad j = 1, \cdots, d \tag{3.2}$$

with

$$\begin{aligned} \rho_j^0 &= ((u_x)_j - (u_x)_{j-1})^{-1} + ((u_x)_{j+1} - (u_x)_j)^{-1}, \\ \rho_j^{-1} &= -((u_x)_j - (u_x)_{j-1})^{-1}, \qquad \rho_j^1 = -((u_x)_{j+1} - (u_x)_j)^{-1}. \end{aligned} \tag{3.3}$$

Since χ_j and $(u_x)_j$ are time independent, the system (3.1), (3.2) yields a system of ODEs for L_j's, so that its initial value problem is solvable locally in time at least when $F \in \mathcal{F}_T$ is C^1.

We shall prove fundamental properties for admissible solutions.

Theorem 3.1 (Maximum principle for u_t). *Assume that $F = F(t,p,X) \in C([0,T) \times P \times \mathbf{R})$ satisfies the following: (a) $F_X \in C([0,T) \times P \times \mathbf{R})$ (so that (3.1)-(3.2) is locally solvable); (b) F satisfies the monotonicity condition (1.2) with respect to X; (c) $F_t \in C([0,T) \times P \times \mathbf{R})$. Let C_0 be $\sup\{|F_t|; [0,T_0] \times P \times \mathbf{R}\}$ with $0 < T_0 < T$. Let $u \in C([0,T_0) \times \mathbf{T})$ be an admissible solution of (1.5) with an admissible crystal $u_0 \in C(\mathbf{T})$. Then*

(i) $\min_{1\le i\le d}(u_t)_i(0) - C_0 t \le (u_t)_j(t) \le \max_{1\le i\le d}(u_t)_i(0) + C_0 t$
for all $1 \le j \le d, t \in [0,T_0)$.

(ii) *If $\chi_j \neq 0$, then j-th faceted region does not disappear at $t = T_0$, i.e., $x_{j+1}(t) - x_j(t)$ does not converges to zero as $t \uparrow T_0$ provided that $\lim_{X\to\pm\infty} F(t,p,X) = \mp\infty$ for all $t \in [0,T), p \in P$.*

Remark 3.2. (i) Results in Theorem 3.1 generalize the corresponding results in [GK1] when $F(t,p,X) = -X$. At some time, say $t = T_0 > 0$ some facet may disappear but by (ii) only a faceted region with $\chi_j = 0$ may disappear. Thus as in [GK1] there is no chance that three consecutive faceted regions disappear simultaneously and $u(T_0 - 0, \cdot)$ is still admissible; see [GG2] for detail. One can solve (3.1)-(3.2) with initial data $u(T_0 - 0, \cdot)$ which has less numbers of faceted regions. We repeat this procedure renumbering indices of faceted region I_j and construct u globally in time. Such u is called a *weakly admissible* solution of (1.5).
(ii) Assumption (a) on F can be weakened. For regularity in X we only need to assume that F is locally Lipschitz in X with a uniform bound for $|F_X|$ on $[0,T'] \times P \times K$ for any $T' < T$ and any compact set K in $\mathbf{R}$ (so that (3.1)-(3.2) is uniquely locally solvable). For assumption (c) of regularity in t we only need to assume Lipschitz continuity in t and a bound for F_t in $[0,T'] \times P \times \mathbf{R}$ for any $T' < T$. In this case $(u_t)_j(t)$ may not be C^1 but is still locally Lipschitz. However, the proof below can be easily adjusted under this weaker assumption.

Proof of Theorem 3.1. The idea of the proof is similar to that of [GK1]; we give it for completeness. Part (ii) follows from (i). Indeed, if a faceted region I_j with $\chi_j \neq 0$ disappeared, then $F(t, (u_x)_j, \chi_j\ \Delta((u_x)_j)/L_j(t)) \to +\infty$ or $-\infty$ as $t \to T_0$. By (3.1) this contradicts (i).

Since $(u_t)_j$ solves (3.1)-(3.2), $(u_t)_j(t) \in C^1[0,T_0)$. For $U(t) := \max_{1\le j\le d} (u_t)_j(t)$ and $\hat{t} \in [0,T_0)$, let $\hat{j} = j(\hat{t})$ be a number that satisfies $(u_t)_{\hat{j}}(\hat{t}) = U(\hat{t})$. Then

$$d(u_t)_{\hat{j}}/dt \le C_0 \quad \text{at} \quad t = \hat{t}. \tag{3.4}$$

Indeed, if $\chi_j = 1$ (resp. $\chi_j = -1$) then $(u_x)_{j-1} < (u_x)_j < (u_x)_{j+1}$ (resp. $(u_x)_{j-1} > (u_x)_j > (u_x)_{j+1}$), so that $\rho_j^0 > 0, \rho_j^{-1} < 0, \rho_j^1 < 0$ (resp. $\rho_j^0 < 0, \rho_j^{-1} > 0, \rho_j^1 > 0$) for ρ_j's in (3.3). By (3.2) and maximality of $(u_t)_j(\hat{t})$ we now observe that

$$\begin{aligned}(dL_j/dt)(\hat{t}) &= \rho_j^0(u_t)_j(\hat{t}) + \rho_j^{-1}(u_t)_{j-1}(\hat{t}) + \rho_j^1(u_t)_{j+1}(\hat{t}) \\ &\geq (\text{resp. } \leq)\rho_j^0(u_t)_j(\hat{t}) + \rho_j^{-1}(u_t)_j(\hat{t}) + \rho_j^1(u_t)_j(\hat{t}) = 0,\end{aligned}$$

since $\rho_j^{-1} + \rho_j^1 + \rho_j^0 = 0$. Thus

$$(d\Lambda_j/dt)(\hat{t}) \leq 0 \quad \text{for} \quad \Lambda_j(t) = \chi_j \Delta((u_x)_j)/L_j(t). \tag{3.5}$$

Differentiating (3.1) in t and using (3.5) we obtain

$$(d(u_t)_j/dt)(\hat{t}) = -F_t(\hat{t}, (u_x)_j, \Lambda_j(\hat{t})) - F_X(\hat{t}, (u_x)_j, \Lambda_j(\hat{t}))(d\Lambda_j/dt)(\hat{t}) \leq C_0 + 0$$

since $F_X \leq 0$ by monotonicity of F in X. This yields (3.4).

The next lemma (which is not explicit in [GK1]) with $f_j(t) := (u_t)_j(t) - C_0 t$ implies that $U(t) - C_0 t$ is a nonincreasing function. Thus $(u_t)_j(t) \leq \max_{1 \leq i \leq d} (u_t)_i(0) + C_0 t$ for all $j, 1 \leq j \leq d$, $t \in [0, T_0)$. A symmetric argument gives the estimate from below for $(u_t)_j$ in (i). □

Lemma 3.3. *For $t_0 > 0$ and $j = 1, 2, \cdots d$ let f_j be a real-valued locally Lipschitz function on $[0, t_0)$. Let f be $f(t) = \max_{1 \leq j \leq d} f_j(t)$. For each j let Σ_j be of form $\Sigma_j = \{t \in [0, t_0); f(t) = f_j(t)\}$. Assume that $f_j'(\hat{t}) \leq 0$ for all $j \in \{1, \cdots, d\}$ and a.e. $\hat{t} \in \Sigma_j$. Then f is nonincreasing in $[0, t_0)$.*

Proof. If f is not nonincreasing, then $f(a) < f(b)$ for some $a < b$, $a, b \in [0, t_0)$. Since f is locally Lipschitz continuous, it is almost differentiable and $0 < f(b) - f(a) = \int_a^b f'(x)dx$. Thus there is a point $\hat{s} \in (a, b)$ such that f together with all f_j's is differentiable at $\hat{s}$ and $\hat{p} := f'(\hat{s}) > 0$.

There exist $i \in \{1, \cdots d\}$ and $x_\ell \downarrow \hat{s}$, $x_\ell \in (\hat{s}, b)$ that satisfies $f(x_\ell) = f_i(x_\ell)$ for all $\ell = 1, 2, \cdots$. If not, for each i there is δ_i such that $0 < x - \hat{s} \leq \delta_i$ implies $f(x) \neq f_i(x)$. However, this implies $f(x) \neq f_i(x)$ for all $i \in \{1, \cdots d\}$ and x with $0 < x - \hat{s} \leq \min_{1 \leq i \leq d} \delta_i$. This contradicts the definition of f.

Since f and f_i are continuous, $f(\hat{s}) = f_i(\hat{s})$. Since $f_i'(\hat{s}) \leq 0$, for any $0 < q < \hat{p} = f'(\hat{s})$ it holds that $f_i(x) - f_i(\hat{s}) \leq q(x - \hat{s})$ for $x > \hat{s}$ sufficiently close to $\hat{s}$. We take $x = x_\ell$ for sufficiently large ℓ to get

$$q(x_\ell - \hat{s}) \geq f_i(x_\ell) - f_i(\hat{s}) = f(x_\ell) - f(\hat{s}),$$

which contradicts that $f'(\hat{s}) = \hat{p} > q > 0$. $\square$

It is easy to derive a priori estimates like (2.1), (2.2) for weakly admissible solutions.

Lemma 3.4. *Assume the same hypotheses of Theorem 3.1 (including (ii)) concerning F. Let $u \in C([0,T) \times \mathbf{T})$ be a weakly admissible solution of (1.5) such that u is admissible on $[t_\ell, t_{\ell+1})$ for some $t_0 = 0 < t_1 < \cdots < t_h < t_{h+1} = T$ $(\ell = 0, \cdots, h-1)$ and some faceted region disappears as $t \uparrow t_\ell (\ell \geq 1)$.*

(i) *$\|u_x\|_\infty(t)$ is constant in $[t_\ell, t_{\ell+1})(\ell \geq 0)$ and may decrease at $t = t_\ell(\ell \geq 1)$. In particular,*

$$\|u_x\|_\infty(t) \leq \|u_{0x}\|_\infty, \quad t \in [0,T). \tag{3.6}$$

(ii) *$\|u_{xx}\|_1(t)$ is constant in $[t_\ell, t_{\ell+1})(\ell \geq 0)$ and $\|u_{xx}\|_1(t_\ell) = \|u_{xx}\|_1 (t_\ell + 0) < \|u_{xx}\|_1(t_\ell - 0)$ for $\ell \geq 1$. In particular,*

$$\|u_{xx}\|_1(t) \leq \|u_{0xx}\|_1, \quad t \in [0,T). \tag{3.7}$$

(iii) *$\|\Lambda(u)\|_1(t)$ is constant in $[t_\ell, t_{\ell+1})(\ell \geq 0)$ and $\|\Lambda(u)\|_1(t_\ell) = \|\Lambda(u)\|_1(t_\ell + 0) \leq \|\Lambda(u)\|_1(t_\ell - 0)$ for $\ell \geq 1$, where $\Lambda(u)(t,x) = \Lambda_j(t)$ in (3.5) for $x \in I_j(t)$. In particular,*

$$\|\Lambda(u)\|_1(t) \leq \|\Lambda(u_0)\|_1, \quad t \in [0,T). \tag{3.8}$$

Proof. (i) This is clear since $(u_x)_j$ does not depend upon $t \in [t_\ell, t_{\ell+1})$ and no faceted regions are created as $t \uparrow t_\ell(\ell \geq 1)$.
(ii) Since $\|u_{xx}\|_1(t) = \sum_{j=1}^d |(u_x)_{j+1} - (u_x)_j|$, $t \in [0, t_1)$, it is easy to see that $\|u_{xx}\|_1(t)$ is independent of $t \in [t_\ell, t_{\ell+1})(\ell \geq 0)$. As $t \uparrow t_\ell(\ell \geq 1)$ jumps of $u_x(t, \cdot)$ of u actually disappear and $u(t_\ell, \cdot)$ is still an admissible crystal. Thus $\|u_{xx}\|_1$ actually drops at $t = t_\ell(\ell \geq 1)$ as $t \uparrow t_\ell$.
(iii) Since $\|\Lambda(u)\|_1(t) = \sum_{j=1}^d L_j(t)|\Lambda_j(t)| = \sum_{j=1}^d |\chi_j| \Delta((u_x)_j)$, $t \in [0, t_1)$ it is easy to see that $\|\Lambda(u)\|_1(t)$ is independent of $t \in [t_\ell, t_{\ell+1})(\ell \geq 0)$. Examining the pictures of possible disappearance of faceted region in [GG2], which is still applicable for our general function F, we conclude that $\|\Lambda(u)\|_1$ may drop at $t = t_\ell(\ell \geq 1)$ as $t \uparrow t_\ell$; note that in the case that our two consecutive faceted regions disappears $\|\Lambda(u)\|_1(t)$ does not drop at $t = t_\ell$ as $t \uparrow t_\ell(\ell \geq 1)$. $\square$

Remark 3.5. The estimates (3.6) and (3.7) follow from Theorem 2.1 once we know that a weakly admissible solution is a generalized solution. This idea can be implemented if we modify the value of $F(t,p,0)$ for $p \neq P$ so that the next proposition applies. We note that our proof for (3.6)-(3.8) does not depend on the theory of partial differential equations. It is not known whether (3.8) follows from Theorem 2.1 since $\Lambda(u)$ is not a local quantity.

Proposition 3.6 (Consistency). *Assume the same hypothesis of Theorem 3.1 (including (ii)) concerning F. Then a weakly admissible solution $u \in C([0,T) \times \mathbf{T})$ of (1.5) is a generalized solution of (1.5) provided that*

$$F(t,p,0) = \theta F(t,p_k,0) + (1-\theta)F(t,p_{k+1},0) \tag{3.8}$$

for $p = \theta p_k + (1-\theta)p_{k+1}$ with $0 \leq \theta \leq 1$, $k = 1, \cdots, m-1$.

This is a main topic of [GG2] for $F(t,p,X) = -a(p)(X - C(t))$ with a nonnegative continuous function a and a continuous function C. However, the proof can be easily extended in our general setting. We remark that assumptions on F in Lemma 3.4 and Proposition 3.6 can be weakened as in Remark 3.2 (ii).

A priori estimates by crystalline algorithm. We shall prove (2.1) and (2.2) with $\xi(\sigma) = \sigma$ by Theorem 1.1 without using the theory of uniformly parabolic equations (Proposition 2.3). For this purpose we approximate $u_0 \in C(\mathbf{T})$ by admissible crystals; this approximation is not intended for numerical computation.

Lemma 3.7 (Approximation by admissible crystals). *Assume that $u_0 \in C(\mathbf{T})$ and $||u_{0xx}||_1 < \infty$. Then there is a sequence of finite set P_n in $\{p; |p| \leq ||u_{0x}||_\infty + 1\}(n = 1, 2, \cdots)$ and a sequence of u_0^n which is an admissible crystal with respect to P_n such that $u_0^n \to u_0$ in $C(\mathbf{T})$ with $||u_{0x}^n||_\infty \to ||u_{0x}||_\infty$ and*

$$\lim_{n\to\infty} ||u_{0xx}^n||_1 = ||u_{0xx}||_1, \tag{3.9}$$

$$\lim_{n\to\infty} |P_n| = 0 \quad \text{with} \quad |P_n| = \max\{|r-s|,\ r,s \in P_n,\ r < s,\ (r,s) \cap P_n = \emptyset\}. \tag{3.10}$$

Proof. There is a sequence of smooth functions v_ℓ such that $v_\ell \to u_0$ in $C(\mathbf{T})$ with $||v_{\ell x}||_\infty \to ||u_{0x}||_\infty$ and $||v_{\ell xx}||_1 \to ||u_{0xx}||_1$ as $\ell \to \infty$ (cf. [Giu]). By a diagonal argument we may assume $u_0 \in C^\infty(\mathbf{T})$. Similarly, we may still assume that u_0 is real analytic by truncating ω-periodic Fourier expansion of u_0.

If $u_{0x} \not\equiv 0$ is real analytic, the set of points $Z = \{x \in \mathbf{T}; u_{0xx}(x) = 0\}$ is finite (and nonempty by periodicity). We express the set Z by $Z = \{0 \leq z_1 < z_2 < \cdots < z_{i_0} < \omega\}$ and set $q_i = u_{0x}(z_i)(i = 1, 2, \cdots, i_0)$, so that

$$||u_{0xx}||_1 = \sum_{j=1}^{i_0} |q_{j+1} - q_j| \quad \text{with} \quad q_{i_0+1} = q_1. \tag{3.11}$$

For $r_+ := ||u_{0x}||_\infty + 1$ and $r_- := -r_+$, we set

$$P_n = \{p_j \in \mathbf{R}; p_j = (r_+ - r_-)\cdot j/2^n + r_-,\ j = 0, 1, 2, \cdots, 2^n\} \cup \{q_i; i = 1, 2, \cdots, i_0\},$$

so that (3.10) holds. If $u_{0x} \equiv 0$ we take P_n as above by interpreting that the set of q_i's is empty and we set $u_0^n = u_0$. If $u_{0xx} > 0$ in (z_i, z_{i+1}) then we approximate u_{0x} by a piecewise constant nondecreasing functions f_n in $[z_i, z_{i+1}]$ with values in P_n that satisfies

$$\lim_{n\to\infty} \sup_{z_i \leq x \leq z_{i+1}} |u_{0x} - f_n|(x) = 0, \tag{3.12}$$

$$\begin{cases} f_n \text{ takes all values of } P_n \cap [q_i, q_{i+1}] \text{ and its inverse } f_n^{-1}(r) \\ \text{for } r \in P_n \cap [q_i, q_{i+1}] \text{ has an interior in } [z_i, z_{i+1}]; \\ \text{moreover } f_n(z_i) = q_i, f_n(z_{i+1}) = q_{i+1}, \end{cases} \tag{3.13}$$

$$\int_{z_i}^{z_{i+1}} f_n(x)dx = \int_{z_i}^{z_{i+1}} u_{0x}(x)dx. \tag{3.14}$$

If $u_{xx} < 0$ in (z_i, z_{i+1}) we take f_n in $[z_i, z_{i+1}]$ as a piecewise constant nonincreasing function satisfying (3.12)-(3.14). In each piece $[z_i, z_{i+1}]$ we assign $u_0^n(x) = \int_{z_i}^x f_n(y)dy + u_0(z_i)$. Then, (3.13) and (3.14) guarantee that u_0^n is an admissible crystal with respect to P_n. The convergence $u_0^n \to u_0$ in $C(\mathbf{T})$ with $||u_{0x}^n||_\infty \to ||u_{0x}||_\infty$ follows from (3.12). Since $||u_{0xx}^n||_1 = \sum_{i=1}^{i_0} |q_{i+1} - q_i|$ by construction we have $||u_{0xx}^n||_1 = ||u_{0xx}||_1$ for all n. Thus (3.9) follows from (3.11). $\square$

We introduce our crystalline algorithm to approximate solutions of (1.5).
(i) For given $u_0 \in C(\mathbf{T})$ with $||u_{0xx}||_1 < \infty$ we take u_0^n and P_n as in Lemma 3.7.
(ii) For $W_0 \in \mathcal{E}$ we take a piecewise linear convex function W_n such that $W_n = W_0$ on P_n and W_n is affine outside P_n with $W_n'(p_1 - 0) = W_0'(p_1 - 0)$, $W_n'(p_m + 0) = W_0'(p_m + 0)$, where $p_1 = \min P_n, p_m = \max P_n$.
(iii) For $F_0 \in \mathcal{F}_{\#T}$, we may assume that F_0 is C^∞ by mollifying. We take $F_{0n}(t, p, X) = F_0(t, p, X) + G_n(X)$ with $G_n(X) = G(X/n)$ with a nonincreasing

C^∞ function G (with bounded derivative) that satisfies $G(X) = 0$ for $|X| \leq 1$ and $\lim_{X\to\pm\infty} G(X) = \mp\infty$. We then take

$$F_n(t,p,X) = \begin{cases} F_{0n}(t,p,X), \quad p \in P_n \text{ or } p \notin [p_1,p_m], \\ \theta F_{0n}(t,p_k,X) + (1-\theta)F_{0n}(t,p_{k+1},X), \\ \qquad\qquad p = \theta p_k + (1-\theta)p_{k+1}, 1 \leq k \leq m-1, 0 \leq \theta \leq 1. \end{cases}$$

For such F_n and W_n we consider a weakly admissible solution u^n of (1.4) with $F_\varepsilon = F_n$, $W_\varepsilon = W_n$, $u_0^\varepsilon = u_0^n$, $\varepsilon = 1/n, n \geq 1$; modification by G guarantees the global existence of weakly admissible solutions. By definition F_n satisfies (3.8). Thus, by Proposition 3.6 u^n is a generalized solution of (1.4). Since (3.10) implies that $W_n \to W_0$ uniformly in $[r_-, r_+]$, $F_n \to F_0$ locally uniformly and $u_0^n \to u_0$ in $C(\mathbf{T})$ as $n \to \infty$, Theorem 1.1 guarantees the local uniform convergence u^n to the generalized solution u of (1.5) in $[0,T) \times \mathbf{T}$. Note that the limit of W_n may not agree with W_0 outside $[p_1, p_m](= [r_-, r_+])$. However, since $||u_x^n||_\infty < r_+$, our u actually solves (1.5) with original W_0. Since u^n approximates u and u^n is constructed by solving ODEs (3.1)-(3.2), the method (i)-(iii) is a kind of crystalline algorithm. Our way of approximation of W_0 is different from [GK1, 2] since our W_n and P_n also depend on initial data u_0.

It is by now clear that (2.1) and (2.2) with $\xi(\sigma) = \sigma$ are obtained by Lemma 3.4 and Theorem 1.1 since we have $||u_{0xx}^n||_1 \to ||u_{0xx}||_1$ and $||u_{0x}^n||_\infty \to ||u_{0x}||_\infty$ as $n \to \infty$ by Lemma 3.7. The convergence (Theorem 2.5) also follows since it needs (2.2) only for $\xi(\sigma) = \sigma$.

4. Counterexample when F depends on the space variable

As we observed by Proposition 3.6 and Theorem 1.1 a weakly admissible solution is obtained by approximation by a smooth strict parabolic problem.

When F depends on the space variable, it is still possible to derive a system like (3.1)-(3.2) if we admit the ansatz that a facet stays as a facet with same slope. However, such an 'admissible solution' may not fulfills a comparison principle. We give an example below. This observation shows that such a solution cannot be obtained by approximation by smoother strict parabolic problem. In [R] and [GG4] a reasonable way to interpret solutions is proposed.

Let us consider an example given in [GG4] when F depends on the space variable. We consider an equation of form

$$u_t - \{W'(u_x)_x - C(x)\} = 0, \ x \in \mathbf{R}, \ t > 0, \tag{4.1}$$

where $W(p) = \Delta|p|/2$, $C(x) = \max\{0, 1-|x|\}$ with positive number Δ. We take

$$u_0(x;\alpha_0) = \begin{cases} 0 & |x| \le \alpha_0, \\ q(x-\alpha_0) & x > \alpha_0, \\ -q(x+\alpha_0) & x < -\alpha_0 \end{cases} \tag{4.2}$$

as an initial data, where q is a positive number and $\alpha_0 > 1$. Let us recall that the exact solution u of this initial value problem in [GG4] obtained by subdifferential formulation.

(i) the case $\Delta \ge 1$.

$$u(t,x;\alpha_0) = \begin{cases} u_0(\alpha(t;\alpha_0)) & |x| < \alpha(t;\alpha_0), \\ u_0(x) & \text{otherwise} \end{cases} \tag{4.3}$$

with $\alpha(t;\alpha_0) = (\alpha_0^2 + (\Delta-1)t/q)^{1/2}$. For later convenience we set the right hand side of (4.3) as $f(t,x;\alpha_0)$.
(ii) the case $\Delta < 1$.

$$u(t,x;\alpha_0) = \begin{cases} -C(\Delta^{1/2})t & |x| \le \Delta^{1/2}, \\ -C(x)t & \Delta^{1/2} < |x| < 1, \\ u_0(x;\alpha_0) & \text{otherwise.} \end{cases}$$

We observe that in the latter case a facet of u_0 on $[-\alpha_0, \alpha_0]$ is split into five facets. We also observe that for any $\overline{\alpha}_0 > \alpha_0 > 1$ we have $u(t,x;\overline{\alpha}_0) \le u(t,x;\alpha_0)$ for $t > 0$ and $x \in \mathbf{R}$ for the both cases.

We shall discuss what would happen if we assume 'facet-stay-facet assumption.' Let us assume that a function $v(t,x)$ *solves* (4.1) under facet-stay-facet assumption with initial data $u_0(x;\alpha_0)$. Integrating v_t from $\alpha_0 - \sigma$ to $\alpha_0 + \sigma$ at $t = +0$ with small positive σ, we have

$$\int_{-\alpha_0-\sigma}^{\alpha_0+\sigma} v_t \, dx = \int_{-\alpha_0-\sigma}^{\alpha_0+\sigma} \{W'(u_x)_x - C(x)\} dx = W'(u_x)|_{x=-\alpha_0-\sigma}^{\alpha_0+\sigma} - 1$$

at $t = +0$, since $\alpha_0 > 1$. Facet-stay-facet assumption implies that the left hand side equals $2(\alpha_0+\sigma)v_t(+0,x)$ which is independent of $x \in (-\alpha_0, \alpha_0)$. Sending σ to 0 implies that

$$v_t(+0,x) = \frac{\Delta - 1}{2\alpha_0} = \Lambda_W(v(+0,\cdot),0) - \frac{1}{2\alpha_0} \quad \text{for} \quad x \in (-\alpha_0, \alpha_0).$$

We observe that the right hand side equals weighted curvature minus the average of $C(x)$ on the faceted region $(-\alpha_0, \alpha_0)$. Thus we obtain that $v(t,x;\alpha_0) = f(t,x;\alpha_0)$ for $x \in \mathbf{R}$ and $t > 0$ with $\alpha(t;\alpha_0) > 0$.

When $\Delta < 1$, comparison principle is not valid for v. In fact, for $\overline{\alpha}_0 > \alpha_0 > 0$ it holds that $u_0(x;\overline{\alpha}_0) \leq u_0(x;\alpha_0)$ for $x \in \mathbf{R}$. However, we have $v(t,x;\overline{\alpha}_0) > v(t,x;\alpha_0)$ at least for $t \in (0, \alpha_0^2 q/(1-\Delta))$ and $x \in (-\alpha(t;\alpha_0),\ \alpha(t;\alpha_0))$, since $\alpha(t;\overline{\alpha}_0) - \overline{\alpha}_0 > \alpha(t;\alpha_0) - \alpha_0$ implies that $u_0(\alpha(t;\overline{\alpha}_0)) > u_0(\alpha(t;\alpha_0))$.

When C is independent of x, the subdifferential formulation is consistent with our generalized solutions [GG5]. It is desirable that the notion of generalized solutions is extended to the case when C depends on x so that it is compatible with the subdifferential formulation as suggested in [GG4].

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Department of Mathematics
Hokkaido University
Sapporo 060, Japan