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Series #392. October 1997

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1997.8.27

Abstract

The category of simulations of nondeterministic dynamical systems is shown to be symmetric monoidal closed category with a subobject classifyer. (AMS Classification:18D15,68Q10,03F50)

Introduction

1 The transition systems are playing the role of a universal model of behaviors of concurrent systems. They give background to the semantics of various frameworks of concurrent systems, such as process algebras, Petri nets, etc.. This framework can also be regarded as a first approximation to describe the concurrent distributed aspects of living systems.

2 In the categorical study of transition systems, there are two possible choices of arrows: either graph maps or simulations.

Usually graph maps are adopted as the arrows[4, 12]. This category is the presheaf category over the finite category

$$\ell \xleftarrow{\lambda} e \underset{d_1}{\overset{d_0}{\rightrightarrows}} v.$$

where e, v, ℓ stand for the sets of edges, nodes and labels respectively. In particular it is a topos.

The concept of subobjects in this category, however, does not match our intution of processes. In fact, let P be a process, i.e. a transition system, and S be a subset of states of P. When we say that S with a set τ of transitions forms a subprocess of P, we intend to mean that not only the transitions in τ

come from the total process P, but also every transition of the total process P starting from a state of S results in a state of S. But this is not the case in the category with graph maps even when S with τ is a subgraph of P. Another reason is as follows: Suppose a process P is a dynamical system, i.e., every state have a unique transition from it. A subobject of P in the category with graphs maps may not be a dynamical system, whereas in the category of simulations, it is.

This leads us to consider the category of transition systems whose arrows are simulations maps. With this choice of arrows, the subobjects describe stable properties, their characteristic maps take values in the universe of hypersets and the truth value of each state says precicely how the system in that state will behave in the future with respect to that property.

This category of simulations is introduced from a point of views different from ours in [8]. It is studied as a mathematical framework of concurrent constrained programming.

3 For the sake of simplicity, we take the singleton set as the labeling set and call such transition systems **nondeterministic dynamical systems.** Moreover we consider only systems all of whose nodes have positive out-degrees.

Our main result is that this category $\mathcal{NDyn}$ is an autonomous category, i.e. a closed symmetric monoidal category, with a subobject classifier. This might enable us to regard this category as a universe of a typed linear set theory, similar to the untyped version developed by Shirahata in [9, 10].

4 We give in §1 basic definitions on nondeterministic dynamical systems. In §2 we study basic properties of the category of simulations, where concompleteness is proved and a tensor product is introduced. In §3 we construct nondeterministic dynamical system from each presheaf over the subcategory of trees. Monoidal closedness is proved in §4 and existence of subobject classifier is proved in §5 using the theory of hypersets. A purely categorical proof of the existence of the subobject classifier is given by the second author in [11].

We denote $\mathcal{G}raph$ the category of nondeterministic dynamical systems with the graph maps as arrows. Construction of objects in this topos will be often utilized but the resulting objects and diagrams will have usually different categorical meanings in $\mathcal{NDyn}$.

1 Definitions

1.1 Nondeterministic dynamical system

A **nondeterministic dynamical system** D is a pair $(|D|, \tau_D)$ of a set $|D|$ and a binary relation $\tau_D \subseteq |D| \times |D|$. Elements of $|D|$ are called **states** and τ_D the **transition relation.**

When $(s_1, s_2) \in \tau_D$, we write $s_1 \rightarrow_D s_2$, which we interprete as the possibility that the system goes from state s_1 to state s_2. We write $s_1 \rightarrow^*_D s_2$ if there is a path from s_1 to s_2 when D is regarded as a directed graph.

A nondeterministic dynamical system D is called **deterministic**, or **a dynamical system** if the transition relation is the graph of a selfmap, which is called **the transition map** and is also denoted by τ_D. Hence $s \rightarrow_D t \iff t = \tau_D(s)$.

1.2 Graph maps and Simulations

Let D, D' be nondeterministic dynamical systems.

A map $\varphi : |D| \rightarrow |D'|$ is called a **graph map** if

$$\varphi_* \mathrm{Child}_D(s) \subseteq \mathrm{Child}_{D'}(\varphi(s))$$

for all $s \in |D|$. Here

$$\mathrm{Child}_D(s) := \{\, x \in |D| \mid s \rightarrow_D x \,\}.$$

A map $\varphi : |D| \rightarrow |D'|$ is a **simulation** if

$$\varphi_* \mathrm{Child}_D(s) = \mathrm{Child}_{D'}(\varphi(s))$$

for all $s \in |D|$.

Obviously the compositions of graph maps and simulations are respectively graph maps and simulaions.

1.3 Bisimulation

Fix a nondeterministic dynamical system D. A relation $R \subseteq |D| \times |D|$ is called a **bisimulation** if the following two conditions are satisfied.

1. For every $(x, y) \in R$ and $x' \in |D|$ with $x \rightarrow_D x'$, there is an element $y' \in |D|$ with $y \rightarrow_D y'$ and $(x', y') \in R$,
2. For every $(x, y) \in R$ and $y' \in |D|$ with $y \rightarrow_D y'$, there is an element $x' \in |D|$ with $x \rightarrow_D x'$ and $(x', y') \in R$.

A bisimulation which is also an equivalence relation will be simply called a **bisimulation equivalence.**

Some of the basic facts about bisimulation are summarized in the following proposition [3, 1].

Proposition 1.1 *(i) The set of bisimulations is closed under the operations of transpose, union and composition. The diagonal is a bisimulation.*

(ii) For every bisimulation R, there is the smallest bisimulation equivalence $\widetilde{R}$ containing it.

(iii) If R is a bisimulation equivalence on a nondeterministic dynamical system $D = (S, \tau)$, then the quotient set S/R has a unique transition relation which makes the quotient simulation π_R a simulation. This nondeterministic dynamical system will be called **the quotient of** *D* **by** *R and will be denoted by D/R.*

(iv) If $f : D \rightarrow D_1$ is a simulation with $fx = fy \Rightarrow xRy$, then it factors through the quotient simulation π_R, namely there is a unique simulation $\overline{f} : D/R \rightarrow D_1$ satisfying $f = \overline{f} \circ \pi_R$.

(v) If $p : D_1 \rightarrow D_2$ is a simulation, then the relation R_p on $|D_1|$ defined by $xR_py \overset{def}{\Longleftrightarrow} px = py$ is a bisimulation equivalence.

1.4 The cateogory $\mathcal{NDyn}$, $\mathcal{Graph}$, $\mathcal{Tree}$

Let $\mathcal{NDyn}$ be the category whose objects are nondeterministic dynamical systems $(|D|, \tau)$ satisfying the conditions

- $\forall s \in |D| \exists t \in |D| \; [\, s \rightarrow t \,]$,
- D is locally finite, namely, the set $\mathrm{Child}_D(s)$ is finite for all $s \in |D|$.

and whose morphisms are simulations.

This is a subcategory of the topos $\mathcal{Graph}$ of all the graphs with multiple edges and graph maps.

A directed graph is called a **tree** when there is a distinguished node r called **the root** such that there is a unique path from r to every node other than r. The root of a tree T is denoted by r_T.

Let $\mathcal{Tree}$ be the full subcategory of $\mathcal{NDyn}$ whose objects are trees and whose morphisms are simulations. Since there are no leaves, all the branches of our trees are necessarily infinite. In other words, every finite path can be prolonged farther. The inclusion functor will be denoted by $i : \mathcal{Tree} \rightarrow \mathcal{NDyn}$. It will be shown in §3.1that this category is skeltally small.

1.5 unfolding

Let $D = (|D|, \tau) \in \mathcal{NDyn}$. For $x \in |D|$, we denote by $\mathrm{Path}(x) \in \mathcal{T}ree$ the tree whose nodes are finite paths starting from x and whose edges are $\gamma \rightarrow \gamma e$ where γ is a path from x and e is an edge from the end point of γ. We denote by

$$\varpi_x : \mathrm{Path}(x) \rightarrow D$$

the simulation which sends each path to its end point. We denote the root of the tree $\mathrm{Path}(x)$ by r_x. By definition $\varpi_x(r_x) = x$.

Proposition 1.2 *Let $T \in \mathcal{T}ree$ and $D \in \mathcal{NDyn}$. Suppose $\kappa : T \rightarrow D$ is a simulation. Then there is a simulation $\tilde{\kappa} : T \rightarrow \mathrm{Path}(\kappa(r_T))$ which makes the following diagram commutative:*

$$\begin{tikzcd} & \mathrm{Path}(x) \arrow[d, "\varpi_x"] \\ T \arrow[ur, "\tilde{\kappa}"] \arrow[r, "\kappa"'] & D. \end{tikzcd}$$

Proof. For $t \in T$ denote by γ_t the unique path in T from the root to t. Define $\tilde{\kappa}(t) = \kappa_* \gamma_t$, where $\kappa_* \gamma_t$ is the image of the path γ_t by κ and is a finite path from $x := \kappa(r_T)$ to $\kappa(t)$ and hence is a node in $\mathrm{Path}(x)$. Obviously this map $\tilde{\kappa}$ is a simulation and makes the diagram commutative.

∎

1.6 Subsystem

Let $D = (|D|, \tau) \in \mathcal{NDyn}$. The concept of subobjects coincides with that of **subsystems.** A **subsystem** D_0 of D is a nondeterministic dynamical system whose state space $|D_0|$ is a subset of $|D|$ and the inclusion map $|D_0| \rightarrow |D|$ is a simulation, namely,

$$\mathrm{Child}_D(x) = \mathrm{Child}_{D_0}(x)$$

for all $x \in |D_0|$.

For $x \in |D|$, we define a subsystem $D.x \in \mathcal{NDyn}$ by

$$|D.x| := \{\, y \in |D| \mid x \rightarrow^*_D y \,\},$$

and

$$u \rightarrow_{D.x} v \overset{def}{\Longleftrightarrow} u \rightarrow_D v.$$

The inclusion map $|D.x| \rightarrow |D|$ is obviously a simulation. We call $D.x$ **the subsystem of D generated by** x.

2 Basic properties of the category $\mathcal{NDyn}$

2.1 The terminal and initial objects

The object $\mathbf{1} := (\{ 0 \}, \tau = \{ (0,0) \})$ is obviously terminal. The initial object is the empty system $\mathbf{0} = (\emptyset, \emptyset)$.

2.2 Tensor product

Proposition 2.1 *Let $\sigma_i : D_i \to D (i = 1,2)$ be simulations. Let $D \in \mathcal{G}raph$ be defined by*

$$|D| := \{ \langle x_1, x_2 \rangle \in |D_1| \times |D_2| \mid \sigma x_1 = \sigma_2 x_2 \},$$

and

$$\langle x_1, x_2 \rangle \to_D \langle y_1, y_2 \rangle \overset{def}{\iff} x_i \to_{D_i} y_i \quad (i = 1, 2).$$

Define $\pi_i : |D| \to |D_i|$ by $\pi_i \langle x_1, x_2 \rangle = x_i$ for $i = 1, 2$. Then

(1) the following is a pullback diagram in Graph,

$$\begin{tikzcd} D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D_2 \arrow[d, "\sigma_2"] \\ D_1 \arrow[r, "\sigma_1"'] & D_0 \end{tikzcd}$$

(2) $D \in \mathcal{NDyn}$,

(3) π_i are simulations for $i = 1, 2$.

We shall refer to this D by a somewhat inexact notation $D_1 \otimes_{D_0} D_2$.

Proof. It is easy to see that the diagram is a pullback in Graph.

From $\mathrm{Child}_D \langle x_1, x_2 \rangle = \mathrm{Child}_{D_1}(x_1) \times \mathrm{Child}_{D_2}(x_2)$, it follows that D is locally finite.

Let $\langle x_1, x_2 \rangle \in |D|$. We must show that there is a $\langle y_1, y_2 \rangle \in |D|$ with $\langle x_1, x_2 \rangle \to_D \langle y_1, y_2 \rangle$. Let $x_0 := \sigma_1 x_1$. Since $D_0 \in \mathcal{NDyn}$, there is a $y_0 \in |D_0|$ with $x_0 \to_{D_0} y_0$. Since σ_i's are simulations, there are $y_i \in |D_i|$ with $x_i \to_{D_i} y_i$ with $\sigma_i y_i = y_0$ for $i = 1, 2$. Then $\langle y_1, y_2 \rangle \in |D|$ and $\langle x_1, x_2 \rangle \to_D \langle y_1, y_2 \rangle$ as was to be shown.

Thus we have $D \in \mathcal{NDyn}$.

Now we show that π_i's are simulations. Let $\langle x_1, x_2 \rangle \in |D|$ and $x_1 \to_{D_1} y_1$. Then $\sigma_1 x_1 \to_{D_1} \sigma_1 y_1$. Since $\sigma_1 x_1 = \sigma_2 x_2$ and σ_2 is a simulation, there is a $y_2 \in |D_2|$ with $x_2 \to_{D_2} y_2$ and $\sigma_2 y_2 = \sigma_1 y_1$. Hence we have found $\langle y_1, y_2 \rangle \in |D|$

with $\langle x_1, x_2\rangle \rightarrow_D \langle y_1, y_2\rangle$ and $\pi_1\langle y_1, y_2\rangle = y_1$. Hence π_1 is a simulation. Similarly, π_2 is a simulation. ∎

Put $D_1 \otimes D_2 := D_1 \otimes_1 D_2$. Define

$$\sigma_1 \otimes \sigma_2 : D_1 \otimes D_2 \rightarrow D_1' \otimes D_2'$$

for simulations $\sigma_i : D_i \rightarrow D_i'$ by

$$\sigma_1 \otimes \sigma_2 \langle x_1, x_2\rangle = \langle \sigma_1 x_1, \sigma_2 x_2\rangle.$$

Then obviously this is a simulation. Hence we have a bifunctor

$$\bullet \otimes \bullet : \mathcal{NDyn} \times \mathcal{NDyn} \rightarrow \mathcal{NDyn}.$$

There are natural isomorphisms giving associativity

$$a_{D_1 D_2 D_3} : (D_1 \otimes D_2) \otimes D_3 \rightarrow D_1 \otimes (D_2 \otimes D_3)$$

and the commutativity

$$r_{D_1 D_2} : D_1 \otimes D_2 \rightarrow D_2 \otimes D_1$$

definte the twist map. These satisfy the coherence conditions obviously.

Form this we conclude

Theorem 2.2 *The category $\mathcal{NDyn}$ has a structure of symmetric monoidal category.*

We note that, although this tensor product has a natural projections:

$$\pi_i : D_1 \otimes D_2 \rightarrow D_i \quad i = 1, 2,$$

it fails to be a product as is shown in the folloiwng remark.

2.2.1 Remark

The diagram

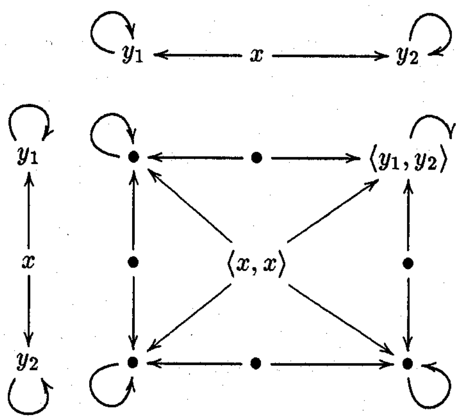
$$\begin{tikzcd} D_1 \otimes_{D_0} D_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D_2 \arrow[d, "\sigma_2"] \\ D_1 \arrow[r, "\sigma_1"'] & D_0 \end{tikzcd}$$

may not be a pullback diagram in $\mathcal{NDyn}$. For example, suppose $D_0 = \mathbf{1}$, and consider

$$D \otimes D := D \otimes_1 D.$$

If this were a pullback, it would be a product of D with itself in $\mathcal{NDyn}$.

However we can show that when D has a state x with two distinct children y_1, y_2 then $D \otimes D$ is not a product. In fact if $D \otimes D$ were a product, there is a unique simulation $\varphi : D \to D \otimes D$ with $\pi_1 \circ \varphi = \pi_2 \circ \varphi = \mathrm{id}_D$. Since φ is a graph map and $D \otimes D$ is a product in Graph, we must have $\varphi = \Delta$, the diagonal map. However, we have $\varphi x = \langle x, x\rangle \to \langle y_1, y_2\rangle$ with no $y \in |D|$ such that $\langle y, y\rangle = \langle y_1, y_2\rangle$. This shows that φ is not a simulation, a contradiction which arised from the assumption that $D \otimes D$ is a product.



2.3 Concreteness

$\mathcal{NDyn}$ is a concrete category, i.e., the forgetful functor $\mathcal{NDyn} \to \mathcal{S}\mathbf{et}$ is faithfull. In particular, a simulation which is injective is a monic. In fact we can show

Proposition 2.3 *A simiulation is a monic if and only if it is injective.*

Proof. Suppose a simulation $\sigma : D \to D'$ is a monic in NDyn and is not injective as a map. Then there are different $x_1, x_2 \in |D|$ with $\sigma x_1 = \sigma x_2$. Let

$$D'' := D.x_1 \otimes_{D'} D.x_2$$

and define simulations $\beta_i : D'' \to D$ $(i = 1, 2)$as the compositions of

$$D'' \xrightarrow{\pi_i} D.x_i \subseteq D.$$

Let $\langle u_1, u_2\rangle \in |D''|$, i.e. $u_i \in D.x_i$ and $\sigma u_1 = \sigma u_2$. We have

$$\sigma \circ \beta_1\langle u_1, u_2\rangle = \sigma u_1 = \sigma u_2 = \sigma \circ \beta_2\langle u_1, u_2\rangle,$$

whence $\sigma \circ \beta_1 = \sigma \circ \beta_2$. However

$$\beta_1\langle x_1, x_2\rangle = x_1 \neq x_2 = \beta_2\langle x_1, x_2\rangle$$

with $\langle x_1, x_2 \rangle \in |D''|$, which implies $\beta_1 \neq \beta_2$. This shows that α is not monic. ∎

From this, we can identify the isomorphic class of subobjects of D as a subsystem, i.e. a subset D_0 of $|D|$ whose inclusion map is a simulation.

Since a surjection is an epic, we have

Proposition 2.4 *Every simulation can be expressed as the composition of an epic followed by a monic.*

Similarly we can show the following proposition using the quotient (2.6) by subobject.

Proposition 2.5 *A simulation is an epic if and only if it is surjective.*

2.4 Pull-back of monics

Although the tensor product is not always a pullback, we can show the following:

Theorem 2.6 *The pullbacks of monic arrows in $\mathcal{G}raph$ are also pullbacks in $\mathcal{NDyn}$. Namely, if σ_2 is monic, then*

$$\begin{tikzcd} D_1 \otimes_{D_0} D_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D_2 \arrow[d, "\sigma_2"] \\ D_1 \arrow[r, "\sigma_1"'] & D_0 \end{tikzcd}$$

is a pullback.

Proof. Suppose $\alpha_i : D \to D_i \; i = 1, 2$ are simulations satisfying

$$\sigma_1 \circ \alpha_1 = \sigma_2 \circ \alpha_2 : D \to D_0.$$

Let $\alpha : D \to D_1 \otimes_{D_0} D_2$ be the graph map with

$$\pi_i \circ \alpha = \alpha_i \quad i = 1, 2.$$

$$\begin{tikzcd} D \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\ & D_1 \otimes_{D_0} D_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D_2 \arrow[d, "\sigma_2"] \\ & D_1 \arrow[r, "\sigma_1"'] & D_0. \end{tikzcd}$$

We show that α is a simulation. Let $x \in |D|$ and suppose

$$\alpha(x) = \langle \alpha_1 x, \alpha_2 x \rangle \longrightarrow \langle y_1, y_2 \rangle.$$

Then we have an $x_1 \in |D|$ with $x \rightarrow_D x_1$ and $\sigma_1 x_1 = y_1$. From

$$\begin{aligned} \alpha_2 \sigma_2 x_1 &= \alpha_1 \sigma_1 x_1 \\ &= \alpha_1 y_1 \\ &= \alpha_2 y_2, \end{aligned}$$

we have $\sigma_2 x_1 = y_2$ since σ_2 is injective by Proposition2.3. Hence $\alpha(x_1) = \langle y_1, y_2 \rangle$. This proves that α is a simulation. ∎

This theorem enables us to define a functor

$$\mathrm{Sub} : \mathcal{NDyn}^{op} \rightarrow \mathcal{S}\mathbf{et}$$

defined by

$$\mathrm{Sub}(D) := \{ \iota : D_0 \rightarrow D \mid \iota \text{ is a monic } \} / \simeq,$$

and for $f : D_1 \rightarrow D_2$

$$\mathrm{Sub}(f) : \mathrm{Sub}(D_2) \rightarrow \mathrm{Sub}(D_1)$$

is defined by

$$\mathrm{Sub}(f)[\iota] := [f^{-1}\iota],$$

which is defined by the following pullback diagram.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f^{-1}\iota"'] & D \arrow[d, "\iota"] \\ D_1 \arrow[r, "f"'] & D_2 \end{tikzcd} .$$

In [11] it is proved that this functor is representable. In §5, we give intuitive construction of the object which represents this functor.

2.5 Cocompleteness

Theorem 2.7 *$\mathcal{NDyn}$ is cocomplete.*

It suffices to show that sum and coequalizer exist.

Proposition 2.8 *Every small family of objects in $\mathcal{NDyn}$ has a sum.*

Proof. Let $\{ D_k \mid k \in K \}$ be a family of objects in $\mathcal{NDyn}$ indexed by a set K. Then the nondeterministic dynamical system $D = \coprod_k D_k$ defined by

$$\begin{aligned} \left|\coprod_k D_k\right| &= \coprod |D_k| \\ x \rightarrow_D y &\overset{def}{\iff} \exists k\,[x, y \in D_k \ \wedge\ x \rightarrow_{D_k} y\,] \end{aligned}$$

with the inclusion simulations $\iota_k : D_k \rightarrow D$ is obviously a sum. ∎

Proposition 2.9 *Every paralell simulations $f, g : D_1 \rightarrow D_2$ have a coequlizer.*

Proof. Let

$$R_{fg} := \{ \langle fx, gx \rangle \mid x \in |D_1| \} \subseteq |D_2| \times |D_2|.$$

This is a bisimulation on D_2 as is easily seen. Let R be the minimal bisimulation equivalence on $|D_2|$ containing R_{fg}. Let $\pi : D_2 \rightarrow D_2/R$ be the quotient simultaion (Proposition1.1).

We show that this is a coequalizer of f, g. Let $\sigma : D_2 \rightarrow D$ a simulation with $\sigma \circ f = \sigma \circ g$. Let R_σ be the bisimulation equivalence on D_2 defined by

$$x R_\sigma y \overset{def}{\iff} \sigma(x) = \sigma(y)$$

(cf. Proposition1.1). Since $R_\sigma \supseteq R$, there is a unique simulation

$$\overline{\sigma} : D_2/R \rightarrow D$$

with $\overline{\sigma} \circ \pi = \sigma$. This proves that π is a coequalizer of f and g. ∎

Remark. The cocompleteness of $\mathcal{NDyn}$ is a special case of the general fact that the coalgebra categories for endofunctors of the category of sets are cocomplete [2, Proposition 1.1], [5, Proposition 2.1].

In fact $\mathcal{NDyn}$ is the category of $\mathbf{pow}_o$-coalgebras, where

$$\mathbf{pow}_o : \mathcal{S}\mathbf{et} \rightarrow \mathcal{S}\mathbf{et}$$

is the covariant functor with

- $\mathbf{pow}_o(X)$ is the set of finite nonempty subsets of X,
- $\mathbf{pow}_o(f)$ for $f : X \rightarrow Y$ maps the finite subset A of X to the finite subset $\{ f(x) \mid x \in A \}$ of Y.

2.6 Quotient

In particular, for a subobject $i : D_0 \to D$, we can define its **quotient**

$$q : D \to D/iD_0$$

by the following pushout diagram:

$$\begin{tikzcd} D_0 \arrow[r, "i"] \arrow[d, "!"'] & D \arrow[d, "q"] \\ \mathbf{1} \arrow[r, "\top"'] & D/iD_0 \end{tikzcd} \qquad (*)$$

which exists by the cocompletenes of $\mathcal{ND}yn$.

Proposition 2.10 *The diagram (*) is a pull-back.*

Proof. The pushout D/D_0 is defined as the coequalizer of

$$D_0 \underset{b}{\overset{a}{\rightrightarrows}} D\coprod \mathbf{1},$$

where $a(x) := \iota_D \circ i$ and $b := \iota_1 \circ !_D$. The bisimulation equivalence on $D \coprod \mathbf{1}$ used in constructing D/D_0 is the minimal one satisfying

$$a(x) \simeq b(x) \quad \forall x \in |D_0|.$$

This is exactly

$$(|D_0|\coprod\{*\}) \times (|D_0|\coprod\{*\}) \cup \Delta.$$

This shows that for $x \in |D|$,

$$q(x) = \top(*) \iff x = iy \text{ for some } y \in |D_0|.$$

This shows that the diagram (*) is a pullback in the set category and whence in Graph. Since $\top$ is a monic, the diagram (*) is a pullback by Theorem 2.6. ∎

3 Construction from presheaf over the tree category

3.1 Skeltal smallness of the tree category

The objects of $\mathcal{T}ree$ can be parametrized as follows: Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{N}^*$ the set of finite words on $\mathbf{N}$. Let T be a subset of $\mathbf{N}^*$.

- T is called **prefix closed** when $w \in T$ implies $v \in T$ for all prefixes v of w,
- T is called **infinite** if for all $v \in T$ there is at least one $i \in \mathbf{N}$ with $v.i \in T$.
- T is called **locally finite** if for each $w \in T$ the set $\{\ i \in \mathbf{N} \mid w.i \in T\ \}$ is a finite set of the form $\{\ i \mid 1 \leq i \leq n_w\ \}$.

Let $\mathcal{T}$ be the set of the subsets of $\mathbf{N}^*$ which satisfy these three conditions. An element $T \in \mathcal{T}$ defines a tree $S[T]$ with the node set T and with the edge $w \rightarrow w'$ when $w' = w.i$ ($i \in \mathbf{N}$).

Lemma 3.1 *Each object in $\mathcal{T}ree$ is isomorphic to $S[T]$ for some $T \in \mathcal{T}$.*

Proof. Let S be an object of $\mathcal{T}ree$. For each node x we fix a bijection $\alpha_x : \mathrm{Child}(x) \rightarrow \{\ 1, 2, \cdots, n_x\ \}$. Define $s(x)$ inductively by

- $s(r_S) := \varepsilon$,
- $s(c) = s(x).\alpha_x(c)$ for $c \in \mathrm{Child}(x)$.

It is obvious that $s : |S| \rightarrow \mathbf{N}^*$ is injective. Let

$$T[S] := \{\ s(x) \mid x \text{ is a node of } S\ \}.$$

Obviously, $T[S] \in \mathcal{T}$ and that $S[T[S]] \simeq S$. ∎

This implies the following.

Corollary 3.2 *There is a small skelton of*

$$\mathcal{T}ree$$

.

Proof. Let $\mathcal{T}ree_1$ be the full subcategory of $\mathcal{T}ree$ whose objects are $S[T]$ with $T \in \mathcal{T}$. This category is small since the collection of its objects is small and it is locally small being a concrete category. Moreover, by the above lemma, every object of $\mathcal{T}ree$ is isomrophic to an object of $\mathcal{T}ree_1$. So, it suffices to take a skelton $\mathcal{T}ree_0$ of $\mathcal{T}ree_1$, which is obviously small and also is a skelton of $\mathcal{T}ree$.

By the above lemma, every object of $\mathcal{T}ree$ is isomrophic to one of the objects of $\mathcal{T}ree_1$. ∎

In the sequel we fix a small skelton $\mathcal{T}ree_0$ of $\mathcal{T}ree$.

3.2 $\mathcal{T}ree$ is dense in NDyn

Theorem 3.3 *Tree is a dense subcategory of* $\mathcal{ND}yn$.

Proof. Let $D \in \mathcal{ND}yn$. Denote by $\Gamma(D)$ the free category generated by D considered as a graph. Let $E : \Gamma(D)^{op} \to \mathcal{T}ree$ be a functor defined by

- $E(x) := \mathrm{Path}(x)$ for $x \in |D|$,
- $E(x \to y) : \mathrm{Path}(y) \to \mathrm{Path}(x)$, where

  $$E(x \to y)\gamma := e.\gamma.$$

  Here γ is an arrow with domain y, i.e. a finite path starting from y, and e denotes the edge $x \to y$.

Let

$$D' := \mathrm{colim} iE,$$

where $iE : \Gamma(D)^{op} \to \mathcal{ND}yn$, which is constructed as

$$\coprod_{x \in |D|} \mathrm{Path}(x)/\simeq.$$

Here $\simeq$ is the smallest bisimulation equivalence including the relation

$$\gamma \simeq E(x \to y)\gamma$$

for $\gamma \in \mathrm{Path}(y)$ and $x \to y$. It is easily seen that the correspondence

$$\mathrm{Path}(x) \ni \gamma \mapsto \text{the end point of } \gamma$$

induces a bijection $|D'| \to |D|$ which is obviously a simulation. Hence we have $D \simeq \mathrm{colim} iE$, where iE comes from a diagram E in $\mathcal{T}ree$.

Note that E defines a functor from $\Gamma(D)^{op}$ to the comma category $i \downarrow D$ by

$$x \in \Gamma(D)^{op} \mapsto (\varpi_x : i\mathrm{Path}(x) \to D)$$

and Proposition 1.2 means precisely that this functor is **final**. Hence,

$$\mathrm{colim}(i \downarrow D \xrightarrow{\pi} \mathcal{T}ree \to \mathcal{ND}yn) \simeq D, \tag{1}$$

[6, p213, Theorem 1]. This means precisely that $\mathcal{T}ree$ is a dense subcategory of

$$\mathcal{ND}yn$$

[6, p242]. ∎

3.3 Reflection of $\mathcal{NDyn}$ to $\widehat{\mathcal{T}ree_0}$

Define

$$R : \mathcal{NDyn} \to \widehat{\mathcal{T}ree_0},$$

by $RD(T) = \mathcal{NDyn}(iT, D)$. Theorem3.3 implies the following proposition [6, p243].

Proposition 3.4

$$R : \mathcal{NDyn} \to \widehat{\mathcal{T}ree_0}$$

is full and faithfull.

This has a left adjoint

$$L : \widehat{\mathcal{T}ree_0} \to \mathcal{NDyn},$$

defined by

$$LP = \mathrm{colim}(\int P \xrightarrow{\pi} \mathcal{T}ree \subset \mathcal{NDyn})$$

[7, p41 Theorem 2]

Recall that $\int P$ is the category of elements defined by

- The objects are the pairs (T, p), with $T \in \mathcal{T}ree_0$ and $p \in PT$,
- an arrow $\phi : (T,p) \to (T', p')$ is a simulation $\phi : T \to T'$ satisfying $p'\phi = p$, where $p'\phi := P(\phi)(p')$.

The functor $\pi : \int_0 P \to \mathcal{T}ree_0$ is defined by $(T,p) \mapsto T$. So the left hand side can be written explicitely as

$$\{\ (p,t)\ \mid p \in PT, t \in T, T \in \mathcal{T}ree_0\ \}/\simeq,$$

where $\simeq$ is the minimal bisimulation equivalence with

$$(p', \phi t) \simeq (p'\phi, t) \quad (p' \in PT', t \in P, \phi : T \to T').$$

We use the following facts in next section.

Proposition 3.5 *Elements of LP are represented by* $\{\ (p, r_T)\ \mid p \in PT, T \in \mathcal{T}ree\ \}$.

Proof. In fact, let $[(p,t)] \in LP$ with $t \in T \in \mathcal{T}ree_0$ an $p \in PT$. Let $T' := T.t$ be the subtree of T with the root t and $j : T' \to T$ the inclusion. Then

$$(p,t) = (p, jr_{T'}) \sim (pj, r_{T'}).$$

∎

From the denseness of i, we have

Proposition 3.6 *The counit of* $L \vdash R$

$$\epsilon : LR \rightarrow 1_{\mathcal{NDyn}}$$

is a natural isomorphism.

We remark that the counit $\epsilon(D) : LR(D) \rightarrow D$ of the adjunction $L \vdash R$ is an isomorphism natural with respect to D. In fact the comma category $i' \downarrow D$ ($i' : \mathcal{T}ree_0 \rightarrow \mathcal{NDyn}$) is nothing but the category of elements $\int R(D)$, whence by (1),

$$\begin{aligned} & \operatorname{colim}(\int R(E) \rightarrow \mathcal{T}ree_0 \rightarrow \mathcal{NDyn}) \\ \simeq\ & \operatorname{colim}(i' \downarrow D \rightarrow \mathcal{T}ree_0 \rightarrow \mathcal{NDyn}) \\ \simeq\ & \operatorname{colim}(i \downarrow D \rightarrow \mathcal{T}ree \rightarrow \mathcal{NDyn}) \simeq D. \end{aligned}$$

4 Monoidal closedness of $\mathcal{NDyn}$

Define

$$[D_1, D_2] := L(\mathcal{NDyn}[D_1 \otimes \bullet, D_2]).$$

More explicitely,

$$[D_1, D_2] = \{\ (p, x)\ \mid p : D_1 \otimes T \rightarrow D_2, x \in T, T \in \mathcal{T}ree\ \}/\simeq,$$

with

$$(p \circ (1 \otimes \alpha), x') \simeq (p, \alpha x')$$

for $\alpha : T' \rightarrow T, x' \in T'$.

Define **the evaluataion arrow** $\varepsilon : D_1 \otimes [D_1, D_2] \rightarrow D_2$ by

$$\varepsilon(v, [(\kappa, x)]) = \kappa(v \otimes x).$$

This definition does not depend on the choice of representatives and is functorical in D_1.

Define **the position marker** arrow $d : D_2 \rightarrow [D_1, D_1 \otimes D_2]$ by $d(v_2) = [(1 \otimes \varpi_{v_2}, r_{v_2})]$, where

$$D_1 \otimes \mathrm{Path}(v_2) \stackrel{1 \otimes \varpi_{v_2}}{\longrightarrow} D_1 \otimes D_2.$$

This is functorial in D_2.

We can show the commutativity of the following diagrams:

$$\begin{tikzcd} D_1 \otimes D_2 \arrow[r, "1\otimes d"] \arrow[dr, "1"'] & D_1 \otimes [D_1, D_1 \otimes D_2] \arrow[d, "\epsilon"] \\ & D_1 \otimes D_2 \end{tikzcd} ,$$

$$\begin{tikzcd} [D_1, D_2] \arrow[r, "d"] \arrow[dr, "1"'] & [D_1, D_1 \otimes [D_1, D_2]] \arrow[d, "{[1,\epsilon]}"] \\ & [D_1, D_2] \end{tikzcd} .$$

Proof.

Commutativity of the first diagram. Let $v_1 \otimes v_2 \in D_1 \otimes D_2$. Then

$$(1 \otimes d)(v_1 \otimes v) = v_1 \otimes [(1 \otimes \varpi_v, r_v)],$$

whence

$$\begin{aligned} \epsilon(1 \otimes d)(v_1 \otimes v) &= \epsilon(v_1 \otimes [(1 \otimes \varpi_v, r_v)]) \\ &= (1 \otimes \varpi_v)(v_1 \otimes r_v) \\ &= v_1 \otimes v. \end{aligned}$$

Commutativity of the second diagram. Let $u = [(p, r_T)] \in [D_1, D_2]$ with $p : D_1 \otimes T \to D_2$. Then $d(u) = [(1 \otimes \varpi_u r_u)]$, where

$$\varpi_u : \mathrm{Path}(u) \to [D_1, D_2].$$

Recall that we denoted by $\kappa_p : T \to [D_1, D_2]$ the canonical map defined by $\kappa_p(t) = [(p, t)]$.

By Proposition1.2 there is a simulation

$$\mu_u : T \to \mathrm{Path}(u)$$

which maps the root r_T to r_u and makes the following diagram commutative.

$$\begin{tikzcd} & \mathrm{Path}(u) \arrow[d, "\varpi_u"] \\ T \arrow[ur, "\mu_u"] \arrow[r, "\kappa"'] & {[D_1, D_2]}. \end{tikzcd}$$

Then

$$\begin{aligned}[(1 \otimes \varpi_u, r_u)] &= [((1 \otimes \varphi_u) \circ (1 \otimes \mu_u), r_T)] \\ &= [((1 \otimes \kappa_p), r_T)].\end{aligned}$$

Finally

$$\begin{aligned}[1, \varepsilon](d(u)) &= [(\varepsilon \circ (1 \otimes \kappa_p), r_T)] \\ &= [(p, r_T)] = u,\end{aligned}$$

since

$$\varepsilon(1 \otimes \kappa_p)(v_1 \otimes t) = \varepsilon(v_1 \otimes [(p, t)]) = p(v_1 \otimes t).$$

The above arguments can be summarized in the following commutative diagram.

$$\begin{tikzcd}
D_1 \otimes \mathrm{Path}(u) \arrow[rr, "1\otimes\varpi_u"] & & D_1 \otimes [D_1, D_2] \arrow[d, "\varepsilon"] \\
D_1 \otimes T \arrow[u, "1\otimes\mu_u"] \arrow[urr, "1\otimes\kappa_p"] \arrow[rr, "p"'] & & D_2
\end{tikzcd}$$

∎

From this we obtain that

$$\mathcal{NDyn}(D_1 \otimes D_2, D_3) \simeq \mathcal{NDyn}(D_1, [D_2, D_3]),$$

hence $\mathcal{NDyn}$ is a monoidal closed category.

5 Subobject classifier

5.1 Construction of subobject classifier

Let Ω be the universe of hereditary finite hypersets constructed from the urelement $\{ \top \}$[3] with this urelement added. Let $\perp$ denote the unique set satisfying

$$\{ \perp \} = \perp.$$

We define the graph structure in this set by

- $\top \to \top$

- $a \to b \stackrel{def}{\Longleftrightarrow} b \in a$ if a is a set.

It will be proved in [11] that this universe can be constructed without using the theory of hypersets by proving directly that the functor Sub is representable.

Theorem 5.1 *The following is a bijection for all $D \in \mathcal{NDyn}$:*

$$\nu : \mathcal{NDyn}(D, \Omega) \to Sub(D),$$

where $\nu(f) := f^{-1}(\top)$.

Proof. Let $D_0 \subseteq D$ be a sub $\mathcal{NDyn}$. Consider the following set equation in the unknowns $\{ x_d \mid d \in D \setminus D_0 \}$

$$x_d = \begin{cases} \perp & \text{if there are no paths from } d \text{ into } D_0 \\ \{ x_e \mid d \to e \notin D_0 \} \bigcup \{ \top \mid d \to e \in D_0 \} & \text{otherwise.} \end{cases}$$

The solution lemma of AFA modified to this Ω gives us a unique solution $x_d = a_d \in \Omega$ and $f : d \mapsto a_d$ is obviously a simulation which satisfies $f^{-1}(\top) = D_0$. ∎

Now note that for each $D \in \mathcal{NDyn}$, we can define $PD := [D, \Omega]$ from the presheaf

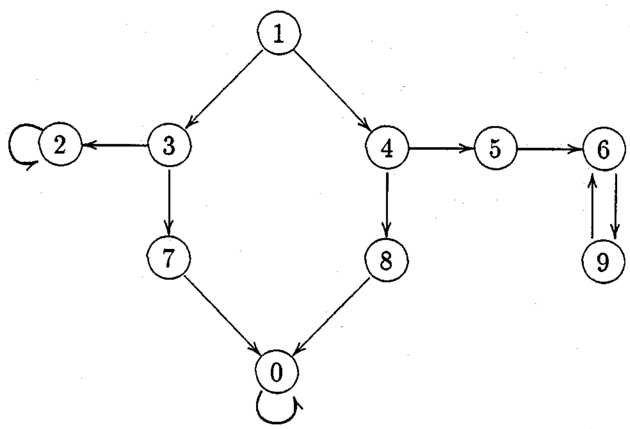
$$T \mapsto Sub(D \otimes T)$$

and we can show

$$Sub(D_1 \otimes D_2) \simeq [D_1, PD_2].$$

5.2 Example 1

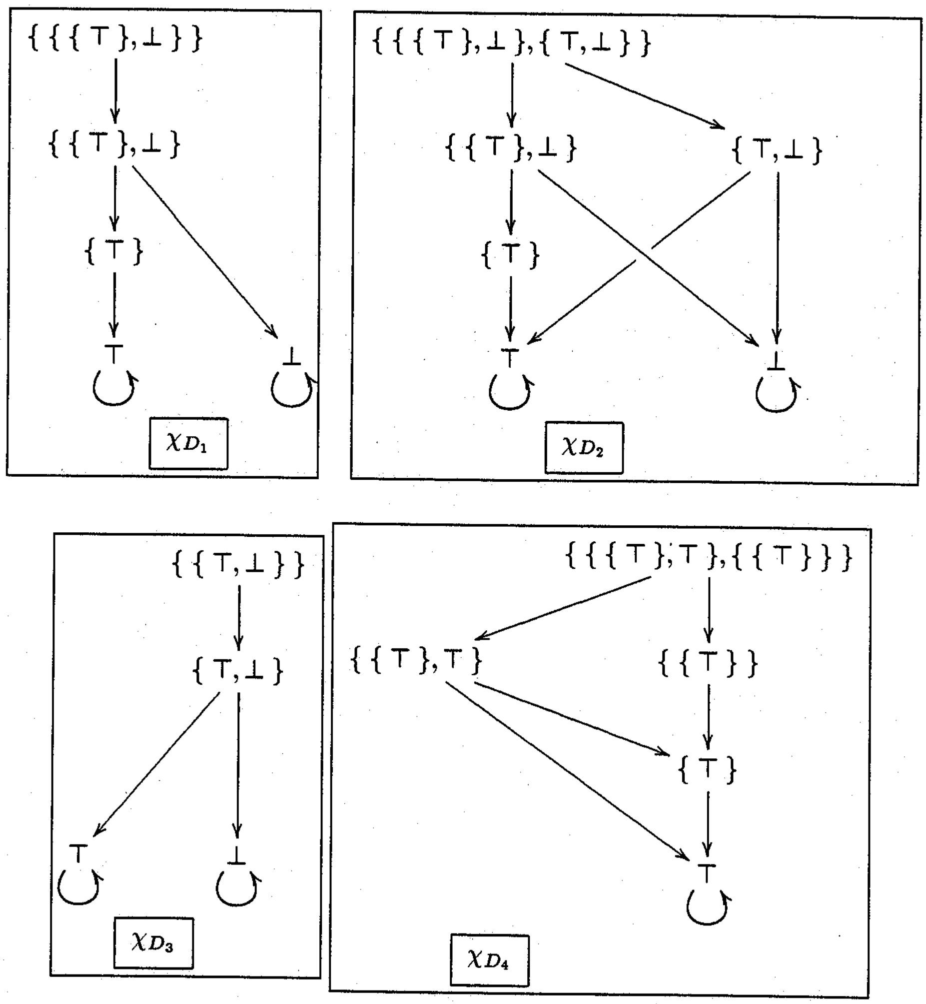
Let D be the following nondeterministic dynamical system:



Let $D_i (i = 1, 2, 3, 4)$ be the subobjects of D with the following nodes:

	nodes
D_1	0
D_2	8, 0
D_3	5, 6, 9
D_4	2, 6, 9, 0

Then the images of χ_{D_i} are as follows:



The truth values of each node with respect to these subobjects are as follows:

node	D_1	D_2
1	$\{\{\{\top\},\bot\}\}$	$\{\{\{\top\},\bot\},\{\top,\bot\}\}$
2	$\bot$	$\bot$
3	$\{\{\top\},\bot\}$	$\{\{\top\},\bot\}$
4	$\{\{\top\},\bot\}$	$\{\top,\bot\}$
5	$\bot$	$\bot$
6	$\bot$	$\bot$
7	$\{\top\}$	$\{\top\}$
8	$\{\top\}$	$\top$
9	$\bot$	$\bot$
0	$\top$	$\top$

node	D_3	D_4
1	$\{\{\top,\bot\}\}$	$\{\{\{\top\},\top\},\{\{\top\}\}\}$
2	$\bot$	$\top$
3	$\bot$	$\{\{\top\},\top\}$
4	$\{\top,\bot\}$	$\{\{\top\}\}$
5	$\top$	$\{\top\}$
6	$\top$	$\{\top\}$
7	$\bot$	$\{\top\}$
8	$\bot$	$\{\top\}$
9	$\top$	$\top$
0	$\bot$	$\top$

From this, we see that the states 3 and 4 have the same truth value with respect to the stable properties D_1 but it is not the case with the stable properties D_2, D_3, D_4.

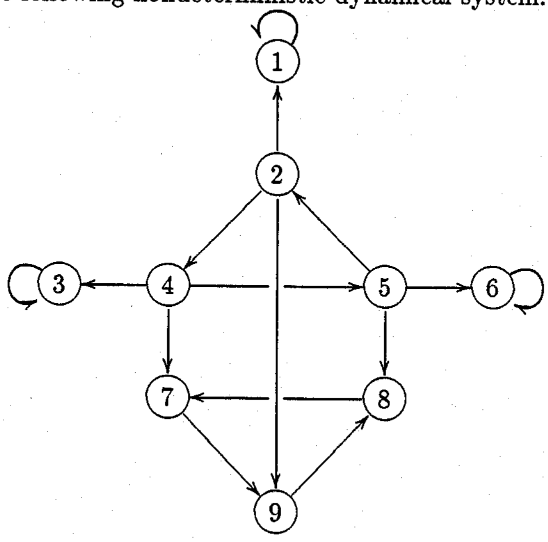
On the other hand, the state 3 has the same truth value $\{\{\top\},\bot\}$ with respect to the stable properties D_1, D_2, D_3 but has different value $\bot$ with respect to D_3.

Note that if the truth value of a state is $\bot$, then it means that the stable property will never hold from that state.

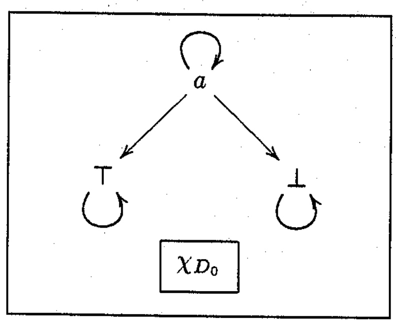
5.3 Example 2

We give now a nondeterministic dynamical system and a stable property which give truth values with non-well founded sets.

Let D be the following nondeterministic dynamical system:



Let D_0 be the subobjects whose nodes are $\{ 7, 8, 9 \}$. Then the image of its classifying map is as follows:



Here a is the hyperset defined uniquely as the solution of the following set equation:

$$a = \{ a, \top, \perp \}.$$

The truth values of states are as follows:

node	truth value
1	$\perp$
2	a
3	$\perp$
4	a
5	a
6	$\perp$
7	$\top$
8	$\top$
9	$\top$

6 Concluding Remarks and Discussions

1. The category $\mathcal{NDyn}$ behaves much like a topos except the product is replaced by a general tensor product. However the logical operations on the subobject classifier seems quite different from that of topos.

2. We note that all the nondeterministic dynamical systems are *bisimilar* to each other. In fact for any nondeterministic dynamical systems X, Y, the trivial correspondence which corresponds any states of X to any states of Y is a bisimulation. This might lead one to think that the category of simulations are meaningless from the point of view of studying processes since bisimilarity is the right notion of equality of processes from many reasons. However when we study a process with respect to stable properties, which amounts to the same thing as to consider the subprocesses, then even bilimilar processes have utterly different behaviour. Indeed only by choosing simulations as arrows, can we obtain subobject classifyier which gives natural **truth values** of stable properties at the current state.

3. Most of our results extend to general labelling sets [11]. However, in this case, the existence of terminal objects is nontrivial and its construction is based on the final coalgebra theorems for endofunctors of the set category [1, 2, 5] and the nontriviality of its structure complicates the theory considerably.

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