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A remark on the quantale structure of multisets

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abstract

The set $\mathcal{P}^*(I)$ of finite multisets on a set I has a natural quantale structure. In this note we study the quantic nuclei of this quantale, and show that they are parametrized by those subsets of the infinite multisets $\mathcal{P}^{**}(I)$ whose elements are mutually uncomparable. We also study the quantic nuclei of another natural quantale structure on $\mathcal{P}^*(I)$ with respect to the opposite order.

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1 Introduction

1.1 Quantale and quantic nucleus

We recall briefly basic notions of quantale and fix notations. We use standard notations and terminologies on lattices [2].

A quantale is a complete lattice Q endowed with an associative binary operation $\otimes$ satisfying

$$\bigvee S \otimes q = \bigvee \{s \otimes q | s \in S\},$$
$$q \otimes \bigvee S = \bigvee \{q \otimes s | s \in S\}.$$

for $S \subseteq Q$ and $q \in Q$. In this note, we consider only commutative quantales, i.e, we assume $\otimes$ is commutative. A quantale has another binary operation "$\rightarrow$" defined by

$$a \rightarrow b = \bigvee \{q | q \otimes a \leq b\}.$$

A quantic nucleus $j : Q \longrightarrow Q$ is a closure operator satisfying

$$j(a \otimes b) \leq ja \otimes jb$$

for $a, b \in Q$. Its fixed point set

$$S_j = \{q \in Q | jq = q\}$$

is meet-closed and satisfies

$$a \to s \in S_j$$

for all $a \in Q$ and $s \in S_j$.
Conversely any subset $S \subseteq Q$ satisfying these two conditions is the fixed point set of the quantic nucleus j_S defined by

$$j_S(q) = \bigwedge \{s \in S | q \leq s\},$$

[3.Proposition 3.2.1].

1.2 The quantales of multisets

In this note we study the quantale $\mathcal{P}^*(I)$ of finite multisets on a set I defined as follows.

Let $\boldsymbol{N}$ be the set of natural numbers and put $\boldsymbol{N}^* = \boldsymbol{N} \cup \{0, \infty\}$. We regard $\boldsymbol{N}^*$ as an ordered set with $0 < 1 < 2 < \cdots < \infty$.

Let I be a finite set. The set $\mathcal{P}^*(I)$ is defined by

$$\mathcal{P}^*(I) = \{(n_i)_{i \in I} | n_i \in \boldsymbol{N}^* \setminus \{\infty\}\} \cup \{\infty\}.$$

This set has a quantale structure defined by

$$\begin{cases} (n \otimes m)_i = n_i + m_i, & \text{if} \quad n, m \neq \top, \\ n \otimes m = \top, & \text{otherwise,} \end{cases}$$

and

$$n \leq m \Longleftrightarrow m = \top \text{ or } n_i \leq m_i \text{ for all } i,$$

where $\top = \infty$. This quantale is called the quantale of finite multisubsets of I.

For $S \subseteq \mathcal{P}^*(I)$

$$(\bigvee S)_i = \sup\{n_i | n \in S\}$$

if $\sup\{n_i | n \in S\}$ is finite for all $i \in I$ and

$$\bigvee S = \top$$

if $\sup\{n_i | n \in S\} = \infty$ for some i.

The meet of S is given by

$$(\bigwedge S)_i = \inf\{n_i | n \in S\}.$$

1.3 The quantale of infinite multisets on I

The product $(N^*)^I$ has a commutative quantale structure defined by

$$n \leq m \Longleftrightarrow n_i \leq m_i \qquad \text{for all } i \in I$$

and

$$(n \otimes m)_i = n_i + m_i \qquad \text{for } i \in I$$

where

$$n + \infty = \infty$$

for all $n \in N^*$. The top is the element $\top$ defined by $(\top)_i = \infty$ for all i. This quantale will be called *the quantale of infinite multisets on* I, and will be denoted by $\mathcal{P}^{**}(I)$.

Note that the inclusion $i : \mathcal{P}^*(I) \hookrightarrow \mathcal{P}^{**}(I)$ is a monoid homomorphism but does not preserve the join. For example, suppose $I = \{a, b\}$ and define $s_k (k = 1, 2, \cdots)$ by

$$(s_k)_a = k, \quad (s_k)_b = 0.$$

Then $\bigvee\{s_k\} = \top$ in $\mathcal{P}^*(I)$, but

$$(\bigvee\{s_k\})_a = \infty, \quad (\bigvee\{s_k\})_b = 0$$

in $\mathcal{P}^{**}(I)$.

2 Quantic nuclei on $\mathcal{P}^*(I)$

Since $n \to m = \bigvee\{x | x \otimes n \leq m\}$ for $n, m \in \mathcal{P}^*(I)$, we have

$$n \to m = \begin{cases} \top, & \text{if} \quad m = \top, \\ 0, & \text{if} \quad m \neq \top \text{ and } m \leq n, \\ m \ominus n, & \text{otherwise,} \end{cases}$$

where $(m \ominus n)_i = [m_i - n_i]_+$, where $[n]_+ = n$ if $n \geq 0$ and 0 if $n < 0$. From this, we can show the following proposition.

Proposition 1 *Let S be a subset of $\mathcal{P}^*(I)$. Then S is the set of the fixed points of a quantic nucleus of $\mathcal{P}^*(I)$ if and only if S contains both $\top$ and the downset $\downarrow s := \{u | u \leq s\}$ of every $s \in S \setminus \{\top\}$, namely,*

$$S \supseteq \{\top\} \cup \bigcup_{s \in S \setminus \{\top\}} \downarrow s. \tag{1}$$

Proof. For $s \in S$

$$\{q \to s | q \in \mathcal{P}^*(I)\} = \begin{cases} \downarrow s, & \text{if} \quad s \neq \top, \\ \{\top\}, & \text{if} \quad s = \top. \end{cases}$$

In fact $u = (s \ominus u) \to s$ if $u \leq s \neq \top$, and $q \to \top = \top$. From this, for the subset $S \subseteq \mathcal{P}^*(I)$ which contains $\top$,

$$\bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\} = \{\top\} \cup \bigcup_{s \in S \setminus \{\top\}} \downarrow s. \tag{2}$$

Let S be the fixed point set of a quantic nucleus. Then S is meet-closed and

$$S \supseteq \bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\}.$$

The meet-closedness of S implies S contains $\top = \bigwedge \emptyset$, and hence the equation (2) holds. Therefore, S satisfies the condition (1).

Conversely, let S be the set which satisfies the condition (1). Since S contains $\top$, the equation (2) holds, whence

$$S \supseteq \bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\}.$$

Since the inverse inclusion of condition (1) holds for any $S \subseteq \mathcal{P}^*(I)$, we obtain

$$S = \bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\}.$$

This implies S is meet closed. Therefore, S is the fixed point set of a quantic nucleus of $\mathcal{P}^*(I)$. ∎

Corollary 2 *Let S_j be the fixed point set of a quantic nucleus j of $\mathcal{P}^*(I)$. Then*

$$j(n) = \begin{cases} n, & n \in S_j, \\ \top, & \textit{otherwise}. \end{cases}$$

Proof. Recall

$$j(n) = \bigwedge \{s \in S_j | n \leq s\}.$$

But $n \leq s \quad (s \in S_j)$ implies $n \in S_j$. Hence if $n \notin S_j$ then

$$j(n) = \bigwedge \emptyset = \top.$$

∎

Let $\mathcal{D}(\mathcal{P}^{**}(I))$ be the set of those subsets of the quantale $\mathcal{P}^{**}(I)$ whose elements are mutually uncomparable, i.e., $M \in \mathcal{D}(\mathcal{P}^{**}(I))$ if and only if no two elements $u, v \in M$ satisfies either $u \leq v$ or $v \leq u$. To prove the set $M \subseteq \mathcal{D}(\mathcal{P}^{**}(I))$ is finite, we show next lemma.

Lemma 3 *There is no infinite sequence* $x_1, x_2, \cdots \in (\boldsymbol{N}^*)^n$ *satisfying*

$$i < j \Longrightarrow x_i \not\leq x_j, \qquad \textit{for all } i, j \in \boldsymbol{N}^*.$$

Proof. Suppose $n = 1$. Because $x \not\leq y$ if and only if $x > y$ for $n = 1$, there is no infinite sequence $x_1, x_2, \cdots \in \boldsymbol{N}^*$ satisfying

$$i < j \Longrightarrow x_i > x_j, \qquad \text{for all } i, j \in \boldsymbol{N}^*.$$

Now we assume there is no infinite sequence $x_1, x_2, \cdots \in (\boldsymbol{N}^*)^n$ satisfying

$$i < j \Longrightarrow x_i \not\leq x_j, \qquad \text{for all } i, j \in \boldsymbol{N}^*$$

for $n \leq k$. Suppose there is an infinite sequence $x_1, x_2, \cdots \in (\boldsymbol{N}^*)^{k+1}$ satisfying

$$i < j \Longrightarrow x_i \not\leq x_j, \qquad \text{for all } i, j \in \boldsymbol{N}^*.$$

Write $x_j \in (\boldsymbol{N}^*)^{k+1}$ as

$$x_j = (x'_j, p_j), \text{ with } x`_j \in (\boldsymbol{N}^*)^k \text{ and } p_i \in \boldsymbol{N}^*.$$

Put $I = \{p_j | j = 1, 2, \cdots\}$.
If I is finite, then there is a $p \in I$ with $J = \{j | p_j = p\}$ is infinite. Let $J = \{j_1 < j_2 < \cdots\}$ and put $y_i = x'_{j_i}$. Then the infinite sequense $y_1, y_2, \cdots \in (\boldsymbol{N}^*)^k$ satisfies

$$i < j \Longrightarrow x_i \not\leq x_j, \qquad \text{for all } i, j \in \boldsymbol{N}^*.$$

This contradicts the assumption.
If I is infinite, then there is an infinite sequence

$$j_1 < j_2 < \cdots$$

with

$$p_{j_1} < p_{j_2} < \cdots.$$

Put $y_1 = x'_{j_i}$, then the infinite sequence $y_1, y_2, \cdots \in (\boldsymbol{N}^*)^k$ satisfies

$$i < j \Longrightarrow x_i \not\leq x_j, \qquad \text{for all } i, j \in \boldsymbol{N}^*.$$

This contradicts the assumption. ∎

Proposition 4 *The set* $M \subseteq \mathcal{D}(\mathcal{P}^{**}(I))$ *is finite.*

Proof. Suppose M is infinite. Then there is an infinite sequence $x_1, x_2, \cdots \in M$, which satisfies

$$i < j \Longrightarrow x_i \not\leq x_j, \qquad \text{for all } i, j \in \boldsymbol{N}^*$$

since all elements of M are mutually uncomparable. This contradicts the conclusion of Lemma 3. Hence M must be finite. ∎

Because for $M \in \mathcal{D}(\mathcal{P}^{**}(I))$

$$S_M = (\{\top\} \cup \bigcup_{m \in M} \downarrow m) \cap \mathcal{P}^*(I)$$

satisfies the condition of the Proposition 1, we can define the quantic nucleus j_M by

$$j_M(n) = \begin{cases} n, & \text{if} \quad n \in S_M, \\ \top, & \text{otherwise.} \end{cases}$$

Theorem 5 *let $\mathcal{N}(\mathcal{P}^*(I))$ be the set of all the quantic nucleus of $\mathcal{P}^*(I)$. Then the map*

$$\pi : \mathcal{D}(\mathcal{P}^{**}(I)) \longrightarrow \mathcal{N}(\mathcal{P}^*(I)),$$

which sends $M \in \mathcal{D}(\mathcal{P}^{**}(I))$ to the quantic nucleus j_M is a bijection.

Proof. For the subset $S \subseteq \mathcal{P}^*(I)$ we define $\bar{S} \subseteq \mathcal{P}^{**}(I)$ as the set of

$$\bar{S} = \{\bigvee i(A) | A \subseteq S \setminus \{\top\},\ A \text{ is directed}\}$$

where $i : \mathcal{P}^*(I) \longrightarrow \mathcal{P}^{**}(I)$ is the inclusion map (cf §1.3). Let

$$\mathcal{M}(S) = \{m | m \text{ is the maximal element of } S\}.$$

Let S_j be the fixed point set of a quantic nucleus j. From the above definition,

$$S_{\mathcal{M}(S_j)} = (\{\top\} \cup \bigcup_{m \in \mathcal{M}(S_j)} \downarrow m) \cap \mathcal{P}^*(I) = \{\top\} \cup \bigcup_{s \in S_j \setminus \{\top\}} \downarrow s = S_j.$$

This implies

$$\pi(\mathcal{M}(\bar{S}_j)) = j,$$

hence, π is surjection.

On the other hand,

$$\mathcal{M}(\bar{S}_M) = M,$$

In fact, we have $M \subseteq \mathcal{M}(\bar{S}_M)$ since $m = \bigvee(\downarrow m \cup \mathcal{P}^*(I))$.
We show now $M \supseteq \mathcal{M}(\bar{S}_M)$. For any $n \in \bar{S}_M$ there is a directed set A such that

$$n = \bigvee i(A)$$

where $i : \mathcal{P}^*(I) \longrightarrow \mathcal{P}^{**}(I)$ is the inclusion map. We show that there is an upperbound of A in M. Suppose that for any $m \in M$ there is an $a_m \in A$ such that $a_m \not\leq m$. Since M is finite and A is directed, there is $a \in A$ such that for all $m \in M$, $a_m \leq a$. Because every $m \in M$ is maximal element of $\bar{S}_M$, we have there is a $m' \in M$ such that $a < m'$. Fron these $a_m \leq m'$, and this is contradiction. Therefore, for all $a \in A$ there is $m \in M$ such that m is comparable with a, and so $a \leq m$.

This implies π is injection. In fact, $S_M = S'_M$ implies $\mathcal{M}(\bar{S}_M) = \mathcal{M}(\bar{S}'_M)$, so $M = M'$. ∎

3 The opposite order

The set $\mathcal{P}^*(I)$ has another quantale structure with the same binary product $\otimes$ and with the opposite order $\preceq$ defined by

$$n \preceq m \iff n = \perp \text{ or } n_i \geq m_i \text{ for all } i$$

where we write ∞ by $\perp$ now. This type of quantales arises as the ideal quantales of the principal ideal domains [1]. The binary operator $\rightarrow$ is the same as before but can be written now as

$$n \rightarrow m = \begin{cases} \perp, & \text{if} \quad m = \perp,\ n \neq \perp, \\ 0, & \text{if} \quad m = n = \perp \text{ or } n \preceq m \neq \perp, \\ m \ominus n, & \text{otherwise.} \end{cases}$$

This implies as before the following proposition.

Proposition 6 *Let S be a subset of $\mathcal{P}^*(I)$. Then S is the fixed point set of a quantic nucleus of $\mathcal{P}^*(I)$ if and only if S contains the all upsets $\uparrow s := \{u | u \succeq s\}$ of $s \in S \setminus \{\perp\}$, namely,*

$$S \supseteq \bigcup_{s \in S \setminus \{\perp\}} \uparrow s,$$

and S is meet-closed.

Proof. Because

$$\{q \rightarrow s | q \in \mathcal{P}^*(I)\} = \begin{cases} \uparrow s, & \text{if} \quad s \neq \perp, \\ \{0, \perp\}, & \text{if} \quad s = \perp, \end{cases}$$

we have

$$\bigcup_{s \in S} \{q \rightarrow s | q \in \mathcal{P}^*(I)\} = \begin{cases} \displaystyle\bigcup_{s \in S \setminus \{\perp\}} \uparrow s, & \text{if} \quad \perp \notin S, \\ \{\perp\} \cup \displaystyle\bigcup_{s \in S \setminus \{\perp\}} \uparrow s, & \text{if} \quad \perp \in S. \end{cases}$$

From this,

$$S \supseteq \bigcup_{s \in S \setminus \{\perp\}} \uparrow s \iff S \supseteq \bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\}. \tag{3}$$

Therefore,

$$S \supseteq \bigcup_{s \in S \setminus \{\perp\}} \uparrow s \quad \text{and } S \text{ is meet-closed,}$$

if and only if

$$S \supseteq \bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\} \quad \text{and } S \text{ is meet-closed,}$$

that is, S is the set of the fixed points of a quantic nucleus of $\mathcal{P}^*(I)$. ∎

Note that the set S of the fixed points of a quantic nucleus is written by

$$S = \begin{cases} \displaystyle\bigcup_{s \in S \setminus \{\perp\}} \uparrow s, & \text{if } \perp \notin S, \\ \displaystyle\{\perp\} \cup \bigcup_{s \in S \setminus \{\perp\}} \uparrow s, & \text{if } \perp \in S. \end{cases}$$

since the inverse inclusion of the right hand side of (3), i.e.,

$$S \supseteq \bigcup_{s \in S} \{q \to s | q \in \mathcal{P}^*(I)\},$$

holds for any $S \subseteq \mathcal{P}^*(I)$.

There are two types of sets which satisfy the conditions of the proposition 3. The one is

$$S^1_m = (\{\perp\} \cup \uparrow m) \cap \mathcal{P}^*(I),$$

for $m \in \mathcal{P}^{**}(I)$. We can define the quantic nucleus j^1_m with the fixed point set S^1_m by

$$j^1_m(n) = \begin{cases} n \vee m, & \text{if } n \neq \perp, \\ \perp, & \text{if } n = \perp, \end{cases}$$

for $n \in \mathcal{P}^*(I)$. The other one is

$$S^2_m = \uparrow m,$$

for $m \in \mathcal{P}^*(I)$. We can define the quantic nucleus j^2_m with the fixed point set S^2_m by

$$j^2_m(n) = n \vee m,$$

for $n \in \mathcal{P}^*(I)$.

Lemma 7 *Define the maps*

$$\xi_1 : \mathcal{P}^{**}(I) \longrightarrow \mathcal{N}(\mathcal{P}^*(I)), \quad \xi_2 : \mathcal{P}^*(I) \longrightarrow \mathcal{N}(\mathcal{P}^*(I))$$

by

$$\xi_1(m) = j^1_m, \quad \xi_2(m) = j^2_m.$$

Then these maps are injection.

Proof. Let $m \neq m'$, then

$$S^1_m \neq S^1_{m'}, \qquad \textit{and} \qquad S^2_m \neq S^2_{m'}.$$

∎

Theorem 8 *Then the map*

$$\eta : \mathcal{P}^{**}(I) \amalg (\mathcal{P}^*(I) \setminus \{\perp\}) \longrightarrow \mathcal{N}(\mathcal{P}^*(I))$$

defined by

$$\eta = \xi_1 \amalg \xi_2$$

is a bijection.

Proof. Let S_j be the fixed point set of a quantic nucleus j, and $m = \bigwedge_{s \in S_j}(S_j \setminus \{\perp\})$.
If $\perp \notin S_j$, $m \in \mathcal{P}^*(I)$. Then

$$S_j = \bigcup_{s \in S_j \setminus \{\perp\}} \uparrow s = \uparrow m \quad = S^2_m.$$

If $\perp \in S_j$, $m \in \mathcal{P}^{**}(I)$. Then

$$S_j = \{\perp\} \cup \bigcup_{s \in S_j \setminus \{\perp\}} \uparrow s = (\{\perp\} \cup \uparrow m) \cap \mathcal{P}^*(I) \quad = S^1_m.$$

Hence, η is a surjection. Since ξ_1, ξ_2 are injection, η is a injection too. ∎

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