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K. Hirata and K. Sugano

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Kazuhiko Hirata and Kozo Sugano

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Throughout this paper A is always a ring with identity 1, and B is a subring of A containing 1. In their previous paper [4] the authors introduced the notion of the semisimple extension of a ring. A ring A is called to be a left semisimple extension of B in the case where every left A-module M is (A,B)-projective, that is, the map π of $A\otimes_B M$ to M, defined by $\pi(a\otimes m) = am$ for any $a \in A$ and $m \in M$, splits as left A-homomorphism, or equivalently, for every left A-module M, every A-submosule N which is a B-direct summand of M is always an A-direct summand. (See Theorem 1.1 [4]). The right semisimple extension is similarly defined, and both left and right semisimple extension is called semisimple extension. Till now some trivial examples of the semisimple extension are known, for example, each semisimple ring is a semisimple extension of each subring if it, and each separable extension is a semisimple extension. However, since the semisimplicity is a quite abstract condition, it is very dificult to research the structure of the semisimple extension or find proper examples of it.

In this paper we will give some structure theorem of semisimple extensions of (two-sided) uni-serial local rings. It is already a well known fact that, if a ring R is serial, R satisfies the following two conditions;

(1) Each left R-module is a direct sum of indecomposable submodules

(2) A left R-module is indecomposable if and only if it is a homomorphic image of some Re, where e is a primitive idempotent of R.

If R satisfies the condition (1), the decomposition of each module into a direct sum of indecomposable submodule is unique up to isomorphism by Corollary 2 to Theorem A [5] and Theorem 12.4 [1]. Also it can be easily proved that, under the condition (2), for each primitive idempotent e of R Re has the unique maximal left subideal, and each epimorphism of Re to M is a projective cover of a indecomposable left R-module M. Therefore each left R-module has the projective cover in case R satisfies the both conditions. Under these preparations we have;

Theorem 1. Let both A and B satisfy the above conditions (1) and (2), and suppose that A is a left semisimple extension of B. Then for each left ideal I of A and each primitive idempotent e of A, there exist a left ideal l of B and a primitive idempotent e' of R such that there is an R-isomorphism of Re to Re' whose restricion on Ie is an isomorphism of Ie to Ale'.

Proof. Suppose that A is a left semisimple extension of B and let I and e be as in the theorem. Then Ae/Ie is A-indecomposable and Ae/Ie is an A-direct summand of $A\otimes_B Ae/Ie$. On the other hand B satisfies the same condition. Therefore there exist classes $\{m_\alpha\}$ of left ideals of B and $\{f_\alpha\}$ of primitive idempotents of B such that $Ae/Ie = \Sigma\oplus Bf_\alpha/m_\alpha f_\alpha$ as B-module. Then we have $A\otimes_B Ae/Ie = \Sigma\oplus A\otimes_B Bf_\alpha/m_\alpha f_\alpha = \Sigma\oplus Af_\alpha/Am_\alpha f_\alpha$. Write $f = f_\alpha$ and $m = m_\alpha$ for a fixed α, and let $Af/Amf = M_1 \oplus M_2$ be a decomposition of Af/Amf with M_1 indecomposable. As is stated above we have the projective covers p_i: $P_i \rightarrow M_i$ $(i = 1,2)$ and the following commutative diagram

$$\begin{array}{ccc} Af & \xrightarrow{\mu} & M_1 \oplus M_2 \\ \rho \downarrow & \downarrow p & \| \\ P_1 \oplus P_2 & \longrightarrow & M_1 \oplus M_2 \end{array}$$

where $p = p_1 + p_2$ is the projective cover of $M_1 \oplus M_2$ and μ is the canonical epimorphism, ρ is an epimorphism such that $p\,\rho = \mu$. Then there exists a monomorphism λ of P to Af such that $\rho\,\lambda =$ identity on P where $P = P_1 \oplus P_2$. Write $\bar{P}_i = \lambda(P_i)$ for each i. Clearly we have $p = \mu\,\lambda$ and $\mu(\bar{P}_i) = p(P_i) = M_i$ for each i. We have also $Af = \bar{P}_1 \oplus \bar{P}_2 \oplus \mathrm{Ker}\,\rho$, and $\mu(\ker\rho) = 0$. Hence we have $\mu(\bar{P}_2 \oplus \mathrm{Ker}\,\rho) = M_2$. Now there exist mutually orthogonal idempotents e' and e" of A such that f = e' + e", $\bar{P}_2 \oplus \mathrm{Ker}\,\rho = Ae''$ and $\bar{P}_1 = Ae'$. Then by the above urguments we have the following commutative diagram, where all rows and cololums are exact

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Ae'' & \xrightarrow{\mu''} & M_2 & \rightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & Amf & \rightarrow & Af & \xrightarrow{\mu} & M_1 \oplus M_2 & \rightarrow & 0 \\ & & & & \pi \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathrm{Ker}\,\mu' & \rightarrow & Ae' & \xrightarrow{\mu'} & M_1 & \rightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

In the above diagram μ' and μ'' are the restrictions of μ on Ae' and Ae" respectively, and π is the projection of Af (= Ae'$\oplus$Ae") to Ae', which is given by the right multiplication of e'. Then the exactness of the above commutative diagram yields the epimorphism of Amf to Ker μ', which is the restriction of π to Amf. Hence we have Ker μ' = Amfe', and M_1 = Ae'/Amfe'. Thus we have shown that each indecomposable direct summand of Af/Amf is of the form Ae'/Ale', with l (= mf) a left ideal of B and e' a primitive idempotent of A. This fact together with the condition (1) shows that $Af_\alpha/Am_\alpha f_\alpha = \Sigma\oplus Ae_{\alpha i}/Al_{\alpha i}e_{\alpha i}$ for some left ideals $l_{\alpha i}$ of B and primitive idempotents $e_{\alpha i}$ of A, and consequently

$$Ae/Ie <\oplus A\otimes_B Ae/Ie = \Sigma\oplus A\otimes_B Bf_\alpha/m_\alpha f_\alpha = \Sigma\oplus Af_\alpha/Am_\alpha f_\alpha = \Sigma\oplus Ae_{\alpha i}/Al_{\alpha i}e_{\alpha i}$$

Then by the uniqueness of the decompositon , we have $Ae/Ie = Ae_{\alpha i}/Al_{\alpha i}e_{\alpha i}$ for some α and i. Since as is stated above the canonical maps

$$Ae \rightarrow Ae/Ie \quad \text{and} \quad Ae_{\alpha i} \rightarrow Ae_{\alpha i}/Al_\alpha e_{\alpha i}$$

are projective covers respectively, there exists an isomorphism ϕ of Ae to $Ae_{\alpha i}$ such that

$$\begin{array}{ccc} Ae & \longrightarrow & Ae_{\alpha i} \\ \downarrow & & \downarrow \\ Ae/Ie & = & Ae_{\alpha i}/Al_\alpha e_{\alpha i} \end{array}$$

is commutative. Obviously we have $\phi(Ie) = Al_\alpha e_{\alpha i}$. Thus we have proved the theorem..

Now we will apply Theorem 1 to two cases. One is the case where B is a commutative local serial ring and A is a B-algebra, the other is the case where both A and B are local serial rings. In either case the converse of Theorem 1 are true.

Proposition 1. Let B be a commutative ring and A a B-algebra, and suppose that both A and B satisfy the conditions (1) and (2). Then A is a left semisimple extension of B if and only if A and B satisfy the condition of Theorem 1.

Proof. The 'only if' part is due to Theorem 1. In order to prove the converse we need only to prove that each indecomposable left A-module is (A,B)-projective. But this is almost clear, since each indecomposable left A-mosule is the form Ae/Ale for some primitive idempotent e of A and an ideal l of B, and is isomorphic to $Ae\otimes_B B/l$. The latter is (A,B)-projective, since it is an A-direct summand of $A\otimes_B B/l$.

In what follows we denote the radicals of A and B by N and n, respectively.

Theorem 2. Let B be a commutative local serial ring, and A a B-algebra which is finitely generated as B-module. If A is a left semisimple extension of B, A is a (twe-sided) serial ring. In addition the length of the composition series of Ae coinsides with that of B for each primitive idempotent e of A.

Proof. By the assumption we see that A is artinian, An is a nilpotent ideal of A, and consequently contained in N. Since B is local, we have $An \cap B = n$. Then A/An is a left semisimple extension of a field B/n, and A/An is a semisimple ring (See Propposition 1.2 and Corollary 1.7 [4]). Then we have $N = An$ and $N^i = An^i$ for each number i. Let r be such that $n^r = 0$ and $n^{r-1} \neq 0$, and e a primitive idempotent of A. Since A is artinian, Ae/Ne is simple. It is obvious that $Ae/Ne = Ae/Ane = B/n\otimes_B Ae$. On the other hand we have $B/n = n^i/n^{i+1}$ for each $1 \leqq i \leqq r-1$. Therefore each $n^i/n^{i+1}\otimes_B Ae$ is also simple. Now consider the following commutaive diagram

$$\begin{array}{ccccccc}
n^{i+1}\otimes_B Ae & \rightarrow & n^i\otimes_B Ae & \rightarrow & n^i/n^{i+1}\otimes_B Ae & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
n^{i+1}Ae & \rightarrow & n^iAe & \rightarrow & n^iAe/n^{i+1}Ae & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where all rows and cololums are exact. Then there exists an epimorphism

$$n^i/n^{i+1}\otimes_B Ae \quad \rightarrow \quad n^iAe/n^{i+1}Ae \quad \rightarrow \quad 0$$

which means that each $n^iAe/n^{i+1}Ae = N^ie/N^{i+1}e$ is simple. Thus we see that

$$0 = N^re < N^{r-1}e < \cdots \quad < Ne < N^0e = Ae$$

is a composition series. Now assume that there exist left A-submodules of Ae other than $\{N^ie \mid 0 \leqq i \leqq r\}$. Then there exists a maximal member I among them, and the smallest number k such that $I \subset N^ke$. If $N^{k+1}e \subset I \subset N^ke$, it contradicts to the fact that $N^ke/N^{k+1}e$ is simple. Hence we have $N^{k+1}e \subsetneq I$. Then we have $I + N^{k+1}e = N^ke$, since I is a maximal submodule of N^ke. But this is a contradiction, since we have $I = N^ke$ by Nakayama's lemma. Therefore $\{N^ie\}$ are the only submodules of Ae, and the above composition series is the unique composition series of Ae. Thus A is a left serial ring, and similarly it is right serial. We have also shown that the composition serieses of Ae and B are of the same length.

Remark. Note that in the proof of Theorem 2 we used only the conditions that A is artinian and N = An. Therefore Theorem 2 holds under a weaker condition that A is a semisimple B-algebra in the sence of Hattori [3].

By Theorem 1 and a part of the proof of Theorem 2 we have

Theorem 3. Let B be a commutative local serial ring and assume that A is a serial B-algrbra. Then A is a left semisimple extension of B if and only if N = nA

Proof. Assume N = An, and let I be a left ideal of A and e a primitive idempotent of A. Since A is left serial, $0 = N^r e < N^{r-1}e < \cdots < Ne < N^0 e = Re$ is the unique composition series of Re. Hence we have $Le = N^i e$ for some i, and $Le = An^i e$. Then by Proposition 1 A is a left semisimple extension of B. The converse can be prooved by the same way as Theorem 2.

Now we will consider the case where B is noncommutative. Assume again that A and B satisfy the conditions (1) and (2). If furthermore A has no nonzero idempotent except for 1, then each indecomposable left A-module (resp. B-module) is isomorphic to A/I (resp. B/l) for some left ideal I of A (resp. l of B). Then the same methods as the proofs of Theorem 1 and Proposition 1 can be applied to A and B. In addition each left A-isomorphism of A to A is given by the right multiplication of some unit element of A. Therefore we have;

Theorem 4. Let A and B satisfy the conditions (1) and (2), and assume A has no idempotent except for 1 and 0. Then A is a left semisimple extension of B if and only if for each left ideal I of A there exist a left ideal l of B and a unit u of A such that Iu = Al.

Note that the condition of Theorem 4 is satisfied in the case where A and B are local serial rings. In this case also the same results as Theorems 2 and 3 hold as follows;

Theorem 5. Let A and B be local serial rings. Then the following conditions are equivalent;

(i) A is a left semisimple extension of B

(ii) N = An

(iii) The lengths of the composition serieses of the left A-module A and the left B-module B are same.

(vi) A is a right semisimple extension of B

Proof. First we will show $n \subseteq N$. Suppose $n \subsetneq N$. Then $nA \subsetneq N$, and we have nA = A, since A is local. Then $A = nA = n^2A = \cdots\cdots = n^rA = 0$, which is a contradiction. Thus we have $n \subseteq N$. Now assume (i). Then by Theorem 4 there exist a number $i \geqq 1$ and a unit u of A such that $N = Nu = An^i$, while we have $An^i \subseteq An \subseteq N$, which imply (ii). Conversely asssume (ii). Then we have $N^i = An^i$ for each i. Then again by Theorem 4 we have (i) since $\{N^i\}$ is the set of all left ideals of A. We have allso (iii), since $N^r = 0$ if and only if $n^r = 0$. Lastly assume (iii). Then $N^r = n^r = 0$ and $N^{r-1} \neq 0 \neq n^{r-1}$. If $N^k = An$ for r>k>1, $N^{k(r-1)} = An^{r-1} \neq 0$. But in this case we have $k(r-1) = rk - k \geqq r + r - k > r$, and $N^{k(r-1)} = 0$, a contradiction. Thus we have (ii). Since the condition (iii) is left and right symmetry for (two-sided) local serial rings A and B, the cnditions (i) to (iii) are equivalent to (vi).

Theorem 6. Let B be a local serial ring and A a local ring, and assume that A is finitely generated as left B-module. Then if A is a left semisimple extension of B, A is a left serial ring.

Proof. By the assumption A is left artinian and A/N is an artinian left B-module. Hence A/N is a finite direct sum of indecomposable modules, and we can write $A/N = \Sigma \oplus B/n^{\alpha_i}$ (finite) with $\alpha_i \geqq 1$. Then since A is left semisimple over B, we have $A/N <\oplus A\otimes_B A/N = \Sigma \oplus A\otimes_B B/n^{\alpha_i}$

$= \Sigma\oplus A/An^{\alpha_i}$ as left A-module. On the other hand as is shown above we have $n \subseteq N$. Then each A/An^{α_i} is A-indecomposable and artinian, since $An^{\alpha_i} \subseteq N$ and A is local left artinian. Then we can apply Krull-Remak-Schmidt-Azumaya's Theorem to obtain $A/N = A/An^{\alpha_i}$ for some α_i. Now by comparing the lenghts of composition serieses we have $N = An^{\alpha_i}$. But B is a both left and right principal ideal ring. Hence N is principal as left ideal. But N is the uniqe prime ideal of A, since N is nilpotent and the unique maximal ideal of A. Then A is a left serial ring. (See for example Theorem 2. 51 [2]).

Finally we will treat with the H-separable extension of a uni-serial ring which is indecomposable as ring. A left uni-serial ring is defined to be a finite direct sum of primary left serial rings. It is also well known that a ring is a primary left serial ring if and only if it is a matrix ring over a local left serial ring.

Let B be a primary serial ring and r be such that $n^r = 0$ and $n^{r-1} \neq 0$. Then by the condition (1) we can write $A \cong \Sigma\oplus Bf/n^{r_i}f$ with a fixed primitive idempotent f of B. If $r_i \leqq r\text{-}1$ for each r_i, then we have $n^{r-1}A = \Sigma\oplus n^{r-1}f/n^{r_i}f = 0$, which is a contradiction, since A is faithful as B-module. Therefore we have $r_i = r$ for some i, and $Bf <\oplus A$ as left B-module. Then $B \cong \Sigma\oplus Bf <\oplus \Sigma\oplus A$, and we see that A is a left B-generator, which implies that B is a left B-direct summand of A. Similarly B is a right B-direct summand of A.

Proposition 2. Let B be a primary serial ring and A an H-separable extension of B. Then A is also a primary serial ring. In addition A is finitely generated as left (as well as right) B-module, and B is a left (as well as right) B-direct summand of A

Proof. As is stated above B is a left (resp. right) B-direct summand of A. Hence A is right (resp. left) B-finitely generated by (2) Proposition (2. 2) [7]. Hence A is an aritinian ring. Now for any prime ideal P of A we have $P = A(P \cap B)$ by (3) Proposition (2. 2) [7]. But $P \cap B$ is principal as left ideal of B. Hence P is also left principal. Thus all prime ideals of A are principal as left ideal, and A is semiprimary. Therefore A is left uni-serial (See e.g., Theorem 2. 51 [2]). Similarly A is right uni-serial. In addition since $B <\oplus A$ as left B-module, B coinsides with $V_B(V_B(B))$, the double centralizer of B in A by Proposition 1.2 [6]. Hence the center of A is contained in the center of B, which has no idempotent except for 0 and 1. Therefore A has no central idempotent except for 0 and 1, and A is indecomposable as ring.

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Kozo Sugano
Department of Mathematics
Hokkaido University
Sapporo 060, Japan

Kazuhiko Hirata
Department of Mathematics
Chiba University
Chiba 263, Japan