A simple proof for monotonicity of entropy in the quadratic family

Masato Tsujii

Series #400. January 1998

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#375 L. Solomon and H. Terao, The double coxeter arrangement, 21 pages. 1997.

#376 G. Ishikawa and T. Morimoto, Solution surfaces of Monge-Ampère equations, 15 pages. 1997.

#377 G. Ishikawa, A relative transversality theorem and its applications, 16 pages. 1997.

#378 J. Inoue and T. Nakazi, On the zeroes of solutions of an extremal problem in H^1, 14 pages. 1997.

#379 Y. Giga and K. Ito, On pinching of curves moved by surface diffusion, 12 pages. 1997.

#380 F. Hiroshima, Weak coupling limit removing an ultraviolet cut-off for a Hamiltonian of particles interacting with a scalar field, 39 pages. 1997.

#381 Y. Giga, S. Matsui and Y. Shimizu, On estimates in Hardy spaces for the Stokes flow in a half space, 13 pages. 1997.

#382 A. Arai, A new estimate for the ground state energy of Schrödinger operators, 12 pages. 1997.

#383 M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order, 24 pages. 1997.

#384 K. Ito, Asymptotic stability of planar rarefaction wave for scalar viscous conservation law, 8 pages. 1997.

#385 A. Arai, Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: canonical commutation relations, quantum algebras, and reduction to lattice quantum systems, 32 pages. 1997.

#386 P. Aviles and Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields, 21 pages. 1997.

#387 T. Nakazi and T. Yamamoto, Norms of some singular integral operators and their inverse operators, 28 pages. 1997.

#388 M.-H. Giga and Y. Giga, Remarks on convergence of evolving graphs by nonlocal curvature, 18 pages. 1997.

#389 T. Tsukada, Reticular Lagrangian singularities, 41 pages. 1997.

#390 M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space, 12 pages. 1997.

#391 Y. Giga, M. Ohnuma and M.-H. Sato, On strong maximum principle and large time behaviour of generalized mean curvature flow with the Neumann boundary condition, 24 pages. 1997.

#392 T. Tsujishita and H. Watanabe, Monoidal closedness of the category of simulations, 24 pages. 1997.

#393 T. Arase, A remark on the quantale structure of multisets, 10 pages. 1997.

#394 N. H. Bingham and A. Inoue, Extension of the Drasin-Shea-Jordan theorem, 16 pages. 1997.

#395 N. H. Bingham and A. Inoue, Ratio Mercerian theorems with applications to Hankel and Fourier transforms, 30 pages. 1997.

#396 Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, 27 pages. 1997.

#397 K. Hirata and K. Sugano, On semisimple extensions of serial rings, 6 pages. 1997.

#398 D. Pei, Singularities of $\mathbb{R}P^2$-valued Gauss maps of surfaces in Minkowski 3-space, 15 pages. 1997.

#399 T. Mikami, Markov marginal problems and their applications to Markov optimal control, 28 pages. 1997

A simple proof for monotonicity of entropy in the quadratic family

Masato TSUJII

December 22, 1997

Abstract

We give a simple proof for monotonicity of topological entropy in the quadratic family. We use a spectral property of a Ruelle operator.

1 Introduction

In this paper, we give a simple proof of the following theorem due to Sullivan, Douady-Hubbard and Milnor ([6]) for the dynamics of real quadaratic maps $Q_t(x) = t - x^2$, $\mathbf{R} \to \mathbf{R}$.

Theorem 1 *The topological entropy $h_{top}(Q_t)$ of Q_t is monotone increasing with respect to the parameter t.*

First we reduce theorem 1 to the following claim by the kneading theory.

Theorem 2 *If the orbit of the critical point 0 is periodic with prime period n for Q_t (i.e. $Q_t^n(0) = 0$ while $Q_t^i(0) \neq 0$ for $0 < i < n$), then*

$$\frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))} > 0$$

where D denote the derivative w.r.t. the space coordinate x.

Then we prove this claim by making use of a spectral property of a Ruelle operator. We also show a similar inequality for Collet-Eckmann case.

Theorem 3 *If Q_t satisfies Collet-Eckmann condition, that is,*

$$\liminf_{n\to\infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > 1, \tag{1}$$

then we have

$$\lim_{n\to\infty} \frac{\frac{\partial}{\partial s}\{Q_s^n(0)\}|_{s=t}}{DQ_t^{n-1}(Q_t(0))} > 0. \tag{2}$$

This, together with a version of Jakobson's theorem in [8], yields

Corollary 4 *If $1/4 < t < 2$ is a parameter for which Q_t satisfies (1) and*

$$\lim_{n\to\infty} n^{-1}\log|Q_t^n(0)| = 0, \qquad (3)$$

then t is a density point of the set $\mathcal{S}$ of parameters s for which Q_s satisfies (1) and (3). Moreover,

$$\lim_{\epsilon\to+0}\frac{\log \mathrm{Leb}([t-\epsilon, t+\epsilon]\backslash\mathcal{S})}{\log\epsilon} = 2$$

where Leb *denotes the Lebesgue measure.*

Another aim of this paper is to clearify the correspondance between monotonicity of quadaratic family and that of the family of tent maps $T_t(x) = 1 - t|x|$. As we shall show in the last section, we can prove the monotonicity for the family of tent maps by quite analogous manner.

Acknowledgement: The author thanks Duncan Sands for his suggesting theorem 3.

2 Proof of therorem 1 and 2

Let us remember some definitions from the kneading theory[1, 6]. Let $f : \mathbf{R} \to \mathbf{R}$ be a unimodal map with 0 its turning point, *i.e.*, a continuous map that is strictly monotone increasing on the left of 0 and strictly monotone decreasing on the right of 0. The kneading invariant of f is an infinite sequence $K(f) = (e_1, e_2, \cdots)$ of symbols L, C and R, defined by

$$e_k = \begin{cases} L, & \text{if } f^k(0) < 0; \\ C, & \text{if } f^k(0) = 0; \\ R, & \text{if } f^k(0) > 0, \end{cases}$$

For sequences of symbols L, C and R, we define the so-called signed lexicographical order $\prec$ in the following manner: for three sequences whose first n-entries are same,

$$\begin{aligned} I_L &= (e_1, e_2, \cdots, e_n, L, \cdots), \\ I_C &= (e_1, e_2, \cdots, e_n, C, \cdots), \\ I_R &= (e_1, e_2, \cdots, e_n, R, \cdots), \end{aligned}$$

we define $I_L \prec I_C \prec I_R$ if the number of the symbol R in $\{e_1, e_2, \cdots, e_n\}$ is even, and define $I_R \prec I_C \prec I_L$ otherwise. The topological entropy of a unimodal map depends only on its kneading invariant, and the dependence is monotone with respect to the order above. Thus theorem 1 follows from

Theorem 5 *$K(Q_t)$ is monotone increasing with respect to t.*

But this follows from theorem 2. In fact, let $K_n(Q_t)$ be the truncated sequence of the kneading invariant $K(Q_t)$ of length n. Then $K_n(Q_t)$ is locally constant with respect to t unless $Q_t^i(0) = 0$ for some $0 < i \le n$. When $Q_t^i(0) = 0$ for some $0 < i \le n$, the derivative $DQ_t^{i-1}(Q_t(0))$ is positive if and only if the number of the symbol R in $K_{i-1}(Q_t)$ is even.

From the definition of the signed lexicographical order, theorem 2 implies that $K_n(Q_t)$ is increasing at each of such parameters. Hence $K_n(Q_t)$ is monotone increasing with respect to t. Since this holds for every n, the kneading invariant $K(Q_t)$ is monotone increasing.

Now we prove theorem 2. Assume $Q_t^n(0) = 0$ and $Q_t^i(0) \neq 0$ for $0 < i < n$. Let $Q : \mathbf{C} \to \mathbf{C}$ be the complex extension of Q_t. A Ruelle operator R is defined by

$$R(\psi)(z) = \sum_{Q(y)=z} \frac{\psi(y)}{DQ(y)^2}$$

for a function ψ on the complex plane. Let $\chi_k(z) = \frac{1}{(z-Q^k(0))}$ and let S be the linear space of functions spanned by χ_k, $k = 1, 2, \cdots, n$. By simple calculation, we can check

$$R(\chi_i) = \frac{\chi_{i+1}}{DQ(Q^i(0))} - \frac{\chi_1}{DQ(Q^i(0))}. \tag{4}$$

Hence the Ruelle operator R preserves S. The representation matrix for $R|_S$ with respect to the basis $\{\chi_i\}_{i=1}^n$ is of the form

$$\Phi = \begin{pmatrix} -\frac{1}{DQ(Q(0))} & -\frac{1}{DQ(Q^2(0))} & -\frac{1}{DQ(Q^3(0))} & -\frac{1}{DQ(Q^4(0))} & \cdots \\ \frac{1}{DQ(Q(0))} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{DQ(Q^2(0))} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{DQ(Q^3(0))} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

We find

$$\begin{aligned} &\det(Id - \Phi) \\ &= \det \begin{pmatrix} 1 + \frac{1}{DQ(Q(0))} & \frac{1}{DQ(Q^2(0))} & \frac{1}{DQ(Q^3(0))} & \cdots \\ -\frac{1}{DQ(Q(0))} & 1 & 0 & \cdots \\ 0 & -\frac{1}{DQ(Q^2(0))} & 1 & \cdots \\ 0 & 0 & -\frac{1}{DQ(Q^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &= \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{1}{DQ(Q^j(0))} = \sum_{k=1}^{n} \frac{1}{DQ^k(Q(0))} = \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^n(Q_t(0))}. \end{aligned} \tag{6}$$

Since Φ is a matrix with real entries, theorem 2 follows from

Lemma 6 *The spectral radius of Φ is smaller than 1.*

[proof] Let us consider the Perron-Frobenius operator

$$P(\psi)(x) = \sum_{Q(y)=x} \left| \frac{\psi(y)}{DQ(y)^2} \right|.$$

Obviously we have $|R^k(\psi)(x)| \leq P^k(|\psi|)(x)$ for $k > 0$. Take a large disk D such that $Q^{-1}(D) \subset D$, and put $A = D - f^{-1}(D)$. Then

$$\|\psi\| = \int_A |\psi||dz|^2$$

is a norm on the finite dimensional vector space S. We have

$$\sum_{k=0}^{\infty} \int_A P^k(|\psi|)|dz|^2 = \sum_{k=0}^{\infty} \int_{f^{-k}(A)} |\psi||dz|^2 \leq \int_D |\psi||dz|^2 < \infty.$$

This implies $\|R^k(\psi)\| \to 0$ and, hence, the lemma. *Q.E.D.*

We finished the proof of theorem 1 and 2. Here we give a remark on the relation between the above proof and that in [6]. In the original proof in [6], they consider the so-called Thurston map which acts on the Teichmuller space of finitely-pointed sphere. The point of the proof is that Thurston map is a contraction with respect to Teichmuller metric. The space S in our proof above corresponds to the cotangent space of the Theichmuller space, and the operator R corresponds to the action of Thurston map on the cotangent space at the fixed point. (cf. [3], [7]) Hence our proof is a local version of that in [6] in a sense. We showed that the local contraction was enough.

The Ruelle operator R in our proof is intresting in itself. For example, Levin, Sodin and Yuditski studied, in [4] and [5], the properties of R in more general setting, and get some formulae similar to (6).

3 Proof of theorem 3

The proof of theorem 3 is similar to that for theorem 2 except for the point that the space of function we consider is infinite dimensional. Let us assume that Q_t satisfies Collet-Eckmann condition (1) and choose $r > 1$ such that

$$\liminf_{n\to\infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > r.$$

Below we assume that the critical point is not pre-periodic. The case when the critical point is pre-periodic is treated at the end of this section. Let us consider a Banach space

$$S = \left\{ (a_k)_{k=1}^{\infty} \;\middle|\; \sum_{k=1}^{\infty} |a_k| \cdot |DQ_t^k(Q_t(0))| \cdot r^{-k} < \infty \right\}$$

with a norm $|||(a_k)_{k=1}^{\infty}||| = \sum_{k=1}^{\infty} |a_k| \cdot |DQ_t^k(Q_t(0))| \cdot r^{-k}$. Let $\chi_k(z) = 1/(z - Q_t^k(0))$. For each $(a_k)_{k=1}^{\infty} \in S$, we consider a holomorphic function defined on $\mathbf{C}\backslash\mathbf{R}$,

$$\Psi((a_k)_{k=1}^{\infty})(z) = \sum_{k=1}^{\infty} a_k \chi_k(z).$$

Lemma 7 *Ψ is a injective map from S to the set of holomorphic functions defined on $\mathbf{C}\backslash\mathbf{R}$.*

[proof] Assume that $(a_k)_{k=1}^{\infty} \in S$ is not null. Take $a_n \neq 0$ and m such that $\sum_{k>m} |a_k| < |a_n|$. Then, for $y \in \mathbf{R} - \{0\}$,

$$\left|\Psi((a_k)_{k=1}^{\infty})(Q_t^n(0) + y\sqrt{-1})\right| \geq \frac{|a_n|}{y} - \sum_{k>m} \frac{|a_k|}{y} - \sum_{k\leq m, k\neq n} \frac{|a_k|}{|Q_t^k(0) - Q_t^n(0)|}$$

Since the last term is bounded as $t \to 0$, we see that $\Psi((a_k)_{k=1}^{\infty})$ is not constant 0.

Q.E.D.

We consider the Ruelle operator R as in the last section. From (4) and the definition of the norm $|||\cdot|||$, it is easy to see that R preserve the image $\Psi(S)$ and induce a continuous linear map $\Phi : S \to S$ such that $\Psi \circ \Phi = R \circ \Psi$. Remark that Φ is of the form (5).

Lemma 8 *The spectrum of Φ contained in $\{|z| > r^{-1}\}$ is discrete and the corresponding eigenspace is finite dimeisional.*

[proof] Let $\pi_1 : S \to S$ be the projection to the first component: $\pi_1((a_k)_{k=1}^{\infty}) = (a_1, 0, 0, \cdots)$. We have $\|(id - \pi_1) \circ \Phi\| \leq r^{-1}$ from the definition of the norm. Since $\pi_1 \circ \Phi$ has one dimensional range, the lemma follows from the general spectral theory. ([2], pp 709).

Q.E.D.

As in the last section, the operator Φ contract the norm induced from

$$\|\psi\| = \int_{A-R} |\psi||dz|^2.$$

on $\Psi(S)$. (A is the annulus taken in the last section.) Hence the discrete spectrum in the above lemma should be contained in the unit disk $\{|z| < 1\}$. Therefore we get

Proposition 9 *The spectral radius of Φ is smaller than 1.*

Let $\pi_n : S \to S$ be the projection to the first n components. We approximate Φ by finite dimensional operators $\Phi_n = \pi_n \circ \Phi \circ \pi_n$.

Lemma 10 *There exists $N > 0$, $M > 0$ and $0 < \mu < 1$ such that $|||\Phi_n^k||| < M\mu^k$ for any $n > N$ and $k > 0$.*

[proof] Let $B_n = \Phi_n + (Id - \pi_n) \circ \Phi$. Since $\Phi_n^m = \pi_n \circ B_n^m$, we have $|||\Phi_n^m||| \leq |||B_n^m|||$. For $a = (a_k)_{k=1}^{\infty} \in S$, we have

$$\begin{aligned} |||\Phi(a) - B_n(a)||| &= |||\pi_1 \circ \Phi \circ (Id - \pi_n)(a)||| \\ &\leq \sum_{k>n} \left| a_k DQ_t^k(Q_t(0)) \cdot r^{-k} \cdot \frac{r^k}{DQ_t^{k+1}(Q_t(0))} \right|. \\ &\leq |||a||| \cdot \max_{k>n} \left\{ \left| \frac{r^k}{DQ_t^{k+1}(Q_t(0))} \right| \right\}. \end{aligned}$$

Hence $|||\Phi - B_n||| \to 0$ as $n \to \infty$. Now the lemma follows from

$$\lim_{m\to\infty} \sqrt[m]{|||\Phi^m|||} < 1,$$

which is equivalent to the claim of proposition 9.([2, pp.567]) *Q.E.D.*

By calculation, we see

$$\det(Id - \Phi_n) = \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))}. \tag{7}$$

On the other hand, for sufficiently large n, we have

$$\det(Id - \Phi_n) = \exp(\operatorname{Trace}\log(\Phi_n))) = \exp\left(-\sum_{k=1}^{\infty} k^{-1}\operatorname{Trace}(\Phi_n^k)\right).$$

Notice that p-th diagonal element of the matrix Φ_n^k is null for $p > k$ because of the special form of Φ. Hence we have

$$|\operatorname{Trace}(\Phi_n^k)| \leq k|||\Phi_n^k||| < kM\mu^k$$

for $n \geq N$. Therefore we obtain

$$\lim_{n\to\infty} \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))} \geq \exp\left(\sum_{k=1}^{\infty} -M\mu^k\right) > 0.$$

We finished the proof of theorem 3 in the case when the critical point is not pre-periodic. When the critical point is pre-periodic, the space S of functions spanned by χ_k, $k = 1, 2, \cdots$, is finite dimensional. By simple calculation, we can check

$$\det(Id - R|_S) = \lim_{n\to\infty} \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))}.$$

By arguing as in the last section, we see that the spectral radius of $R|_S$ is smaller than 1 and obtain (2).

4 On the family of tent maps

Monotonicity of the entropy is well-known for the family of tent maps $T_t(x) = 1 - t|x|$, $(1 < t \leq 2)$.

Theorem 11 *The kneading invariant $K(T_t)$ is monotone increasing with respect to t. Hence, so is the topological entropy.*

Here we give a proof which is analogous to that of theorem 2. Assume that $T_t^n(0) = 0$. We write $T = T_t$. Notice that the interval $[T^2(0), T(0)]$ is invariant for T and contains 0. We consider the Perron-Frobenius operator P for T:

$$P(\psi)(x) = \sum_{T(y)=x} \left|\frac{\psi(y)}{DT(y)}\right|.$$

Let $\chi_k(x)$ be a function defined by

$$\chi_k(x) = \begin{cases} 1, & x > T^k(0); \\ 0, & x = T^k(0); \\ -1, & x < T^k(0). \end{cases}$$

Then we have the following formula, which is similar to (4):

$$P(\chi_k) = \frac{\chi_{k+1}}{DT(T^k(0))} - \frac{\chi_1}{DT(T^k(0))}.$$

Let S be a linear space of functions spanned by $\{\chi_k(0)\}_{k=1}^n$. Let S' be the set of functions $\varphi \in S$ such that $\varphi(x) = 0$ for $x \notin [T^2(0), T(0)]$. Obviously, $P(S) = S'$. Since P preserves the integrals of functions, P also preserve the subspace

$$S'' = \left\{ \varphi \in S' \;\middle|\; \int_{\mathbf{R}} \varphi dx = 0 \right\}.$$

Spectral radius of $P|_S$ is not greater than 1. Moreover, from the ergodic properties of T, we know that 1 is a simple eigenvalue for $P|_S$ whose eigenvector corresponds to the unique absolutely continuous ergodic measure for T. Remark that $P|_{S''}$ do not have 1 as its eigenvalue. Therefore we have $\det(Id_S - P|_S) = 0$ and $\det(Id_{S''} - P|_{S''}) > 0$. Theorem 11 follows from

Lemma 12 $\det(Id_{S''} - P|_{S''}) = t \cdot \dfrac{\frac{\partial}{\partial s}(T_s^{n-1}(0))|_{s=t}}{DT^n(1)}.$

[proof] Let us define $\pi : S' \to S$ by

$$\pi(\psi) = \psi - \frac{\int_{\mathbf{R}} \psi dx}{2} \cdot \chi_1.$$

Obviously $\pi \circ P|_{S''} = P|_{S''}$. Since $\int_{\mathbf{R}} P(\chi_1)dx = 2$, we see $P \circ \pi(S') \subset S''$ and $(\pi \circ P)^2(S) \subset S''$. Hence $\det(Id_{S''} - P|_{S''}) = \det(Id_S - \pi \circ P|_S)$. We can check

$$\pi \circ P(\chi_k) = \frac{\chi_{k+1}}{DT(T^k(0))} - \frac{1 - (T(0) - T^{k+1}(0))}{DT(T^k(0))}\chi_1.$$

Hence, the representation matrix of $\pi \circ P|_S$ w.r.t. the basis $\{\chi_k\}_{k=1}^n$ is

$$\Phi = \begin{pmatrix} -\frac{1-(T(0)-T^2(0))}{DT(T(0))} & -\frac{1-(T(0)-T^3(0))}{DT(T^2(0))} & -\frac{1-(T(0)-T^4(0))}{DT(T^3(0))} & \cdots \\ \frac{1}{DT(T(0))} & 0 & 0 & \cdots \\ 0 & \frac{1}{DT(T^2(0))} & 0 & \cdots \\ 0 & 0 & \frac{1}{DT(T^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since

$$\det(Id_S - P|_S) = \det \begin{pmatrix} 1 + \frac{1}{DT(T(0))} & \frac{1}{DT(T^2(0))} & \frac{1}{DT(T^3(0))} & \cdots \\ -\frac{1}{DT(T(0))} & 1 & 0 & \cdots \\ 0 & -\frac{1}{DT(T^2(0))} & 1 & \cdots \\ 0 & 0 & -\frac{1}{DT(T^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

equals to 0, we get

$$\det(Id_S - \pi \circ P|_S) = \det \begin{pmatrix} -\frac{T(0)-T^2(0)}{DT(T(0))} & -\frac{T(0)-T^3(0)}{DT(T^2(0))} & -\frac{T(0)-T^4(0)}{DT(T^3(0))} & \cdots \\ -\frac{1}{DT(T(0))} & 1 & 0 & \cdots \\ 0 & -\frac{1}{DT(T^2(0))} & 1 & \cdots \\ 0 & 0 & -\frac{1}{DT(T^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Calculating the left hand side, we obtain the lemma. *Q.E.D.*

References

[1] P.Collet and J.P.Eckmann, Iterated maps on the interval as dynamical system, Birkhauser

[2] M.Dunford, J.T.Schwartz, Linear operators 1, Interscience, New York,(1958)

[3] J.H.Hubbard and D.Schleicher, *The spider algorithm*, Proc. Sympos. Appl. Math.,49, 155–180 (1994)

[4] G.Levin, M.Sodin, P.Yuditski, *A Ruelle operator for a real Julia set*, Commun. Math. Phys.141, 119-132 (1991)

[5] G.Levin, M.Sodin, P.Yuditski, *Ruelle operators with rational weight for Julea sets*, Journal d'Analyse Math.,vol.63,303–331(1994)

[6] J.Milnor and W.Thurston,*On iterated maps of the interval*, in Lect. Note in Math. 1342, 465–563 (1988)

[7] M. Tsujii, *A note on Milnor and Thurston's monotonicity theorem*,Kokyuroku 863, 51–52, RIMS (1994)

[8] M. Tsujii, Positive Lyapunov exponent in families of one dimensional dynamical systems, *Invent. math.* vol.111 (1993), 113–137