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ρ-Contraction And 2×2 Matrix

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ρ-Contraction And 2×2 Matrix

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Abstract. In this paper, the following is proved. When $|a| \leq 1$ and $|b| \leq 1$, $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ is a ρ-contraction if and only if

$$\begin{aligned} & |c|^2 + |a-b|^2 \\ \leq & \inf_{\zeta \in D} \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2 \end{aligned}$$

where D is the open unit disc.

§1. Introduction

For $\rho > 0$, a bounded linear operator A on a Hilbert space $\mathcal{H}$ is a ρ-contraction if and only if the powers of A admit a representation

$$A^n h = \rho P U^n h \quad (h \in \mathcal{H}\ ;\ n = 1, 2, \cdots),$$

where U is a unitary operator (called a unitary ρ-dilation) on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and where P is the projection from $\mathcal{K}$ to $\mathcal{H}$. A is a contraction, that is, the norm $\|A\| \leq 1$ if and only if A is a 1-contraction. This is a theorem of Sz.-Nagy [4]. The numerical radius $w(A)$ of A is not bigger than 1 if and only if A is a 2-contraction. This is a theorem of Berger [1]. Sz.-Nagy and Foias [5] gave intrinsic characterizations of operators of ρ-contractions. Unfortunately, even if $\mathcal{H}$ is of two dimension, it is difficult to determine when A is a ρ-contraction except $\rho = 1$. In this paper, we assume that $\mathcal{H}$ is a two dimensional Hilbert space. Assuming A is triangular, we determine when A is a ρ-contraction. In fact, for $\rho = 2$, the numerical radius is given.

In this paper, D is the open unit disc and $\bar{D}$ is its closure. $\mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ denotes a set of all holomorphic functions from $\bar{D}$ into $\bar{D}$ which vanish at $\alpha \in D$ and $\mathcal{H}ol(\bar{D})$ denotes a set of all holomorphic contractions on $\bar{D}$.

In the proof of Theorem, we use three elementary well known results.

(1) $\sup\{|f'(\alpha)|\ ;\ f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})\} = \dfrac{1}{1 - |\alpha|^2}$.

(2) $\sup\{|f(\beta)|\ ;\ f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})\} = \left|\dfrac{\alpha - \beta}{1 - \bar{\alpha}\beta}\right|$ when $\beta \in D$.

(3) $f(A)$ is a contraction for any $f \in \mathcal{H}ol(\bar{D})$ if and only if $f(A)$ is a contraction for any $f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})$.

Proof. (1) If $f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ then $|(f \circ \phi_{-\alpha})'(0)| \leq 1$ by the Schwarz's lemma where $\phi_{-\alpha}(z) = (z + \alpha)/(1 + \bar{\alpha}z)$. Hence the supremum is less than equal to $1/(1 - |\alpha|^2)$. The supremum is attained by $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. (2) If $f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ then $f(z) = \phi_\alpha(z)g(z)$ and $|g(z)| \leq 1$. Hence the supremum is attained by $\phi_\alpha(z)$. (3) The 'only if' part is clear. If $f \in \mathcal{H}ol(\bar{D})$ and $f(\alpha) = \beta$, $g = \phi_\beta \circ f$ belongs to $\mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ and it follows that $\|g(A)\| \leq 1$ or, equivalently $\|f(A)\| = \|\phi_{-\beta}(g(A))\| \leq 1$(*see* [3]).

§2. Theorem

In order to give Theorem, we use a theorem of Misra [3, Theorem 1.1] and a result of Ando and the second author [6]. Corollary 1 is known. Corollary 2 is new and with a theorem of Berger [1], it gives a necessary and sufficient condition for $w(A) \leq 1$.

Theorem. *Suppose a and b are complex numbers in the closed unit disc $\bar{D}$. Then,* $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ *is a ρ-contraction if and only if*

$$\begin{aligned} &|c|^2 + |a-b|^2 \\ &\leq \inf_{\zeta \in D} \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2 \end{aligned}$$

Proof. By [6], A is a ρ-contraction if and only if for any $\zeta \in D$, $g_\zeta(A)$ is a contraction where $g_\zeta(z) = z\zeta/\{\rho + (1-\rho)z\zeta\}$. If $a = b$, by [3, Theorem 1.1], $g_\zeta(A)$ is a contraction if and only if

$$\begin{aligned} &|c| \cdot |g'_\zeta(a)| \\ &\leq (\sup\{|f'(g_\zeta(a))| \ ; \ f \in \mathcal{H}ol(\bar{D}, g_\zeta(a), \bar{D})\})^{-1}. \end{aligned}$$

Hence by (1)

$$\begin{aligned} |c| &\leq \frac{1 - |g_\zeta(a)|^2}{|g'_\zeta(a)|} \\ &= \frac{|\rho + (1-\rho)a\zeta|^2 - |a\zeta|^2}{|\zeta\rho|}. \end{aligned}$$

Thus A is a ρ-contraction if and only if

$$|c| \leq \inf_{\zeta \in D} \left| \frac{|\rho + (1-\rho)a\zeta|^2 - |a\zeta|^2}{\zeta\rho} \right|.$$

This implies the theorem when $a = b$.

Suppose $a \neq b$. Note that $g_\zeta(a) \neq g_\zeta(b)$ for $\zeta \neq 0$. Hence A is a ρ-contraction if and only if

$$\begin{bmatrix} f \circ g_\zeta(a) & c\, \dfrac{f \circ g_\zeta(a) - f \circ g_\zeta(b)}{a-b} \\ 0 & f \circ g_\zeta(b) \end{bmatrix}$$

is a contraction for any $f \in \mathcal{H}ol(\bar{D}, g_\zeta(b), \bar{D})$ and for any $\zeta \in \bar{D}$ by the von Neumann's inequality and (3). As in [3, Theorem 1.1], this is equivalent to

$$|c|^2/|a-b|^2 \leq \frac{1}{|f \circ g_\zeta(a)|^2} - 1$$

for any $f \in \mathcal{H}ol(\bar{D}, g_\zeta(b), \bar{D})$ and for any $\zeta \in \bar{D}$. It is easy to see that by (2)

$$\begin{aligned} &\sup\{|f \circ g_\zeta(a)| \ ; \ f \in \mathcal{H}ol(\bar{D}, g_\zeta(b), \bar{D})\} \\ &= \left| \frac{g_\zeta(a) - g_\zeta(b)}{1 - \overline{g_\zeta(a)}g_\zeta(b)} \right|^{-1} \\ &= \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{(a-b)\rho\zeta} \right|. \end{aligned}$$

This implies the theorem when $a \neq b$.

Corollary 1. *In Theorem, A is a 1-contraction if and only if*

$$\begin{aligned} |c|^2 &\le \inf_{\zeta\in D} \left| \frac{1-\bar{a}b|\zeta|^2}{\zeta} \right|^2 - |a-b|^2 \\ &= (1-|a|^2)(1-|b|^2). \end{aligned}$$

Corollary 2. *In Theorem, A is a 2-contraction if and only if*

$$\begin{aligned} |c|^2 &\le \inf_{|\zeta|=1} |2-(\bar{a}\bar{\zeta}+b\zeta)|^2 - |a-b|^2 \\ &= 4\inf_{|\zeta|=1} \{(1-Re(a\zeta))(1-Re(b\zeta))\}. \end{aligned}$$

§3. Remarks

1. An operator A on H is called a quadratic operator if A satisfies quadratic polynomial, that is, $A^2+rA+sI=0$ for some complex numbers r and s. In the finite dimentional case, if $t^2+rt+s=(t-a)(t-b)$ then A is unitarily similar to a matrix of the form

$$\begin{bmatrix} aI_l & C \\ 0 & bI_m \end{bmatrix} \qquad (*)$$

where C is an $l\times m$ matrix. Since the ρ-contraction is invariant under the unitary similarity, we may assume that a quadratic operator A has a form of $(*)$. Then we can show A is a ρ-contraction if and only of

$$\begin{aligned} &||C||^2 + |a-b|^2 \\ &\le \inf_{\zeta\in D} \left| \frac{\{\rho+(1-\rho)\bar{a}\bar{\zeta}\}\{\rho+(1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2. \end{aligned}$$

In fact, by the singular value decomposition of C(cf. [2]), there is an $l\times m$ matrix $\Sigma=[\sigma_{ij}]$ where $\sigma_{ij}=0$ for all $i\neq j$, and $\sigma_{11}\ge\sigma_{22}\ge\cdots\ge\sigma_{qq}\ge 0$ ($q=\min\{l,m\}$) and there are unitary matrices $U\in M_l$, $V\in M_m$ such that $UCV^*=\Sigma$. Here $\sigma_{11},\sigma_{22},\cdots,\sigma_{qq}$ are the decerasing ordered singlar values of A, specially, $\sigma_{11}=||C||$. Then A is unitarily similar to a matrix $\begin{bmatrix} aI_l & \Sigma \\ 0 & bI_m \end{bmatrix}$, and this matrix is unitarily similar to the direct sum of 2×2 matrices of form $C_i = \begin{bmatrix} a & \sigma_{ii} \\ 0 & b \end{bmatrix}$ and possibly with aI_k or bI_k with $k = |l - m|$. It follows that A is a ρ-contraction if and only if C_1 is. From Theorem in this paper our assertion is led.

2. In the rest of this paper, let $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$. For some special a and b, we compute the infimums in Theorem and Corollary 2 as that in Corollary 1.

(1) When $|a| = 1$ or $|b| = 1$, A is a ρ-contraction if and only if $c = 0$.

(2) When $a = b$, A is a ρ-contraction if and only if $|c| \leq (\rho-2)|a|^2+2(1-\rho)|a|+\rho$.

(3) When $b = 0$, A is a ρ-contraction if and only if $|c|^2 \leq \rho(\rho - 2)|a|^2 + 2\rho(1 - \rho)|a| + \rho^2$.

(4) When $a \geq 0$ and $b \geq 0$, A is a ρ-contraction if and only if $|c|^2 \leq \{\rho + (1 - \rho)(a + b) + (\rho - 2)ab\}^2 - (a - b)^2$.

(5) When $a = -b$ is real, A is a ρ-contraction if and only if $|c|^2 + 4a^2 \leq \{\rho - (2 - \rho)a^2\}^2$.

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