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Global Existence of Classical Solutions to Systems of Wave Equations with Critical Nonlinearity in Three Space Dimensions

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Abstract

We discuss the existence of a global small solution to the Cauchy problem for a system of quasilinear wave equations in three space dimensions, when its nonlinear term have a critical exponent. Global existence is established on the null condition which is extended to the condition for systems of wave equations with different propagation speeds.

1 Introduction.

We consider the Cauchy problem for a system of quasilinear wave equations

$$\Box_i u^i = F^i(\partial u, \partial^2 u) \quad \text{in } [0,\infty)\times \mathbf{R}^3, \tag{1.1}$$

$$u^i(0,\cdot) = \varepsilon f^i,\ \partial_t u^i(0,\cdot) = \varepsilon g^i \quad \text{in } \mathbf{R}^3, i = 1,\ldots,m, \tag{1.2}$$

where $\Box_i = \partial_0^2 - c_i^2 \sum_{j=1}^3 \partial_j^2, c_i > 0$, $\partial_\alpha = \partial/\partial x^\alpha, t = x^0$, $x = (x^1, x^2, x^3)$ and $u^i = u^i(t,x)$ are real-valued unknown functions. We denote by ∂ the space-time derivatives, i.e.

$$\partial u = (\partial_\alpha u^i)_{\alpha,i}, \partial^2 u = (\partial_\alpha \partial_\beta u^i)_{\alpha,\beta,i},$$

where α, β range over $0,1,2,3$ and i over $1,\ldots,m$. f^i and g^i belong to $C_0^\infty(\mathbf{R}^3)$ and ε is a positive small parameter. We also assume that each $F^i(\partial u, \partial^2 u)$ takes the form

$$F^i(\partial u, \partial^2 u) = \sum_{j=1}^m \sum_{\alpha,\beta=0}^3 C_{\alpha\beta}^{ij}(\partial u)\partial_\alpha \partial_\beta u^j + D^i(\partial u), \tag{1.3}$$

$$C_{\alpha\beta}^{ij}(\partial u) = C_{\beta\alpha}^{ij}(\partial u), \tag{1.4}$$

$$C_{\alpha\beta}^{ij}(0) = 0, D^i(0) = \partial D^i/\partial(\partial_\alpha u^j) = 0, \tag{1.5}$$

where $C_{\alpha\beta}^{ij}, D^i$ are C^∞-functions near $\partial u = 0$.

We give a condition of global existence for the initial value problem (1.1)-(1.2) in three space dimensions. Small solutions exist globally when $F^i(\partial u, \partial^2 u)$ do not have quadratic parts, but in general we cannot expect global solutions when they have quadratic parts even if ε is small. However, Klainerman [7] introduced the null condition for the quadratic parts of single wave equations (or systems of wave equations with the same propagation speeds) and proved a global existence theorem on that condition. We extend the Klainerman's null condition to the case where the propagation speeds are different. In the two spatial dimensions, it was shown in [1] that the null condition for the systems with different propagation speeds could be derived by applying John-Shatah observation provided $D^i(\partial u) = 0$. We can prove the following null condition by applying similar argument to our case. That is,

$$\begin{aligned} &\sum_{\alpha,\beta,\gamma=0}^3 C_{\alpha\beta\gamma}^{iii} X_\alpha^i X_\beta^i X_\gamma^i = 0, \sum_{\alpha,\beta=0}^3 D_{\alpha\beta}^{iii} X_\alpha^i X_\beta^i = 0 \quad (i = 1,\ldots,m) \\ &\text{for all real vector } X^i = (X_0^i, X_1^i, X_2^i, X_3^i) \text{ satisfying} \\ &(X_0^i)^2 - c_i^2 \sum_{j=1}^3 (X_j^i)^2 = 0. \end{aligned} \tag{1.6}$$

Here we have set

$$C_{\alpha\beta\gamma}^{ijk} = \frac{\partial C_{\alpha\beta}^{ij}}{\partial(\partial_\gamma u^k)}(0), D_{\alpha\beta}^{ijk} = \frac{\partial^2 D^i}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^k)}(0).$$

The aim of this paper is to prove a global existence theorem on the null condition (1.6).

Theorem 1.1 *Let c_i to be different from each other. Assume that the nonlinear terms $F^i(\partial u, \partial^2 u)$ given by (1.3)-(1.5) satisfy the null condition (1.6) and*

$$C^{ij}_{\alpha\beta}(\partial u) = C^{ji}_{\alpha\beta}(\partial u). \tag{1.7}$$

Then there exits a positive constant ε_0 such that the initial value problem (1.1)-(1.2) has a unique C^∞-solution in $[0,\infty) \times \mathbf{R}^3$ for ε with $0 \le \varepsilon < \varepsilon_0$.

In the two spatial dimensions, Hoshiga and Kubo have proved in [2] a corresponding global existence theorem. They obtained necessary estimates with the use of only the angular derivatives and the scaling operator to avoid the difficulty coming from the difference of speeds. We improve their method to prove our theorem. Comparing the two spatial dimensions, the weighted L^∞-estimates involve '$\log t$' (Proposition 3.1), and it is shown that the null condition is used to take away '$\log t$' from our estimates (section 3.2). Further, by virtue of the null condition, we can prove L^2-boundedness of the derivatives of the solution (section 4), which leads to the global existence.

2 Notation.

Set

$$\begin{aligned}
&\partial_\alpha = \partial/\partial x^\alpha,\ x^0 = t,\\
&\partial = (\partial_0, \partial_1, \partial_2, \partial_3),\\
&\nabla = (\partial_1, \partial_2, \partial_3),\\
&r = |x| \quad \text{for } x \in \mathbf{R}^3.
\end{aligned}$$

We denote by $\Gamma = (\Gamma_0, \dots, \Gamma_7)$ the collection of differential operators ∂, Ω, S where

$$\Omega = x \wedge \nabla, \tag{2.1}$$

$$\Gamma_7 = \quad S = t\partial_t + r\partial_r, \tag{2.2}$$

$$\partial_r = \frac{x}{r} \cdot \nabla. \tag{2.3}$$

Then we find that the bracket $[\Gamma_\alpha, \Gamma_\beta]$ of any Γ_α and Γ_β is written by another Γ_γ. Moreover, we have

$$[\Gamma_\alpha, \Box_i] = 0 \text{ for } 0 \le \alpha \le 6 \text{ and } [\Gamma_7, \Box_i] = -2\Box_i. \tag{2.4}$$

We also note that

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega. \tag{2.5}$$

For $a = (a_1, \dots, a_k)$ $(a_i \in \{0, \dots, 7\}, 1 \le i \le k)$ we define

$$\Gamma^a = \Gamma_{a_1} \cdots \Gamma_{a_k} \text{ and } |a| = k. \tag{2.6}$$

Let $u = {}^t(u^1, \dots, u^m)$ be a vector and set

$$w_i(t, r) = (1 + r)(1 + |c_i t - r|) \quad (i = 1, \dots, m). \tag{2.7}$$

Then we define

$$[\partial u]_{k,t} = \sum_{|a|\le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \sup_{0\le s\le t} \sup_{x\in\mathbf{R}^3} \left| w_i(s, |x|)\Gamma^a \partial_\alpha u^i(s, x)\right|, \tag{2.8}$$

$$\|\partial u(t)\|_k = \sum_{|a|\le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \|\Gamma^a \partial_\alpha u^i(s, \cdot)\|_{L^2(\mathbf{R}^3)} . \tag{2.9}$$

$$\|\partial u\|_{k,t} = \sum_{|a|\le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \sup_{0\le s\le t} \|\Gamma^a \partial_\alpha u^i(s, \cdot)\|_{L^2(\mathbf{R}^3)} . \tag{2.10}$$

3 Weighted L^∞-estimates.

3.1 Estimates for solutions of scalar wave equations.

Let $v = v(t, x)$ be the smooth solution of the Cauchy problem

$$\partial_t^2 v - c_0^2 \Delta v = F \quad \text{in} \quad [0, T) \times \mathbf{R}^3, \tag{3.1}$$

$$v(0, \cdot) = \partial_t v(0, \cdot) = 0 \quad \text{in} \quad \mathbf{R}^3, \tag{3.2}$$

where $F \in C^\infty([0, T) \times \mathbf{R}^3)$ and $F(t, \cdot) \in C_0^\infty(\mathbf{R}^3)$ for each t. In this subsection we present the decay estimates for v that we will use later on.

Proposition 3.1 *Let v be the solution of (3.1)-(3.2). For $1 \le \mu$, $0 < \nu$ and $0 \le c$, we set*

$$z_{\mu,\nu}(s,\lambda) = (1 + |cs - \lambda|)^{\mu}(1 + s + \lambda)^{\nu},$$

$$\Phi_\theta(t) = \begin{cases} \log(2+t) & (\theta = 0) \\ 1 & (\theta > 0), \end{cases} \tag{3.3}$$

$$M_{\mu,\nu;k}(F) = \sum_{|a|\le k} \sup_{0\le s\le t} \sup_{y\in\mathbf{R}^3} |y| z_{\mu,\nu}(s,|y|)|\Gamma^a F(s,y)|. \tag{3.4}$$

Then we have

$$(i) \quad |v(t,x)| \le C(1+t+|x|)^{-1}\Phi_{\mu-1}(t)\Phi_{\nu-1}(t)M_{\mu,\nu;0}(F) \tag{3.5}$$

for $1 \le \mu, 1 \le \nu$,

$$(ii) \quad |\partial v(t,x)| \le C(1+|x|)^{-1}(1+|c_0 t - |x||)^{-\nu}\Phi_{\mu-1}(t)M_{\mu,\nu;1}(F) \tag{3.6}$$

for $1 \le \mu, 0 < \nu, c \ne c_0$, *and*

$$\begin{aligned}(iii) \quad |\partial v(t,x)| \le C(1+|x|)^{-1}\{&(1+|c_0 t - |x||)^{-\nu}\Phi_{\mu-1}(t) \\ &+(1+|c_0 t - |x||)^{-\mu}\Phi_{\nu-1}(t)\}M_{\mu,\nu;1}(F)\end{aligned} \tag{3.7}$$

for $1 \le \mu, 1 \le \nu, c = c_0$.

Proof. By a change of coodinates, the proof can be reduced to the case where $c_0 = 1$. So we let $c_0 = 1$. Set $|x| = r$. In appendix in [3], F. John showed that the solution of (3.1)-(3.2) could be expressed in the form

$$v(t,x) = (4\pi r)^{-1} \int_0^t ds \int_{|r-t+s|}^{r+t-s} \lambda d\lambda \int_0^{2\pi} F(s, \lambda\Theta) d\varphi, \tag{3.8}$$

where

$$\begin{aligned}&\Theta = \Theta(s,\lambda,\varphi) = R(\sin\psi\cos\varphi, \sin\psi\sin\varphi, \cos\psi), \\ &R \text{ is an orthogonal transformation with } R(0,0,r) = x, \\ &\cos\psi = (2r\lambda)^{-1}(r^2 + \lambda^2 - (t-s)^2),\ \sin\psi = (1-\cos^2\psi)^{1/2}.\end{aligned} \tag{3.9}$$

Thus, v can be written as

$$v(t,x) = (4\pi r)^{-1} \int_D \lambda d\lambda ds \int_0^{2\pi} F(s,\lambda\Theta) d\varphi, \tag{3.10}$$

where

$$D = \{(s,\lambda) \,|\, 0 < s < t,\ \lambda_1 < \lambda < \lambda_2\},$$
$$\lambda_1 = |r - t + s|,\ \lambda_2 = r + t - s. \tag{3.11}$$

We first prove (3.5). By (3.10),

$$|v(t,x)| \leq C I_0 M_{\mu,\nu;0}(F)$$

where

$$I_0 = r^{-1} \int_D z_{\mu,\nu}(s,\lambda)^{-1} d\lambda ds.$$

We obtain (3.5) from

$$I_0 \leq C(1+t+r)^{-1}\Phi_{\mu-1}(t)\Phi_{\nu-1}(t), \tag{3.12}$$

which we will prove now. In order to prove (3.12), four cases

1. $r \leq 1$,

2. $1 \leq r$ and $(2+c)t \leq r$,

3. $1 \leq r$ and $2^{-1}t \leq r \leq (2+c)t$,

4. $1 \leq r$ and $r \leq 2^{-1}t$

are considered separately.

In the first case, we have

$$\lambda_2 - \lambda_1 \leq 2r \, (\leq 2),$$

and hence

$$\begin{aligned} I_0 &\leq Cr^{-1} \int_0^t ds \int_{\lambda_1}^{\lambda_2} z_{\mu,\nu}(s,\lambda_2)^{-1} d\lambda \\ &\leq C \int_0^t (1 + |cs - \lambda_2|)^{-\mu} (1 + s + \lambda_2)^{-\nu} ds \\ &\leq C(1+t+r)^{-\nu}\Phi_{\mu-1}(t). \end{aligned}$$

Therefore, (3.12) is proved for the case 1.

In the second case, the inequalities

$$\lambda - cs \geq \min\{r-t, r-ct\} \geq (2+c)^{-1}(1+\min\{1,c\})r \tag{3.13}$$

hold for $(s,\lambda) \in D$. Hence it follows from

$$z_{\mu,\nu}(s,\lambda)^{-1} \leq C(1+r)^{-\mu-\nu} \tag{3.14}$$

that

$$\begin{aligned} I_0 &\leq Cr^{-1}\int_D (1+r)^{-\mu-\nu} d\lambda ds \\ &\leq C(1+t+r)^{1-\mu-\nu}. \end{aligned}$$

Consequently, we have (3.12) for the case 2.

In the third and the fourth case, we introduce the new variables of integration

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -c & 1 \end{pmatrix} \begin{pmatrix} s \\ \lambda \end{pmatrix}. \tag{3.15}$$

Set

$$\alpha_0 = 2^{-1}\{(1-c)\alpha + (1+c)(r-t)\}. \tag{3.16}$$

If $|t-r| < \alpha < t+r$, then $-c\alpha < \alpha_0 < \alpha$. Hence,

$$\begin{aligned} &\int_D z_{\mu,\nu}(s,\lambda)^{-1} d\lambda ds \\ &\leq \int_D (1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-\nu} d\lambda ds \\ &\leq \int_{|r-t|}^{r+t} (1+\alpha)^{-\nu} d\alpha \int_{\alpha_0}^{\alpha} (1+|\beta|)^{-\mu} d\beta \\ &\leq \int_{|r-t|}^{r+t} (1+\alpha)^{-\nu}\Phi_{\mu-1}(\alpha) d\alpha. \end{aligned}$$

Therefore in the third case, it follows that

$$\begin{aligned} I_0 &\leq C(1+r)^{-1}\Phi_{\mu-1}(t)\int_{|r-t|}^{r+t}(1+\alpha)^{-\nu} d\alpha \\ &\leq C(1+t+r)^{-1}\Phi_{\mu-1}(t)\Phi_{\nu-1}(t), \end{aligned}$$

and in the fourth case, it follows that

$$\begin{aligned} I_0 &\leq Cr^{-1}(1+t-r)^{-\nu}\Phi_{\mu-1}(t)\int_{t-r}^{t+r} d\alpha \\ &\leq C(1+t+r)^{-\nu}\Phi_{\mu-1}(t). \end{aligned}$$

Hence we have proved (3.12) in all cases.

We next prove (3.6) and (3.7). In order to prove (3.6) and (3.7), we give representation formulae for $\partial_\alpha v$.

Let $(s, \lambda) \in D$ and $0 \leq \varphi \leq 2\pi$. By (2.5),

$$(\nabla F)(s, \lambda\Theta) = \Theta(\partial_r F)(s, \lambda\Theta) - \lambda^{-1}\Theta \wedge (\Omega F)(s, \lambda\Theta). \tag{3.17}$$

Since $\partial_\lambda \Theta \cdot \Theta = 0$, it follows that

$$\begin{aligned}
&(\partial_r F)(s, \lambda\Theta) \\
&= \partial_\lambda\{F(s, \lambda\Theta)\} - \lambda\partial_\lambda\Theta \cdot (\nabla F)(s, \lambda\Theta) \\
&= \partial_\lambda\{F(s, \lambda\Theta)\} - \lambda\partial_\lambda\Theta \cdot \{\Theta(\partial_r F)(s, \lambda\Theta) - \lambda^{-1}\Theta \wedge (\Omega F)(s, \lambda\Theta)\} \\
&= \partial_\lambda\{F(s, \lambda\Theta)\} + \partial_\lambda\Theta \cdot (\Theta \wedge (\Omega F)(s, \lambda\Theta)).
\end{aligned} \tag{3.18}$$

In a similar manner,

$$(\partial_t F)(s, \lambda\Theta) = \partial_s\{F(s, \lambda\Theta)\} + \partial_s\Theta \cdot (\Theta \wedge (\Omega F)(s, \lambda\Theta)). \tag{3.19}$$

If we substitute (3.18) into (3.17), we have

$$\begin{aligned}
(\nabla F)(s, \lambda\Theta) &= \Theta\partial_\lambda\{F(s, \lambda\Theta)\} + \Theta\{\partial_\lambda\Theta \cdot (\Theta \wedge (\Omega F)(s, \lambda\Theta))\} \\
&\quad -\lambda^{-1}\Theta \wedge (\Omega F)(s, \lambda\Theta).
\end{aligned} \tag{3.20}$$

We use these expressions in

$$\partial_\alpha v(t, x) = (4\pi r)^{-1} \int_D \lambda d\lambda ds \int_0^{2\pi} (\partial_\alpha F)(s, \lambda\Theta) d\varphi,$$

which are obtained from (3.10). We split the domain of integration D into D_1 and D_2, where

$$\begin{aligned}
&D_1 = \{(s, \lambda) \in D \,|\, \lambda_1 < \lambda < \lambda_1 + \delta \text{ or } \lambda_2 - \delta < \lambda < \lambda_2\}, \\
&D_2 = D \setminus D_1, \\
&\delta = \min\{1, r\}.
\end{aligned} \tag{3.21}$$

Using (3.19),(3.20) and integrating by parts on the domain D_2, we obtain the following representation formulae.

$$
\begin{aligned}
&4\pi r\partial_t v(t,x)\\
&\quad=\int_{D_1}\lambda d\lambda ds\int_0^{2\pi}(\partial_t F)(s,\lambda\Theta)d\varphi\\
&\qquad+\int_{\partial D_2}n_s d\sigma\int_0^{2\pi}\lambda F(s,\lambda\Theta)d\varphi\\
&\qquad+\int_{D_2}\lambda d\lambda ds\int_0^{2\pi}\partial_s\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta))d\varphi
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&4\pi r\nabla v(t,x)\\
&\quad=\int_{D_1}\lambda d\lambda ds\int_0^{2\pi}(\nabla F)(s,\lambda\Theta)d\varphi\\
&\qquad+\int_{\partial D_2}n_\lambda d\sigma\int_0^{2\pi}\lambda\Theta F(s,\lambda\Theta)d\varphi\\
&\qquad-\int_{D_2}d\lambda ds\int_0^{2\pi}\{\Theta F(s,\lambda\Theta)+\Theta\wedge(\Omega F)(s,\lambda\Theta)\}d\varphi\\
&\qquad+\int_{D_2}\lambda d\lambda ds\int_0^{2\pi}\begin{array}{l}[-\partial_\lambda\Theta F(s,\lambda\Theta)\\ \quad+\Theta\{\partial_\lambda\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta))\}]d\varphi\end{array}
\end{aligned}
\tag{3.23}
$$

Here, (n_s, n_λ) is the unit outer normal vector field on ∂D_2, and $d\sigma$ is the line element on ∂D_2.

Note that $D_2=\emptyset$ if $r<1$. Therefore it follows from (3.22) and (3.23) that

$$
|\partial v(t,x)|\le C\sum_{i=1}^{4} I_i M_{\mu,\nu;1}
\tag{3.24}
$$

where

$$
\begin{aligned}
I_1 &= r^{-1}\int_{D_1} z_{\mu,\nu}(s,\lambda)^{-1}d\lambda ds,\\
I_2 &= (1+r)^{-1}\int_{\partial D_2} z_{\mu,\nu}(s,\lambda)^{-1}d\sigma,\\
I_3 &= (1+r)^{-1}\int_{D_2}(1+\lambda)^{-1}z_{\mu,\nu}(s,\lambda)^{-1}d\lambda ds,\\
I_4 &= (1+r)^{-1}\int_{D_2}\sup_{0\le\varphi\le 2\pi}|\partial\Theta| z_{\mu,\nu}(s,\lambda)^{-1}d\lambda ds.
\end{aligned}
$$

By the definition of the domain D_1, we can easily see

$$
I_1\le CI_2.
\tag{3.25}
$$

Concerning I_2, I_3 and I_4, we will prove

$$I_2 \le C(1+r)^{-1}(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.26}$$

for $1 \le \mu, 0 < \nu, c \neq 1$,

$$I_2 \le C(1+r)^{-1}\{(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) + (1+|t-r|)^{-\mu}\Phi_{\nu-1}(t)\} \tag{3.27}$$

for $1 \le \mu, 1 \le \nu, c = 1$,

$$I_3 \le C(1+r)^{-1}(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.28}$$

for $1 \le \mu, 0 < \nu$, and

$$I_4 \le C(1+r)^{-1}(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.29}$$

for $1 \le \mu, 0 < \nu, c \neq 1$ or $1 \le \mu, 1/2 < \nu, c = 1$. If we have proved (3.26)-(3.29), then (3.6) and (3.7) are obtained through (3.24), (3.25) and these estimates.

(a) Proof of (3.26) and (3.27).

If $r \ge (2+c)t$, (3.26) and (3.27) are easily obtained from (3.13). So we let $r \le (2+c)t$.

By the definition of I_2, we have

$$(1+r)I_2 \le C(I_2' + I_2'' + I_2'''),$$

where

$$\begin{aligned} I_2' &= \int_0^t z_{\mu,\nu}(s,\lambda_1)^{-1}ds, \\ I_2'' &= \int_0^t z_{\mu,\nu}(s,\lambda_2)^{-1}ds, \\ I_2''' &= \int_{|t-r|}^{t+r} z_{\mu,\nu}(0,\lambda)^{-1}d\lambda. \end{aligned}$$

If $c \neq 1$, it follows that

$$\begin{aligned} I_2' &\le \int_0^t (1+|cs-\lambda_1|)^{-\mu}(1+s+\lambda_1)^{-\nu}ds \\ &\le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t), \end{aligned}$$

because $s+\lambda_1 \geq |t-r|$. However, if $c=1$,

$$\begin{aligned}
I_2' &\leq \int_0^t (1+|s-\lambda_1|)^{-\mu}(1+s+\lambda_1)^{-\nu} ds \\
&= \int_0^{(t-r)_+} (1+|s-\lambda_1|)^{-\mu}(1+t-r)^{-\nu} ds \\
&\quad + \int_{(t-r)_+}^t (1+|t-r|)^{-\mu}(1+s+\lambda_1)^{-\nu} ds \\
&\leq C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) + C(1+|t-r|)^{-\mu}\Phi_{\nu-1}(t).
\end{aligned}$$

As for I_2'' and I_2''', we obtain, straightforwardly,

$$\begin{aligned}
I_2'' &\leq C(1+t+r)^{-\nu}\Phi_{\mu-1}(t), \\
I_2''' &\leq C(1+|t-r|)^{1-\mu-\nu}
\end{aligned}$$

for $1 \leq \mu$ and $0 < \nu$. Thus we have proved (3.26) and (3.27).

(b) Proof of (3.28).

In case $r \geq (2+c)t$, (3.28) results from (3.13).

Let $r \leq (2+c)t$. If $c=0$, by the change of variables (3.15),

$$\begin{aligned}
(1+r)I_3 &\leq \int_{|t-r|}^{t+r} (1+\alpha)^{-\nu} d\alpha \int_{\alpha_0}^{\alpha} (1+\beta)^{-1-\mu} d\beta \\
&\leq C\int_{|t-r|}^{t+r} (1+|t-r|)^{-\nu}(1+\alpha_0)^{-\mu} d\alpha \\
&\leq C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t).
\end{aligned}$$

If $c > 0$, set

$$\begin{aligned}
\Lambda_0 &= \{(s,\lambda)|0 \leq \lambda \leq 2^{-1}cs\}, \\
\Lambda_1 &= \{(s,\lambda)|0 \leq 2^{-1}cs \leq \lambda\}.
\end{aligned} \tag{3.30}$$

Since

$$(s,\lambda) \in \Lambda_0 \Rightarrow (1+\lambda)^{-1}z_{\mu,\nu}(s,\lambda)^{-1} \leq C(1+\lambda)^{-1}(1+s+\lambda)^{-\mu-\nu}, \tag{3.31}$$

$$(s,\lambda) \in \Lambda_1 \Rightarrow (1+\lambda)^{-1}z_{\mu,\nu}(s,\lambda)^{-1} \leq C(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-1-\nu}, \tag{3.32}$$

the inequality

$$(1+\lambda)^{-1}z_{\mu,\nu}(s,\lambda)^{-1} \le C\{(1+\lambda)^{-\mu} + (1+|cs-\lambda|)^{-\mu}\}(1+s+\lambda)^{-1-\nu} \tag{3.33}$$

holds. Hence, adapting (3.15) for each term of (3.33), we obtain (3.28).

(c) Proof of (3.29).

We first need estimates of $|\partial\Theta|$. By the definition (3.9) of Θ, we have

$$\begin{aligned} |\partial_\lambda \Theta| &= |\lambda^2 + \bar\lambda_1\lambda_2|\lambda^{-1}\{(\lambda^2-\lambda_1^2)(\lambda_2^2-\lambda^2)\}^{-1/2}, \\ |\partial_s\Theta| &= (\bar\lambda_1+\lambda_2)\{(\lambda^2-\lambda_1^2)(\lambda_2^2-\lambda^2)\}^{-1/2}, \end{aligned}$$

where $\bar\lambda_1 = t-s-r$. Noting

$$\bar\lambda_1 = \begin{cases} -\lambda_1 & \text{for } (t-r)_+ < s < t \\ \lambda_1 & \text{for } 0 < s < t-r, \end{cases}$$

it follows from

$$\begin{aligned} \lambda^2 + \bar\lambda_1\lambda_2 &= \lambda(\lambda+\bar\lambda_1) + \bar\lambda_1(\lambda_2-\lambda), \\ \bar\lambda_1+\lambda_2 &= (\lambda+\bar\lambda_1) + (\lambda_2-\lambda) \end{aligned}$$

that

$$(t-r)_+ < s < t \Rightarrow |\partial\Theta| \le (\lambda^2-\lambda_1^2)^{-1/2} + (\lambda_2^2-\lambda^2)^{-1/2}, \tag{3.34}$$

$$0 < s < t-r \Rightarrow |\partial\Theta| \le (\lambda^2-\lambda_1^2)^{-1/2} + \{(\lambda-\lambda_1)(\lambda_2-\lambda)\}^{-1/2}. \tag{3.35}$$

As before, the case $r \ge (2+c)t$ is easy. So we consider $r \le (2+c)t$. Set

$$\begin{aligned} D_2^{(1)} &= \{(s,\lambda) \in D_2 \,|\, (t-r)_+ < s < t\}, \\ D_2^{(2)} &= \{(s,\lambda) \in D_2 \,|\, 0 < s < t-r\}. \end{aligned} \tag{3.36}$$

Then from (3.34) and (3.35),

$$\begin{aligned} (s,\lambda) \in D_2^{(1)} &\Rightarrow |\partial\Theta| \le C\{(1+\lambda-\lambda_1)^{-1/2} + (1+\lambda_2-\lambda)^{-1/2}\}(1+\lambda)^{-1/2}, \\ (s,\lambda) \in D_2^{(2)} &\Rightarrow |\partial\Theta| \le C\{(1+\lambda)^{-1/2} + (1+\lambda_2-\lambda)^{-1/2}\}(1+\lambda-\lambda_1)^{-1/2}. \end{aligned}$$

Hence, noting (3.31) and (3.32), we obtain the following estimates:

$$(s,\lambda)\in D_2^{(i)} \Rightarrow |\partial\Theta| z_{\mu,\nu}(s,\lambda)^{-1} \le C\{p_{\mu,\nu}^{(i)}(s,\lambda)+q_{\mu,\nu}^{(i)}(s,\lambda)\},\ i=1,2 \tag{3.37}$$

where

$$p_{\mu,\nu}^{(i)}(s,\lambda)=\xi_{\mu,\nu}^{(i),1}(s,\lambda)+\xi_{\mu,\nu}^{(i),2}(s,\lambda),\ i=1,2, \tag{3.38}$$

$$q_{\mu,\nu}^{(i)}(s,\lambda)=\eta_{\mu,\nu}^{(i),1}(s,\lambda)+\eta_{\mu,\nu}^{(i),2}(s,\lambda),\ i=1,2, \tag{3.39}$$

$$\begin{aligned}
\xi_{\mu,\nu}^{(1),1}(s,\lambda)&=(1+\lambda-\lambda_1)^{-1/2}(1+\lambda)^{-1/2-\mu}(1+s+\lambda)^{-\nu},\\
\xi_{\mu,\nu}^{(1),2}(s,\lambda)&=(1+\lambda_2-\lambda)^{-1/2}(1+\lambda)^{-1/2-\mu}(1+s+\lambda)^{-\nu},\\
\eta_{\mu,\nu}^{(1),1}(s,\lambda)&=(1+\lambda-\lambda_1)^{-1/2}(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-1/2-\nu},\\
\eta_{\mu,\nu}^{(1),2}(s,\lambda)&=(1+\lambda_2-\lambda)^{-1/2}(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-1/2-\nu},\\
\xi_{\mu,\nu}^{(2),1}(s,\lambda)&=(1+\lambda)^{-1/2-\mu}(1+\lambda-\lambda_1)^{-1/2}(1+s+\lambda)^{-\nu},\\
\xi_{\mu,\nu}^{(2),2}(s,\lambda)&=(1+\lambda_2-\lambda)^{-1/2}(1+\lambda-\lambda_1)^{-1/2}(1+\lambda)^{-\mu}(1+s+\lambda)^{-\nu},\\
\eta_{\mu,\nu}^{(2),1}(s,\lambda)&=\eta_{\mu,\nu}^{(1),1}(s,\lambda),\\
\eta_{\mu,\nu}^{(2),2}(s,\lambda)&=(1+\lambda_2-\lambda)^{-1/2}(1+\lambda-\lambda_1)^{-1/2}(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-\nu}.
\end{aligned}$$

We change the variables of integration by (3.15). Here, we let $c=0$ to adapt (3.15) for $p_{\mu,\nu}^{(1)}(s,\lambda)$ and $p_{\mu,\nu}^{(2)}(s,\lambda)$. Then we can prove

$$\int_{D_2^{(i)}} \xi_{\mu,\nu}^{(i),j}(s,\lambda)d\lambda ds \le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.40}$$

for $1\le\mu, 0<\nu$,

$$\int_{D_2^{(i)}} \eta_{\mu,\nu}^{(i),j}(s,\lambda)d\lambda ds \le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.41}$$

for $1\le\mu, 0<\nu, (i,j)\ne(1,1)$, and

$$\int_{D_2^{(1)}} \eta_{\mu,\nu}^{(1),1}(s,\lambda)d\lambda ds \le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.42}$$

for $1\le\mu, 0<\nu, c\ne 1$ or $1\le\mu, 1/2<\nu, c=1$. We show the estimate of the integral of $\xi_{\mu,\nu}^{(1),1}(s,\lambda)$ and $\eta_{\mu,\nu}^{(1),1}(s,\lambda)$ here. The others are easy to treat.

We first prove

$$\int_{\alpha_0}^{\alpha}(1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\tau}d\beta$$
$$\leq C\{(1+|\alpha_0|)^{1/2-\tau}+\chi(\alpha_0)(1+|\alpha_0|)^{-1/2}\Phi_{\tau-1}(\alpha)\}, \tag{3.43}$$

for $\tau \geq 1$ and $|t-r|<\alpha<t+r$, where χ is the characteristic function of the interval $(-\infty,0)$. Let $(a,b)\subset(\alpha_0,\alpha)$ to be the interval where $1+\beta-\alpha_0<1+|\beta|$. If $\alpha_0\geq 0$, then $(a,b)=(\alpha_0,\alpha)$, and if $\alpha_0<0$, then $(a,b)=(\alpha_0,\alpha_0/2)$. Integrating by parts,

$$\begin{aligned}
&\int_{a}^{b}(1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\tau}d\beta\\
&=\int_a^b \partial_\beta\{2(1+\beta-\alpha_0)^{1/2}\}(1+|\beta|)^{-\tau}d\beta\\
&\leq 2(1+|b|)^{1/2-\tau}+2\tau\int_a^b(1+|\beta|)^{-1/2-\tau}d\beta\\
&\leq C\{(1+|a|)^{1/2-\tau}+(1+|b|)^{1/2-\tau}\}\\
&\leq C(1+|\alpha_0|)^{1/2-\tau}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\alpha_0}^{\alpha}(1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\tau}d\beta\\
&=\int_a^b+\int_b^\alpha\\
&\leq C(1+|\alpha_0|)^{1/2-\tau}+C\chi(\alpha_0)\int_b^\alpha(1+|\alpha_0|)^{-1/2}(1+|\beta|)^{-\tau}d\beta,
\end{aligned}$$

and we obtain (3.43).

Since $\lambda-\lambda_1=2(1+c)^{-1}(\beta-\alpha_0)$ in $D_2^{(1)}$, it follows from (3.43) that

$$\begin{aligned}
&\int_{D_2^{(1)}}\xi_{\mu,\nu}^{(1),1}(s,\lambda)d\lambda ds\\
&\leq C\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}d\alpha\int_{\alpha_0}^{\alpha}(1+\beta-\alpha_0)^{-1/2}(1+\beta)^{-1/2-\mu}d\beta\\
&\leq C\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}(1+\alpha_0)^{-\mu}d\alpha\\
&\leq C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t),
\end{aligned}$$

which prove (3.40) for $i=j=1$. Concerning $\eta^{(1),1}_{\mu,\nu}(s,\lambda)$, we have

$$
\begin{aligned}
&\int_{D_2^{(1)}} \eta^{(1),1}_{\mu,\nu}(s,\lambda)d\lambda ds \\
&\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}d\alpha\int_{\alpha_0}^{\alpha}(1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\mu}d\beta \\
&\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}(1+|\alpha_0|)^{-1/2}d\alpha\Phi_{\mu-1}(t).
\end{aligned}
\tag{3.44}
$$

If $c=1$, we note that $\alpha_0=r-t$. If $c\neq 1$, let $h(\alpha)$ be a primitive of $(1+|\alpha_0|)^{-1/2}$. Integrating by parts, it follows that

$$
\begin{aligned}
&\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}(1+|\alpha_0|)^{-1/2}d\alpha \\
&\le (1+\alpha)^{-1/2-\nu}|h(\alpha)|\Big|_{\alpha=|t-r|,t+r}+(1/2+\nu)\int_{|t-r|}^{t+r}(1+\alpha)^{-3/2}|h(\alpha)|d\alpha \\
&\le C(1+|t-r|)^{-\nu},
\end{aligned}
$$

since $|h(\alpha)|\le C(1+\alpha)^{1/2}$. Hence we obtain (3.42).

3.2 An estimate for the quasilinear system with the null condition.

Proposition 3.2 *Let $u=(u^1,\dots,u^m)$ be the smooth solution of*

$$\Box_i u^i = F^i(\partial u,\partial^2 u) \quad \text{in } [0,T)\times\mathbf{R}^3, \tag{3.45}$$

$$u^i(0,\cdot)=\varepsilon f^i,\ \partial_t u^i(0,\cdot)=\varepsilon g^i \quad \text{in } \mathbf{R}^3, \tag{3.46}$$

$$i=1,\dots,m.$$

Assume that F^i satisfy (1.6) and c_i are different from each other. If $\varepsilon<1$ and $[\partial u]_{[(N+7)/2],t}<1$, then

$$[\partial u]_{N,t}\le C_N(\varepsilon+\|\partial u\|^6_{N+9,t}). \tag{3.47}$$

Proof. Let u_0^i be the solutions of the homogeneous equations

$$\Box_i u_0^i = 0 \quad \text{in } [0,\infty)\times\mathbf{R}^3, \tag{3.48}$$

$$u_0^i(0,\cdot)=\varepsilon f^i,\ \partial_t u_0^i(0,\cdot)=\varepsilon g^i \quad \text{in } \mathbf{R}^3, \tag{3.49}$$

$$i=1,\dots,m.$$

From (2.4), each $\Gamma^a u_0^i$ satisfies (3.48). Hence by the proof of Theorem 1 in [6] we have

$$
\begin{aligned}
|\Gamma^a u_0^i(t,x)| &\le C_N\varepsilon(1+r)^{-1}(1+|c_i t - r|)^{-1} \\
&\le C_N\varepsilon(1+t+r)^{-1}, && (3.50) \\
|\partial\Gamma^a u_0^i(t,x)| &\le C_N\varepsilon(1+r)^{-1}(1+|c_i t - r|)^{-1}, && (3.51)
\end{aligned}
$$

for $|a| \le N$.

Set

$$
u_1 = u - u_0. \tag{3.52}
$$

Then, each $\Gamma^a u_1^i$ satisfies the equation of the form

$$
\begin{aligned}
&\Box_i \Gamma^a u_1^i = \sum_{b\le a} C_{ab}\Gamma^b F^i(\partial u, \partial^2 u) \quad \text{in } [0,T)\times\mathbf{R}^3, && (3.53) \\
&\Gamma^a u_1^i(0,\cdot) = \partial_t \Gamma^a u_1^i(0,\cdot) = 0 \quad \text{in } \mathbf{R}^3. && (3.54)
\end{aligned}
$$

We apply Proposition 3.1 to a solution of (3.53)-(3.54) by replacing the weight $z_{\mu,\nu}(s,\lambda)$ with

$$
z(s,\lambda) = \min_{1\le i\le m+1} z_{1,1}^{(i)}(s,\lambda),
$$

where

$$
z_{\mu,\nu}^{(i)}(s,\lambda) = (1+|c_i s - \lambda|)^{\mu}(1+s+\lambda)^{\nu},\ c_{m+1} = 0. \tag{3.55}
$$

Then it follows from (3.5)-(3.7) that

$$
\begin{aligned}
&|\Gamma^a u_1^i(t,x)| \le C_N(1+t+r)^{-1}\{\log(2+t)\}^2 M_N(F^i), && (3.56) \\
&|\partial\Gamma^a u_1^i(t,x)| \le C_N(1+r)^{-1}(1+|c_i t - r|)^{-1}\log(2+t) M_{N+1}(F^i), && (3.57)
\end{aligned}
$$

for $|a| \le N$, where

$$
M_k(F^i) = \sum_{|a|\le k} \sup_{0\le s\le t}\sup_{y\in\mathbf{R}^3} |y| z(s,|y|)|\Gamma^a F^i(\partial u, \partial^2 u)(s,y)|.
$$

In order to estimate $M_k(F^i)$, we use the Sobolev inequality (Lemma 4.2 in [8]):

$$
|y||f(y)| \le C\{\sum_{|a|\le 2}\|\Omega^a f\|_{L^2(\mathbf{R}^3)} + \sum_{|a|\le 1}\|\partial_r\Omega^a f\|_{L^2(\mathbf{R}^3)}\}. \tag{3.58}
$$

If $|b|+|c|\le k$ and $0\le s\le t$, we have

$$|y|z(s,|y|)|\partial\Gamma^b u^j(s,y)||\partial\Gamma^c u^l(s,y)|\le C_k[\partial u]_{[k/2],t}\,\|\partial u\|_{k+2,t},$$

since $z(s,|y|)\le Cw_i(s,|y|)$ $(i=1,\ldots,m)$. Therefore,

$$M_k(F^i)\le C_k[\partial u]_{[(k+1)/2],t}\,\|\partial u\|_{k+3,t}, \tag{3.59}$$

provided $|\partial u|$, $|\partial^2 u|<1$. Therefore, it follows from (3.50), (3.51), (3.56), (3.57) and (3.59) that

$$\begin{aligned}
|\Gamma^a u^i(t,x)| &\le C_N(1+t+r)^{-1}\{\log(2+t)\}^2(\varepsilon+[\partial u]_{[(N+1)/2],t}\,\|\partial u\|_{N+3,t}), && (3.60)\\
|\partial\Gamma^a u^i(t,x)| &\le C_N(1+r)^{-1}(1+|c_it-r|)^{-1}\log(2+t)\cdot\\
&\quad\cdot(\varepsilon+[\partial u]_{[(N+2)/2],t}\,\|\partial u\|_{N+4,t}) && (3.61)
\end{aligned}$$

for $|a|\le N$.

Next, we estimate the nonlinear terms by making use of (3.60), (3.61). We separate F^i into three parts:

$$F^i(\partial u,\partial^2 u)=N^i(\partial u,\partial^2 u)+R^i(\partial u,\partial^2 u)+G^i(\partial u,\partial^2 u), \tag{3.62}$$

where

$$\begin{aligned}
N^i(\partial u,\partial^2 u) &= \sum_{0\le\alpha,\beta,\gamma\le 3} C^{iii}_{\alpha\beta\gamma}\partial_\gamma u^i\partial_\alpha\partial_\beta u^i+\sum_{0\le\alpha,\beta\le 3} D^{iii}_{\alpha\beta}\partial_\alpha u^i\partial_\beta u^i, && (3.63)\\
R^i(\partial u,\partial^2 u) &= \sum_{(j,k)\ne(i,i)}\left(\sum_{0\le\alpha,\beta,\gamma\le 3} C^{ijk}_{\alpha\beta\gamma}\partial_\gamma u^k\partial_\alpha\partial_\beta u^j+\sum_{0\le\alpha,\beta\le 3} D^{ijk}_{\alpha\beta}\partial_\alpha u^j\partial_\beta u^k\right), && (3.64)
\end{aligned}$$

and $G^i(\partial u,\partial^2 u)$ are higher order terms. Moreover, by the null condition (1.6), $N^i(\partial u,\partial^2 u)$ can be expressed in the form

$$\begin{aligned}
N^i(\partial u,\partial^2 u) &= \sum_{0\le\alpha\le 3} T^i_\alpha Q^i(u^i,\partial_\alpha u^i)+\sum_{0\le\alpha,\beta,\gamma\le 3} T^i_{\alpha\beta\gamma}Q_{\alpha\beta}(u^i,\partial_\gamma u^i)\\
&\quad+\sum_{0\le\alpha\le 3}\bar{T}^i_\alpha\partial_\alpha u^i\Box_i u^i+T^iQ^i(u^i,u^i), && (3.65)
\end{aligned}$$

where

$$Q^i(u,v) = \partial_t u \partial_t v - c_i^2 \nabla u \cdot \nabla v, \tag{3.66}$$

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \tag{3.67}$$

These forms gain good decay near $c_i t - r = 0$. Indeed, the following estimates hold for $|c_i t - r| < c_i t/2$:

$$\begin{aligned}
|Q^i(u,v)| &\leq C|c_i t - r|(1+t+r)^{-1}|\partial u||\partial v| \\
&\quad + C(1+t+r)^{-1}(|\Gamma u||\partial v| + |\partial u||\Gamma v|), &(3.68)\\
|Q_{\alpha\beta}(u,v)| &\leq C(1+t+r)^{-1}(|\partial u||\Gamma v| + |\Gamma u||\partial v|), &(3.69)\\
|\Box_i u| &\leq C|c_i t - r|(1+t+r)^{-1}|\partial^2 u| \\
&\quad + C(1+t+r)^{-1}(|\partial u| + |\partial \Gamma u|). &(3.70)
\end{aligned}$$

Let us give a proof of (3.68)-(3.70). Following [2], we define

$$S_i^\pm = \partial_t \pm c_i \partial_r. \tag{3.71}$$

Noting

$$S_i^+ u = t^{-1}(c_i t - r)\partial_r u + t^{-1} S u,$$

$$|r^{-1}\Omega u| \leq C|\nabla u|,$$

(3.68) follows from the identity

$$\begin{aligned}
Q^i(u,v) &= 2^{-1}(S_i^+ u S_i^- v + S_i^- u S_i^+ v) \\
&\quad - c_i^2 r^{-2}(\hat{x} \wedge \Omega u) \cdot (\hat{x} \wedge \Omega v), \hat{x} = x/r,
\end{aligned}$$

which is derived from (2.5). If we rewrite $Q_{\alpha\beta}(u,v)$ by using (2.5) and

$$\partial_t u = -r t^{-1} \partial_r u + t^{-1} S u,$$

we can prove (3.69). Finally, (3.70) is the consequence of

$$\Box_i u = S_i^+ S_i^- u - c_i^2(2r^{-1}\partial_r u + r^{-2}\Omega u \cdot \Omega u).$$

Hence, it follows from (3.65), (3.68)-(3.70) that

$$\begin{aligned}|\Gamma^a N^i(\partial u,\partial^2 u)| &\le C|c_it-r|(1+t+r)^{-1}\sum_{|b|+|c|\le|a|+1}|\partial\Gamma^b u^i||\partial\Gamma^c u^i|\\ &\quad +C(1+t+r)^{-1}\sum_{|b|+|c|\le|a|+2}|\Gamma^b u^i||\partial\Gamma^c u^i| \end{aligned}\tag{3.72}$$

for $|c_it-r|<c_it/2$. Here we have used Lemma 1.2 in [7]. Therefore, if $|c_it-r|<c_it/2$, the estimates (3.60), (3.61) and (3.72) yield

$$\begin{aligned}|\Gamma^a N^i(\partial u,\partial^2 u)| &\le C_N(1+t+r)^{-3}(1+|c_it-r|)^{-1}\{\log(2+t)\}^3\cdot\\ &\quad\cdot(\varepsilon+[\partial u]_{[(N+4)/2],t}\,\|\partial u\|^2_{N+6,t})\end{aligned}\tag{3.73}$$

for $|a|\le N$, since $\varepsilon<1$ and $[\partial u]_{[(N+4)/2],t}<1$. In case $|c_it-r|\ge c_it/2$, we find from (3.61) that

$$\begin{aligned}|\Gamma^a N^i(\partial u,\partial^2 u)| &\le C_N\sum_{|b|+|c|\le N+1}|\partial\Gamma^b u^i||\partial\Gamma^c u^i|\\ &\le C_N(1+r)^{-2}(1+t+r)^{-2}\{\log(2+t)\}^2\cdot\\ &\quad\cdot(\varepsilon+[\partial u]_{[(N+4)/2],t}\,\|\partial u\|^2_{N+6,t}).\end{aligned}\tag{3.74}$$

Similarly, by (3.61)

$$\begin{aligned}|\Gamma^a R^i(\partial u,\partial^2 u)| &\le C_N\sum_{(j,k)\ne(i,i)}\sum_{|b|+|c|\le N+1}|\partial\Gamma^b u^j||\partial\Gamma^c u^k|\\ &\le C_N\{(1+r)^{-2}(1+t+r)^{-2}\\ &\quad+\sum_{j\ne i}(1+t+r)^{-2}(1+|c_jt-r|)^{-2}\}\{\log(2+t)\}^2\cdot\\ &\quad\cdot(\varepsilon+[\partial u]_{[(N+4)/2],t}\,\|\partial u\|^2_{N+6,t}),\end{aligned}\tag{3.75}$$

$$\begin{aligned}|\Gamma^a G^i(\partial u,\partial^2 u)| &\le C_N\sum_{1\le j,k,l\le m}\sum_{|b|+|c|+|d|\le N+1}|\partial\Gamma^b u^j||\partial\Gamma^c u^k||\partial\Gamma^d u^l|\\ &\le C_N\{(1+r)^{-3}(1+t+r)^{-3}\\ &\quad+\sum_{1\le j\le m}(1+t+r)^{-3}(1+|c_jt-r|)^{-3}\}\{\log(2+t)\}^3\cdot\\ &\quad\cdot(\varepsilon+[\partial u]_{[(N+4)/2],t}\,\|\partial u\|^3_{N+6,t}),\end{aligned}\tag{3.76}$$

for $|a| \leq N$.

Therefore, it follows from (3.73)-(3.76) that

$$\begin{aligned}
&|\Gamma^a F^i(\partial u, \partial^2 u)| \\
&\leq C_N\{(1+t+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(t,r)^{-1} \\
&\quad +(1+t+r)^{-1}\sum_{j\neq i} z^{(j)}_{2,1-\rho}(t,r)^{-1} \\
&\quad +(1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(t,r)^{-1}\}(\varepsilon + [\partial u]_{[(N+4)/2],t}\ \|\partial u\|^3_{N+6,t}) \qquad (3.77)
\end{aligned}$$

for $0<\gamma<1, 0<\kappa<1-\gamma, 0<\rho<1$ and $|a|\leq N$. (3.77) give estimates of the right-hand side of the equations (3.53), and hence by Proposition 3.1 and (3.51),

$$|\partial\Gamma^a u^i(t,x)| \leq C_N(1+r)^{-1}(1+|c_i t - r|)^{-1+\rho}(\varepsilon + [\partial u]_{[(N+5)/2],t}\ \|\partial u\|^3_{N+7,t}). \qquad (3.78)$$

Using (3.78), we estimate $\Gamma^a R^i(\partial u, \partial^2 u)$ again. Then

$$\begin{aligned}
|\Gamma^a R^i(\partial u, \partial^2 u)| \ \leq\ & C_N\{(1+r)^{-2}(1+t+r)^{-2+2\rho} \\
& +\sum_{j\neq i}(1+t+r)^{-2}(1+|c_j t - r|)^{-2+2\rho}\}\ \cdot \\
& \cdot(\varepsilon + [\partial u]_{[(N+6)/2],t}\ \|\partial u\|^6_{N+8,t}), \qquad (3.79)
\end{aligned}$$

for $|a|\leq N$. We take $\rho<1/2$ and replace the estimate (3.75) with (3.79). Then we have

$$\begin{aligned}
&|\Gamma^a F^i(\partial u, \partial^2 u)| \\
&\leq C_N\{(1+t+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(t,r)^{-1} \\
&\quad +(1+t+r)^{-1}\sum_{j\neq i} z^{(j)}_{1+\gamma,1}(t,r)^{-1} \\
&\quad +(1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(t,r)^{-1}\}(\varepsilon + [\partial u]_{[(N+6)/2],t}\ \|\partial u\|^6_{N+8,t}) \qquad (3.80)
\end{aligned}$$

for some $0<\gamma,\ \kappa<1$. Then, applying Proposition 3.1 again, we have

$$|\partial\Gamma^a u^i(t,x)| \leq C_N(1+r)^{-1}(1+|c_i t - r|)^{-1}(\varepsilon + [\partial u]_{[(N+7)/2],t}\ \|\partial u\|^6_{N+9,t}). \qquad (3.81)$$

Consequently we have finished the proof.

4 Energy estimates.

Proposition 4.1 *Let u be the solution of (3.45)-(3.46). Assume that F^i satisfy (1.6) and c_i are different from each other. Assume moreover that $C^{ij}_{\alpha\beta}(\partial u)$ satisfy (1.7) and $[\partial u]_{[(N+10)/2],t} \leq \omega_N$, where $\omega_N \leq 1$ is a small number depending on N and given functions. Then,*

$$\|\partial u\|_{N,t} \leq C_N \varepsilon. \tag{4.1}$$

The proof of Theorem 1.1 is now easy. Uniqueness is proved by the application of the method in John [4], Appendix. Since the Cauchy problem (1.1)-(1.2) can be converted to a Cauchy problem for a symmetric hyperbolic system, we know the local existence of the smooth solution to (1.1)-(1.2).

Let $N \geq 10$. Fix $C_N \gg 1$ in (3.47) and (4.1), and set $\varepsilon_0 = \omega_N/2C_N$. According to Kato [5] and Majda [10] Chapter 2, we can take $T > 0$ so that there exists a solution to (1.1)-(1.2) in $[0,T] \times \mathbf{R}^3$ with $[\partial u]_{[(N+10)/2],T} \leq 2C_N\varepsilon$. Suppose that $T_* < \infty$ is the maximal of such T. Then, for arbitrary $t < T_*$, $[\partial u]_{[(N+10)/2],t} \leq \omega_N$. Therefore by Proposition 3.2 and Proposition 4.1,

$$\begin{aligned}
[\partial u]_{N,t} &\leq C_N\varepsilon + C_N^6\varepsilon^6 \\
&\leq C_N\varepsilon + C_N\varepsilon 2^{-5} \\
&\leq (3/2)C_N\varepsilon.
\end{aligned}$$

This contradicts the maximality of T_*, and hence $T_* = \infty$. This proves the Theorem 1.1.

Proof of Proposition 4.1. If $v = (v^1, \ldots, v^m)$ satisfies

$$\sum_{0\leq\alpha,\beta\leq 3} \sum_{1\leq j\leq m} a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta v^j = b^i,\ i = 1, \ldots, m \tag{4.2}$$

with

$$a^{ij}_{\alpha\beta} = a^{ij}_{\beta\alpha} = a^{ji}_{\alpha\beta}, \tag{4.3}$$

then we have the energy identity

$$\sum_{0\le\alpha,\beta\le 3}\sum_{1\le i,j\le m}\{\partial_\alpha(a^{ij}_{\alpha\beta}\partial_0 v^i\partial_\beta v^j)-\partial_\alpha a^{ij}_{\alpha\beta}\partial_0 v^i\partial^j_\beta-2^{-1}\partial_0(a^{ij}_{\alpha\beta}\partial^i_\alpha\partial^j_\beta)+2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha v^i\partial_\beta v^j\}$$
$$=\sum_{1\le i\le m} b^i\partial_0 v^i, \tag{4.4}$$

by multiplying both side of (4.2) by $\partial_0 v^i$. Integrating (4.4) on $[0,t]\times\mathbf{R}^3$, we obtain

$$2^{-1}\int_{\mathbf{R}^3}\langle\partial v(t),\partial v(t)\rangle dx-2^{-1}\int_{\mathbf{R}^3}\langle\partial v(0),\partial v(0)\rangle dx$$
$$=\int_0^t ds\int_{\mathbf{R}^3}\{\sum_{0\le\alpha,\beta\le 3}\sum_{1\le i,j\le m}(\partial_\alpha a^{ij}_{\alpha\beta}\partial_0 v^i\partial_\beta v^j-2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha v^i\partial_\beta v^j)+\sum_{1\le i\le m}b^i\partial_0 v^i\}dx, \tag{4.5}$$

where

$$\langle\partial v,\partial w\rangle=\sum_{1\le i,j\le m}(a^{ij}_{00}\partial_0 v^i\partial_0 w^j-\sum_{1\le k,l\le 3}a^{ij}_{kl}\partial_k v^i\partial_l w^j). \tag{4.6}$$

We first set

$$a^{ij}_{\alpha\beta} = \eta^i_{\alpha\beta}\delta^{ij}-C^{ij}_{\alpha\beta}(\partial u), \tag{4.7}$$
$$b^i = \sum_{1\le j\le m}\sum_{0\le\alpha,\beta\le 3}\{a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta u^j)\}+\Gamma^a D^i(\partial u), \tag{4.8}$$

where

$$(\eta^i_{\alpha\beta})_{0\le\alpha,\beta\le 3}=\begin{pmatrix}1&0&0&0\\0&-c_i^2&0&0\\0&0&-c_i^2&0\\0&0&0&-c_i^2\end{pmatrix}. \tag{4.9}$$

Then each $a^{ij}_{\alpha\beta}$ satisfies (4.3), and $\Gamma^a u$ is a solution of (4.2). Therefore, it follows from (4.5) that

$$2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle dx-2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(0),\partial\Gamma^a u(0)\rangle dx$$
$$=\int_0^t ds\int_{\mathbf{R}^3}\{\sum_{0\le\alpha,\beta\le 3}\sum_{1\le i,j\le m}(\partial_\alpha a^{ij}_{\alpha\beta}\partial_0\Gamma^a u^i\partial_\beta\Gamma^a u^j-2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^j)$$
$$+\sum_{1\le i\le m}b^i\partial_0\Gamma^a u^i\}dx.$$

Noting

$$\sum_{1\le j\le m}\sum_{0\le\alpha,\beta\le 3}\{a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta u^j)\}$$
$$=[\Box_i,\Gamma^a]u^i-\sum_{1\le j,k\le m}\sum_{0\le\alpha,\beta\le 3}\{C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^j\},$$

we obtain

$$\begin{aligned}&2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle dx-2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(0),\partial\Gamma^a u(0)\rangle dx\\ &\le C_N\sum_{\substack{|b|+|c|\le N+1\\ |b|,|c|\ne 0}}\sum_{1\le i,j,k\le m}\int_0^t ds\int_{\mathbf{R}^3}|\Gamma^a\partial u^i||\Gamma^b\partial u^j||\Gamma^c\partial u^k|dx\\ &\le C_N[\partial u]_{[(N+1)/2],t}\int_0^t(1+s)^{-1}\,\|\partial u(s)\|_N^2\,ds\end{aligned}$$

for $|a|\le N$. Since

$$\sum_{|a|\le N}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle dx\ge C_N\,\|\partial u(t)\|_N^2$$

if $[\partial u]_{0,t}$ is small, it follows that

$$\|\partial u(t)\|_N^2\le C_N\{\varepsilon^2+[\partial u]_{[(N+1)/2],t}\int_0^t(1+s)^{-1}\,\|\partial u(s)\|_N^2\,ds\}.$$

Hence, by Gronwall's lemma,

$$\|\partial u(t)\|_N^2\le C_N\varepsilon^2(1+t)^{C_N[\partial u]_{[(N+1)/2],t}}. \tag{4.10}$$

Next, we set

$$a^{ij}_{\alpha\beta}=\eta^i_{\alpha\beta}\delta^{ij}, \tag{4.11}$$
$$b^i=[\Box_i,\Gamma^a]u^i+\Gamma^a F^i(\partial u,\partial^2 u). \tag{4.12}$$

Then by (4.5), we have

$$\|\partial u(t)\|_N^2\le C_N(\varepsilon^2+\sum_{|a|,|b|\le N}\sum_{1\le i\le m}\int_0^t ds\int_{\mathbf{R}^3}|\Gamma^b F^i(\partial u,\partial^2 u)||\partial_0\Gamma^a u^i|dx). \tag{4.13}$$

Using (3.80) and (3.81), we have

$$|\Gamma^b F^i(\partial u,\partial^2 u)||\partial_0\Gamma^a u^i|$$

$$
\begin{aligned}
&\le C_N\{(1+s+r)^{-1}z^{(i)}_{1+\gamma,1+\kappa}(s,r)^{-1}\\
&\qquad +(1+s+r)^{-1}\sum_{j\neq i} z^{(j)}_{1+\gamma,1}(s,r)^{-1}\\
&\qquad +(1+r)^{-1}z^{(m+1)}_{1+\gamma,1+\kappa}(s,r)^{-1}\}(1+r)^{-1}(1+|c_i s-r|)^{-1}\cdot\\
&\quad \cdot(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\,\|\partial u\|^{12}_{N+9,s})\\
&\le C_N\{(1+s+r)^{-3-\kappa}(1+|c_i s-r|)^{-2-\gamma}\\
&\qquad +(1+s+r)^{-4}\sum_{j\neq i}(1+|c_j s-r|)^{-1-\gamma}\\
&\qquad +(1+r)^{-3-\gamma}(1+s+r)^{-2-\kappa}\}(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\,\|\partial u\|^{12}_{N+9,s})\\
&\le C_N(1+s)^{-1-\kappa}\sum_{j=1}^{m+1}(1+|c_j s-r|)^{-1-\gamma}r^{-2}(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\,\|\partial u\|^{12}_{N+9,s}). \quad (4.14)
\end{aligned}
$$

Moreover, by (4.10) and (4.14) it follows that

$$
\begin{aligned}
&|\Gamma^b F^i(\partial u,\partial^2 u)||\partial_0\Gamma^a u^i|\\
&\le C_N(1+s)^{-1-\kappa}\sum_{j=1}^{m+1}(1+|c_j s-r|)^{-1-\gamma}r^{-2}(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\varepsilon^{12}(1+s)^{C_N[\partial u]_{[(N+10)/2],s}})\\
&\le C_N\varepsilon^2(1+[\partial u]_{[(N+7)/2],s})(1+s)^{-1-\kappa+C_N[\partial u]_{[(N+10)/2],s}}\sum_{j=1}^{m+1}(1+|c_j s-r|)^{-1-\gamma}r^{-2}. \quad (4.15)
\end{aligned}
$$

Therefore, if $[\partial u]_{[(N+10)/2],t}\le 2^{-1}C_N^{-1}\kappa$, we obtain (4.1) from (4.13) and (4.15).

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