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Generalized Numerical Radius And Unitary ρ-Dilation

by

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Abstract. In this paper, we study an operator A on a Hilbert space H which satisfies one of the following inequalities :
For some λ with $0 \leq \lambda \leq 1$

$$|(Ay, y)| \leq \lambda\|y\|^2 + (1-\lambda)\|Ay\|^2 \quad (y \in H)$$

or

$$\lambda\|Ay\|^2 + (1-\lambda)|(Ay, y)| \leq \|y\|^2 \quad (y \in H).$$

These two inequalities can be regarded as special cases of generalized numerical ranges. If A has a ρ-dilation with $\rho > 0$, then it satisfies one of them. We show that the operator radii $w_\rho(A)$ of A are calculated using $|(Ay, y)|$ and $\|Ay\|$. Several applications are given.

§1. Introduction

According to Sz.-Nagy and Foias [7], a bounded linear operator A on a complex Hilbert space H is said to be of class $\mathcal{C}_\rho$ with $\rho > 0$ if there exists a unitary operator U on some Hilbert space K such that K contains H as a subspace and such that

$$A^n = \rho P_H U^n | H \quad \text{for } n = 1, 2, \cdots,$$

where P_H is the orthogonal projection of K onto H. For $\rho = 2$ it is known (see [7, Chapter I, Proposition 11.2]) that A is of class $\mathcal{C}_2$ if and only if its numerical radius of A :

$$w(T) = \sup\{|(Ay, y)| : \|y\| \leq 1\}$$

is not greater than one. Of course, A is of class $\mathcal{C}_1$ if and only if the usual norm $\|A\|$ of A is not greater than one. Holbrook [3] introduced the operator radii $w_\rho(A)$ of an operator A, relative to $\mathcal{C}_\rho$, by the formula :

$$w_\rho(A) = \inf\{\gamma\ ;\ \gamma > 0, \gamma^{-1}A \in \mathcal{C}_\rho\}.$$

Then $w_1(A) = \|A\|$, $w_2(A) = w(A)$, and $\lim_{\rho\to\infty} w_\rho(A) = r(A)$: the spectral radius of A.

Let S be a positive bounded operator on H. We define $V_S^+(A)$ and $v_S(A)$ as

$$V_S^+(A) = \{(Ay, y)\ ;\ y \in H, (Sy, y) = 1\}.$$

and

$$v_S(A) = \sup\{|(Ay, y)|\ ;\ y \in H, (Sy, y) = 1\}.$$

$V_S^+(A)$ is defined and studied in [4] for a self-adjoint S. If S is the identity operator I on H, then $V_S^+(A)$ is the numerical range of A and $v_S(A) = w(A)$. In Section 2, when $0 < \rho \leq 2$ and $\rho \neq 1$, we show that A is of class $\mathcal{C}_\rho$ if and only if

$$S = \frac{\rho}{2|\rho - 1|} I + \frac{\rho - 2}{2|\rho - 1|} |A|^2$$

is nonnegative and $v_S(A) \leq 1$. As results, several corollaries are given. In Section 3, when $0 < \rho \leq 2$ and $\rho \neq 1$, we give two formulae for $w_\rho(A)$ using $|(Ay, y)|$ and $\|Ay\|$. In Section 4, we try to generalize results in Sections 2 and 3 for $2 < \rho < \infty$.

A result in this paper is the following : For $0 < \rho \leq 2$ and $\rho = 2/(\lambda + 1)$,

$$\sup_{\|y\|=1} \{\lambda \|Ay\|^2 + |1 - \lambda| \cdot |(Ay, y)|\} \leq 1$$

if and only if $w_\rho(A) \leq 1$ (see Corollay 2). This shows the following as $A/w_\rho(A)$. If $w_\rho(A) \leq 1$, then

$$\sup_{\|y\|=1} \{\lambda \|Ay\|^2 + |1 - \lambda| \cdot |(Ay, y)|\} \leq w_\rho(A)$$

and if $w_\rho(A) \geq 1$, then

$$\sup_{\|y\|=1} \{\lambda\|Ay\|^2 + |1-\lambda| \cdot |(Ay,y)|\}^{1/2} \leq w_\rho(A).$$

$w_\rho(A)$ is calculated using $|(Ay,y)|$ and $\|Ay\|$ (see (1) of Theorem 2). This shows the following general inequality (see Corollary 3). For any $w_\rho(A)$,

$$w_\rho(A) \leq \sup_{\|y\|=1} \{\sqrt{\lambda}\|Ay\| + |1-\lambda| \cdot |(Ay,y)|\}.$$

§2. ρ-dilation for $0 < \rho \leq 2$ and generalized numerical radius

In this section, we are interested in operators with $v_S(A) \leq 1$ when S is a special positive operator. If $S = |A|$ and $v_S(A) \leq 1$, then A is normal (*cf.* [2],[8]). We consider A when $S = \lambda I + \mu|A|^2, \lambda + \mu = \pm 1$ and $v_S(A) \leq 1$ where $\lambda \geq 0$ and μ are constants.

For $0 \leq \mu < \infty$

$$\mathbf{w}_\mu(A) = \sup\{\mu\|Ay\|^2 + |1-\mu| \cdot (Ay,y)| \ ; \ \|y\| = 1\}.$$

Then $\mathbf{w}_0(A) = w(A)$ and $\mathbf{w}_1(A) = \|A\|^2$.

Theorem 1. *Suppose* $0 < \rho \leq 2$ *and* $\rho \neq 1$. *Then* A *is of class* C_ρ *if and only if* $S \geq 0$ *and* $v_S(A) \leq 1$ *where* $S = (\rho I + (\rho - 2)|A|^2)/2|\rho - 1|$.

Proof. It is known that A is of class C_ρ if and only if

$$\|y\|^2 + \left(1 - \frac{2}{\rho}\right)|\zeta|^2\|Ay\|^2 - 2\left(1 - \frac{1}{\rho}\right) Re\zeta(Ay,y) \geq 0$$

for $\zeta \in D$ and $y \in H$ where $D = \{z \in \mathbf{C} : |z| < 1\}$. This inequality is equivalent to

$$Re\zeta(Ay,y) \leq \frac{\rho}{2|\rho-1|}\|y\|^2 + \frac{\rho-2}{2|\rho-1|}|\zeta|^2\|Ay\|^2$$

for $\zeta \in D$ and $y \in H$, and

$$|\zeta|(Ay,y)| \leq \frac{\rho}{2|\rho-1|}\|y\|^2 + \frac{\rho-2}{2|\rho-1|}|\zeta|^2\|Ay\|^2$$

for $\zeta \in D$ and $y \in H$. Since $0 < \rho \leq 2$, the last inequality is equivalent to

$$|(Ay,y)| \leq \frac{\rho}{2|\rho-1|}\|y\|^2 + \frac{\rho-2}{2|\rho-1|}\|Ay\|^2$$

for $y \in H$.

Corollary 1. *Suppose $0 < \rho < 2$ and $\rho \neq 1$. A is of class C_ρ if and only if A admits a factorization : $A = S^{1/2}BS^{1/2}$ where $S = (\rho I + (\rho - 2)|A|^2)/2|\rho - 1|$ and $w(B) \leq 1$.*

Proof. Suppose A is of class C_ρ. If there exists $y \in H$ such that $Sy = 0$, then $|A|^2 y = \rho y/(2-\rho)$. Since $\rho w_\rho(A) \geq \|A\|$ [1],

$$1 \geq w_\rho(A) \geq \frac{\|A\|}{\rho} \geq \frac{1}{\sqrt{\rho(2-\rho)}}.$$

and hence $\rho = 1$. This contradiction implies that $\ker S = \{0\}$. Since $\ker S = \{0\}, v_S(A) \leq 1$ if and only if

$$|(S^{-1/2}AS^{-1/2}x, x)| \leq (x, x) \quad (x \in H).$$

Let $B = S^{-1/2}AS^{-1/2}$, then this inequality is equivalent to that

$$A = S^{1/2}BS^{1/2} \text{ and } w(B) \leq 1.$$

Now Theorem 1 implies the corollary.

Corollary 2. *Let A be a bounded linear operator on H and $0 < \rho = \dfrac{2}{\mu + 1} \leq 2$. Then the following conditions are mutually equivalent :*

(1) $w_\rho(A) \leq 1$.

(2) $\mathbf{w}_\mu(A) \leq 1$

and

(3) $w(\mu|A|^2 + |1-\mu|e^{i\theta}A) \leq 1$

for any $\theta \in R$.

Proof. By Theorem 1, $w_\rho(A) \leq 1$ if and only if

$$\frac{2-\rho}{2|\rho-1|}\|Ay\|^2 + |(Ay, y)| \leq \frac{\rho}{2|\rho-1|}\|y\|^2$$

for $y \in H$. The last inequality is equivalent to

$$\mu\|Ay\|^2 + |1-\mu| \cdot |(Ay, y)| \leq \|y\|^2$$

where $\mu = (2-\rho)/\rho$. This implies the equivalence of (1) and (2). The equivalence of (2) and (3) is trivial.

(1),(2) of Corollary 2 implies that if $w_\rho(A) \leq 1$ then $\mathbf{w}_\mu(A) \leq w_\rho(A)$ and if $w_\rho(A) \geq 1$ then $\mathbf{w}_\mu(A)^{1/2} \leq w_\rho(A)$. By (1),(3) of Corollary 2, if $\mu\|A\|^2 + |1-\mu|w(A) \leq 1$

then $w_\rho(A) \leq 1$. However the converse is not true. For if $A = \begin{bmatrix} 0 & \rho \\ 0 & 0 \end{bmatrix}$, then $\|A\| = \rho, w(A) = \rho/2$ and $w_\rho(A) = 1$. Hence $\mu = (2-\rho)/\rho$ and

$$\mu\|A\|^2 + |1-\mu|w(A) = \frac{2-\rho}{\rho} \times \rho^2 + \frac{2|\rho-1|}{\rho} \times \frac{\rho}{2} > 1.$$

§3. Operator radii for $0 < \rho \leq 2$

In this section, we give two exact formulae and useful estimates for $w_\rho(A)$ when $0 < \rho \leq 2$ and $\rho \neq 1$. Put

$$D = D(A, \rho, y) = |(Ay, y)|^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}\|Ay\|^2$$

for a bounded linear operator $A, \rho > 0$ and y in H.

Theorem 2. *Let A be a bounded linear operator on H.*
(1) If $0 < \rho \leq 2$ and $\rho \neq 1$, then

$$w_\rho(A) = \frac{|\rho-1|}{\rho} \sup_{\|y\|=1} \{|(Ay, y)| + \sqrt{D}\}.$$

(2) If $0 < \rho \leq 2$, and $\rho \neq 1$ then

$$w_\rho(A) = \frac{2}{\rho} \sup_{\|y\|=1} \sup_{0 \leq t \leq 1} \{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho-1| \cdot |(Ay, y)|t\}.$$

Proof. (1) By Theorem 1, if $t \geq w_\rho(A)$ then $\lambda\|y\|^2t^2 - |(Ay, y)|t + (1-\lambda)\|Ay\|^2 \geq 0$ for $y \in H$ where $\lambda = \rho/2|\rho-1|$. Hence

$$w_\rho(A) \leq \frac{|(Ay, y)| - \sqrt{D}}{2\lambda\|y\|^2}$$

or

$$w_\rho(A) \geq \frac{|(Ay, y)| + \sqrt{D}}{2\lambda\|y\|^2}.$$

Put

$$t_0 = \sup_{y \neq 0} \frac{|(Ay, y)| + \sqrt{D}}{2\lambda\|y\|^2}.$$

If $t \geq t_0$, then

$$\lambda\|y\|^2t^2 - |(Ay,y)|t + (1-\lambda)\|Ay\|^2 \geq 0.$$

for $y \in H$ and so by Theorem 1 $w_\rho(A) \leq t_0$. When $0 < \rho \leq 2, |(Ay,y)| - \sqrt{D} \leq 0$ and so $w_\rho(A) \geq t_0$. Thus $w_\rho(A) = t_0$.

(2) Put $g(t,y) = \sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho-1| \cdot |(Ay,y)|t$ for each y with $\|y\| = 1$. Then

$$\begin{aligned} & 2\left(\frac{d}{dt}g(t,y)\right)\sqrt{t(1-t)} \\ = \ & \sqrt{\rho(2-\rho)}\|Ay\|(1-2t) + 2|\rho-1|\cdot|(Ay,y)|\sqrt{t(1-t)}. \end{aligned}$$

Hence $\frac{d}{dt}g(t,y)|_{t=t_0} = 0$ and $0 \leq t_0 \leq 1$ if and only if

$$t_0 = \frac{1}{2} + \frac{|(Ay,y)|}{2\sqrt{D}}$$

and so

$$\sqrt{t_0(1-t_0)} = \frac{\sqrt{\rho(2-\rho)}\|Ay\|}{2|\rho-1|\sqrt{D}}.$$

Therefore

$$\begin{aligned} & \frac{2}{\rho}\sup_{\|y\|=1}\sup_{0\leq t\leq 1}\{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho-1|\cdot|(Ay,y)|t\} \\ & = \frac{2}{\rho}\sup_{\|y\|=1}\left\{\sqrt{\rho(2-\rho)}\|Ay\| \times \frac{\sqrt{\rho(2-\rho)}\|Ay\|}{2|\rho-1|\cdot\sqrt{D}} + |\rho-1|\cdot|(Ay,y)| \times \left(\frac{1}{2} + \frac{|(Ay,y)|}{2\sqrt{D}}\right)\right\} \\ & = \frac{|\rho-1|}{\rho}\sup_{\|y\|=1}\{|(Ay,y)| + \sqrt{D}\}. \quad \square \end{aligned}$$

Corollary 3. *If* $0 < \rho \leq 2$, *then*

$$w_\rho(A) \leq \sup_{\|y\|=1}\{\sqrt{\frac{2-\rho}{\rho}}\|Ay\| + 2\left|1-\frac{1}{\rho}\right|\cdot|(Ay,y)|\}.$$

Proof. Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, (a,b > 0)$ for $\rho \neq 1$

$$\sqrt{D} \leq |(Ay,y)| + \frac{\sqrt{\rho(2-\rho)}}{|\rho-1|}\|Ay\|.$$

This inequality and (1) of Theorem 2 imply the corollary.

Corollary 4. *If* $0 < \rho \le 2$, *then*

$$\max\left\{2\left|1-\frac{1}{\rho}\right|w(A), \sqrt{\frac{2-\rho}{\rho}}\|A\|\right\} \le w_\rho(A) \le 2\left|1-\frac{1}{\rho}\right|w(A) + \sqrt{\frac{2-\rho}{\rho}}\|A\|$$

Proof. Since $-\rho(\rho-2)\|Ay\|^2/(\rho-1)^2 \ge 0$. by (1) of Theorem 2

$$w_\rho(A) \ge 2\frac{|\rho-1|}{\rho}|(Ay,y)|, \ \frac{\sqrt{\rho(2-\rho)}}{\rho}\|Ay\|$$

for $y \in H$. Hence

$$w_\rho(A) \ge \max\left\{2\left|1-\frac{1}{\rho}\right|w(A), \sqrt{\frac{2-\rho}{\rho}}\|A\|\right\}.$$

We can get the upper estimate of $w_\rho(A)$ using Corollary 3.

Corollary 5. *If* $0 < \rho \le 2$, *and* $\rho \ne 1$ *then*

$$\begin{aligned} w_\rho(A) &\le \frac{2}{\rho}\sup_{0\le t\le 1}\{\sqrt{\rho(2-\rho)}\|A\|\sqrt{t(1-t)} + |\rho-1|\cdot w(A)t\} \\ &= \frac{|\rho-1|}{\rho}\left\{w(A) + \sqrt{w(A)^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}\|A\|^2}\right\} \end{aligned}$$

and for any $\|y\| = 1$ and $0 \le t \le 1$

$$w_\rho(A) \ge \frac{2}{\rho}\{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho-1|\cdot|(Ay,y)|t\}.$$

Corollary 6. [5] *If* $0 < \rho \le 2$, *then*

$$\rho w_\rho(A) = 2w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)} & A \\ 0 & (1-\rho) & A \end{bmatrix}\right).$$

Proof. For $0 < \rho \le 2$,

$$\begin{aligned} &2w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)} & A \\ 0 & (1-\rho) & A \end{bmatrix}\right) \\ &= 2\sup_{\|x\|^2+\|z\|^2=1}|\sqrt{\rho(2-\rho)}(Az,x) + (1-\rho)(Az,z)| \\ &= 2\sup_{\|x\|^2+\|z\|^2=1}\{\sqrt{\rho(2-\rho)}|(Az,x)| + |1-\rho|\cdot|(Az,z)|\} \\ &= 2\sup_{0\le\lambda\le 1}\sup_{\|z\|=\sqrt{\lambda}}\{\sqrt{\rho(2-\rho)}\|Az\|\sqrt{1-\lambda} + |1-\rho|\cdot|(Az,z)|\} \\ &= 2\sup_{0\le\lambda\le 1}\sup_{\|y\|=1}\{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{\lambda(1-\lambda)} + |1-\rho|\cdot|(Ay,y)|\sqrt{\lambda}\} \\ &= \rho\, w_\rho(A). \end{aligned}$$

We used (2) of Theorem 2 to show the last eqnality. □

§4. The case of $2 < \rho < \infty$

In this section, we consider Theorems 1 and 2 for $2 < \rho < \infty$. Unfortunately the results for $2 < \rho < \infty$ are more complicated than those for $0 < \rho \leq 2$.

For $0 \leq \lambda \leq 1$, put

$$\mathrm{w}'_\lambda(A) = \sup\{|(Ay,y)| \ ; \ \lambda\|y\|^2 + (1-\lambda)\|Ay\|^2 \leq 1\}.$$

Then $\mathrm{w}'_1(A) = w(A)$, and $\mathrm{w}'_0(A) = w(A^{-1})$ if A is invertible.

Proposition 3. *Suppose $2 < \rho < \infty$ and for $0 < t \leq 1$*

$$S_t = \frac{1}{t}\,\frac{\rho}{2|\rho-1|}I + t\,\frac{\rho-2}{|\rho-1|}\,|A|^2.$$

A is of class C_ρ if and only if $S_t \geq 0$ and $v_{S_t}(A) \leq 1$ for $0 < t \leq 1$.

Proof is almost same to that of Theorem 1.

Using Proposition 3 we can show a version of Corollary 1 for $2 < \rho < \infty$. For $0 < \lambda < 1$ and arbitrary bounded operator A on H,

$$\mathrm{w}'_\lambda(A) \leq 1/\sqrt{\lambda(1-\lambda)}.$$

In fact, for any constant $t > 0$

$$|(Ay,y)| \leq (\sqrt{t}\|y\|)(\frac{1}{\sqrt{t}}\|Ay\|) \leq 2t\|y\|^2 + \frac{1}{2t}\|Ay\|^2.$$

Assuming $\lambda \neq 0,1$, if $k \geq 1/\sqrt{\lambda(1-\lambda)}$ then $k\lambda \geq 1/k(1-\lambda)$. Hence if $k\lambda = 2t$ then $k(1-\lambda) \geq 1/2t$. Therefore if $k \geq 1/\sqrt{\lambda(1-\lambda)}$, then

$$|(Ay,y)| \leq k(\lambda|y\|^2 + (1-\lambda)\|Ay\|^2)$$

and so $\mathrm{w}'_\lambda \leq 1/\sqrt{\lambda(1-\lambda)}$.

$\mathrm{w}'_0(A) < \infty$ if and only if there exists a bounded operator B such that $BA = P$ where P is an orthogonal projection to $(\ker A)^\perp$. For if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is the matrix of A with respect to the decomposition $H = (\ker A)^{\perp} \oplus \ker A$, then $A_{12} = A_{22} = 0$. If $k = \mathrm{w}'_0(A) < \infty$, then for $z = (x, y) \in (\ker A)^{\perp} \oplus \ker A$

$$|(A_{11}x, x) + (A_{21}x, y)| \leq k(\|A_{11}x\|^2 + \|A_{21}x\|^2).$$

This implies $A_{21} = 0, A_{11}$ is one to one on $(\ker A)^{\perp}$ and $|(A_{11}x, x)| \leq k\|A_{11}x\|^2$. If $B_{11}A_{11}x = x$, then by the inequality above B_{11} is bounded and so $B = B_{11} \oplus O$ satisfies $BA = P$. The proof is reversible.

Corollary 7. *Let A be a bounded linear operator and let $0 \leq \lambda \leq 1$.*

(1) Suppose $\frac{1}{2} < \lambda \leq 1$ and $\rho = 2\lambda/(2\lambda - 1) \geq 2$. Then $\mathrm{w}'_\lambda(tA) \leq 1$ for any $0 < t \leq 1$ and if and only if $w_\rho(A) \leq 1$

(2) Suppose $0 \leq \lambda < \frac{1}{2}, \rho = 2(\lambda - 1)/(2\lambda - 1) \geq 2$ and A is invertible. Then $\mathrm{w}'_\lambda(tA) \leq 1$ for any $t \geq 1$ if and only if $w_\rho(A^{-1}) \leq 1$.

Proof. It is clear by Proposition 3. □

Proposition 4. *Let A be a bounded linear operator on H*

(1) If $2 < \rho < \infty$, then

$$w_\rho(A) = \frac{\rho - 1}{\rho} \sup_{\|y\|=1} \{|(Ay, y)| + \sqrt{D}\ ;\ D \geq 0\}.$$

(2) If $2 < \rho < \infty$, then

$$w_\rho(A) = \frac{2}{\rho} \sup_{\|y\|=1} \inf_{t \geq 1} \{-\sqrt{\rho(\rho - 2)}\|Ay\|\sqrt{t(t-1)} + (\rho - 1)|(Ay, y)|t\ ;\ D \geq 0\}.$$

Proof. (1) The proof is almost same to that of (1) of Theorem 2.

(2) Put $f(t, y) = -\sqrt{\rho(\rho - 2)}\|Ay\|\sqrt{t(t-1)} + (\rho - 1)|(Ay, y)|t$ for each y with $\|y\| = 1$. Then

$$\begin{aligned} &2\left(\frac{d}{dt}f(t, y)\right)\sqrt{t(t-1)} \\ &\quad = -\sqrt{\rho(\rho - 2)}\|Ay\|(2t - 1) + 2(\rho - 1)|(Ay, y)|\sqrt{t(t-1)}. \end{aligned}$$

Hence $\frac{d}{dt}f(t, y)|_{t=t_0} = 0$ and $t_0 \geq 1$ if and only if

$$t_0 = \frac{1}{2} + \frac{|(Ay, y)|}{2\sqrt{D}}$$

and so

$$\sqrt{t_0(t_0-1)} = \frac{\sqrt{\rho(\rho-2)}\|Ay\|}{2(\rho-1)\sqrt{D}}$$

Therefore

$$\frac{2}{\rho}\sup_{\substack{\|y\|=1\\ D\geq 0}}\inf_{t\geq 1}\{-\sqrt{\rho(\rho-2)}\|Ay\|\sqrt{t(t-1)}+(\rho-1)|(Ay,y)|t\}$$

$$= \frac{2}{\rho}\sup_{\substack{\|y\|=1\\ D\geq 0}}\left\{-\sqrt{\rho(\rho-2)}\|Ay\|\times\frac{\sqrt{\rho(\rho-2)}\|Ay\|}{2(\rho-1)\sqrt{D}}+(\rho-1)|(Ay,y)|\times\left(\frac{1}{2}+\frac{|(Ay,y)|}{2\sqrt{D}}\right)\right\}$$

$$= \frac{\rho-1}{\rho}\sup_{\substack{\|y\|=1\\ D\geq 0}}\{|(Ay,y)|+\sqrt{D}\}$$

Corollary 8. *Suppose* $\rho \geq 2$ *and* $w(A) \geq \dfrac{\sqrt{\rho(\rho-2)}}{\rho-1}\|A\|$. *Then*

$$\begin{aligned}
&2(1-\frac{1}{\rho})w(A)-\sqrt{\frac{\rho-2}{\rho}}\|A\| \\
&\leq\ (1-\frac{1}{\rho})\left\{w(A)+\left(w(A)^2-\frac{\rho(\rho-2)}{(\rho-1)^2}\|A\|^2\right)^{1/2}\right\} \\
&\leq\ w_\rho(A) \\
&\leq\ (1-\frac{1}{\rho})\left\{w(A)+\left(w(A)^2+\frac{\rho(\rho-2)}{(\rho-1)^2}\|A\|^2\right)^{1/2}\right\} \\
&\leq\ 2(1-\frac{1}{\rho})w(A)+\sqrt{\frac{\rho-2}{\rho}}\|A\|.
\end{aligned}$$

References

1. T.Ando and K.Nishio, Convexity properties of operator radii associated with unitary ρ-dilations, Michigan Math. J., 20(1973), 303–307.

2. T.Ando and K.Takahashi, On operators with unitary ρ-dilalions, Ann. Pol. Math. LXVI(1997), 11–14.

3. J.A.R.Holbrook, On the power-bounded operators of Sz.-Nagy and Foias, Acta Sci. Math. (Szeged) 29(1968), 299–310.

4. C.K.Li, N.K. Tsing and F.Uhling, Numerical ranges of operator on an indefinite inner product space, Electronic J.Lin Alg., 1(1996), 1–17.

5. R.Mathias and K.Okubo, The induced norm of the Schur multiplication operator with respect to the operator radius, Linear and Multilinear Algebra 37(1994), 114-124.

6. B.Sz.-Nagy and C.Foias, On certain classes of power bounded operators in Hilbert space, Acta Sci. Math. 27(1966), 17–25.

7. B.Sz.-Nagy and C.Foias, Harmonic Analysis of Operators on Hilbert Space, North-Holland, Amsterdam, 1970.

8. H.Watanabe, Operators characterized by certain Cauchy-Schwarz type inequalities, Publ. Res. Inst. Math. Sci. 30(1994), 249–259.

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