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Loss of convexity of simple closed curves moved by surface diffusion

Dedicated to Professor Herbert Amann
on the Occasion of His 60th Birthday

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Abstract

We rigorously prove that there exists a simple, strictly convex, smooth closed curve which loses convexity but stays simple without developing singularities when it moves by its surface diffusion for a short time.

AMS subject classification. 35K99, 80A22

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1 Introduction

We consider the surface diffusion equation of the form

$$\begin{cases} V = -\kappa_{ss} & \text{on } \Gamma(t),\ t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1}$$

Here t denotes the time variable and $\Gamma(t)$ denotes an unknown evolving closed curve embedded in $\mathbf{R}^2$; Γ_0 is a given initial closed curve. The quantities V, κ and s denote the outward normal velocity, the outward curvature and the arc-length parameter of $\Gamma(t)$ respectively. For consistency of notation we take s so that $\Gamma(t)$ is parametrized clock-wise by s. The subscript s in (1) denotes the partial derivative with respect to s. The main goal of this paper is to prove that there exists a simple strictly convex, smooth closed curve Γ_0

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such that the solution curve $\Gamma(t)$ of (1) loses its convexity for t belonging to some interval (t_0, T) with $t_0 > 0$ while $\Gamma(t)$ stays simple without developing singularities for $t \in (0, T)$. This answers the conjectiure posed by J. Escher in the conference "Nonlinear Evolution Equation" held in the end of June of 1997 in Oberwolfach. Loss of convexity was also suggested by numerical studies by B. D. Coleman, R. S. Falk and M. Moakher [7, 8]. We actually prove a stronger statement. Our Γ_0 (so that $\Gamma(t)$ loses its convexity) is obtained by deforming any strict convex smooth curve C in xy-plane symmetric with respect to y-axis such that Γ_0 agrees with C outside a small neighborhood of one of intersection of C and y-axis. In particular, a curve Γ_0 obtained by a slight modification of a circle C near a point on C leads a loss of convexity in motion by surface diffusion.

Equation (1) was first proposed by Mullins [17] to explain thermal grooving in material sciences. We refer to J. W. Cahn and J. E. Taylor [5] for derivation of (1) as well as other related equations. Recently, J. W. Cahn, C. M. Elliott and A. Novick-Cohen [6] derived (1) from the Cahn-Hilliard equation with a concentration dependent mobility as a singular limit in formal basis. The equation (1) is a fourth order fully nonlinear parabolic equation. C. Elliott and H. Garcke [9] constructed without uniqueness a local-in-time classical solution $\Gamma(t)$ of (1) which is simple for arbitrary smooth, simple closed initial curve Γ_0. The unique existence of local solution allowing that $\Gamma(t)$ may develop self intersection is established by the authors [13]. Their proof is elementary in the sense that they only use the coerciveness property in L^2 Sobolev spaces. A little bit modified version is also presented in the present paper. Recently, J. Escher, U. F. Mayer and G. Simonett [10] established the unique existence of local solution for any immersed initial data not only for (1) but also for higher dimensional version of (1). Their proof, however, uses a sophisticated semi-group theory in small Hölder spaces developed by H. Amann [2]. Besides the parabolicity, the equation (1) has two important structures: preservation of the area $A(t)$ enclosed by $\Gamma(t)$ and decrease of the total length $L(t)$ of $\Gamma(t)$ which is easily observed by

$$\begin{aligned} \frac{dA(t)}{dt} &= \int_{\Gamma(t)} V ds = -\int_{\Gamma(t)} \kappa_{ss} ds = 0, \\ \frac{dL(t)}{dt} &= -\int_{\Gamma(t)} \kappa V ds = -\int_{\Gamma(t)} \kappa_s^2 ds \leq 0. \end{aligned} \tag{2}$$

In fact, C. Elliott and H. Garcke [9] utilized property (2) to establish the global existence of solution $\Gamma(t)$ of (1) if initial data Γ_0 is close to a circle. They also proved that $\Gamma(t)$ converges to a circle with the enclosed area equal to that of $\Gamma(t)$ as $t \to \infty$. These results are extended to higher dimensional version by J. Escher, U. F. Mayer and G. Simonett [10].

The equation (1) is a nonlinear fourth order parabolic equation so there are several phenomena which are different from those of second order model such as the curve shortening equation

$$\begin{cases} V = \kappa \quad \text{on } \Gamma(t), t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{3}$$

For the curve shortening equation (3) if Γ_0 is a simple, closed, smooth curve then $\Gamma(t)$ stays simple and smooth and becomes convex in a finite time (Grayson [14]). Once $\Gamma(t)$ becomes convex, it stays convex until it shrinks to a point (M. Gage and R. Hamilton [12]). For the surface diffusion equation (1) it is conjectured by C. Elliott and H. Garcke [9] that $\Gamma(t)$ may cease to be embedded even if initial data Γ_0 is simple, i.e., embedded.

This conjecture is proved by the authors [13]. In fact, it is shown in [13] that if initial curve Γ_0 is a dumbbell like shape, then $\Gamma(t)$ ceases to be embedded in a finite time before it develops singularities. Our method in [13] yields an expicit example that the order of solution may not be preserved. In fact consider a small circle contained in the neck of dumbbell. Since the circle is a stationary solution of (1) and since our solution pinches its neck, the order of solutions are not preserved. A numerical evidence of such 'pinching' is presented in [10] for various closed curves. In the present paper we show that (1) does not preserve convexity. This shows a strong contrast with order-preserving curvature flow equation

$$V = \kappa - \frac{1}{L}\int_{\Gamma(t)} \kappa ds,$$

where convexity is preserved [11] but embeddedness is not preserved. Loss of convexity has been proved for somewhat nonlocal model such as Mullins-Sekerka problems by U. F. Mayer [15, 16]. We note that convexity may be lost also by the effect of nonlocal lower order term for a class of spatially homogeneous surface evolution equation related to chemotaxis [4].

The loss of both embeddedness and convexity reflects the fact that a fourth order parabolic equation does not fulfill the maximum principle or the comparison principle which are main properties of a second order parabolic equation. We explain why the convexity may not be preserved for a fourth order problem by giving a simple linear example. Consider the initial boundary value problem

$$\begin{cases} u_t = -u_{xxxx}, & \text{in } (0,\infty)\times(-1,1),\\ u(t,\pm 1) = 0, \quad u_{xx}(t,\pm 1) = 0, & t>0,\\ u(0,x) = f(x), & x \in (-1,1). \end{cases}$$

The second derivative $v = u_{xx}$ solves the same problem with initial data $v(0,x) = f_{xx}(x)$ since

$$v_{xx}(t,\pm 1) = u_{xxxx}(t,\pm 1) = -u_t(t,\pm 1) = 0.$$

So the nonpreserving of concavity is reduced to the nonpreserving of negativity of solution v, which is easy to imagine at least heuristically. For $\varepsilon > 0$ we deform concave function f near zero so that $\partial_x^6 f^\varepsilon(0) = -1$ and $0 > \partial_x^2 f^\varepsilon(0) \geq -\varepsilon$ where f^ε denotes the deformed concave function. Let u^ε denote the solution with initial data f^ε and $v^\varepsilon = u^\varepsilon_{xx}$. The mean value theorem implies

$$v^\varepsilon(t,0) = v^\varepsilon(0,0) + v^\varepsilon_t(0,0)t + v^\varepsilon_{tt}(\zeta,0)t^2/2,$$

for some $\zeta \in (0,t)$. Since $v^\varepsilon_t = -v^\varepsilon_{xxxx}$, this implies

$$v^\varepsilon(t,0) \geq -\varepsilon + t - \sup_{0\leq\tau<t} |v^\varepsilon_{tt}(\tau,0)|t^2/2.$$

Thus $v^\varepsilon(t,0)$ becomes positive for some t if ε is sufficiently small provided that the size of v^ε_{tt} is bounded independent of ε. This is a rough idea of proving the loss of concavity. However, one should be afraid that v^ε_{tt} may depend on ε significantlly unless we specify the way of deformation. (For this particular problem v_{tt} depends only on f_{xxxx} so it is easy to specify the way of deformation.) Also if the problem is nonlinear one should be afraid that the maximal existence time T^ε may tend to zero as $\varepsilon \to 0$. We introduce a specific way of deformation so that the local existence results guarantee a positive lower bound for T^ε for (1). To deform f we are tempting to replace f_{xx} by $-\varepsilon - x^4/4!$ and integrate in x twice. However, this simple integration is not good since the deformation affects in the first derivatives in a long range. One should localize the deformation also in the first derivative. Our way of deformation is given in Section 2. In contrast to the fourth order equation if we consider the second order initial-boundary value problem

$$\begin{cases} u_t = u_{xx}, & \text{in } (0,\infty) \times (-1,1), \\ u(t,\pm 1) = 0, & t > 0, \\ u(0,x) = f(x), & x \in (-1,1), \end{cases}$$

then $u(t,\cdot)$ is concave in $(-1,1)$ if f is concave. This is easy to prove by the maximum principle for $v = u_{xx}$ since v solves the same equation with same boundary condition with initial data $v(0,x) = f(x)$.

This paper is organized as follows. In Section 2 we specify a way to deform a concave function and estimate several quantities of deformed functions. In Section 3, we use this deformation to deform a given simple convex closed curve Γ_0 to another simple convex closed curve Γ_0' whose local convexity is sufficiently weak compared with $-\kappa_{ssss}$. In Secton 4 we present the fact that the solution curve $\Gamma'(t)$ of (1) starting from Γ_0' exists in a finite time interval $(0,T)$ depending on Γ_0 which, however, does not shrink by the above deformation of Γ_0. In Section 5 we prove that $\Gamma'(t)$ obtained as above loses its convexity in $(0,T)$ with suitable choice of parameter of deformation. In the last section we give the proof of unique existence results in Section 4.

After this work was completed we were informed of two recent interesting works related to ours. In [1] S. A. Alvarez and C. Liu proved the analytic dependence of solutions of (1) with respect to initial data as well as unique local existence of soluitons. In [3] several interesting self-similar patterns of pinching-off are presented. However, formation of pinching for (1) is not proved there.

2 Deformation of a concave function

We give a way to modify an even strictly concave function defined in the interval $I := (-1,1)$ near zero so that its concavity is weak near zero while the minus of its sixth derivative is not small at zero.

For a given concave function f on I and parameters $\varepsilon, \delta > 0$ we define a new function

$$(M_{\varepsilon,\delta} f)(x) := f(-\frac{1}{2}) + \int_{-1/2}^{x} v_{\varepsilon,\delta}(\xi)d\xi, \quad x \in I$$

with

$$v_{\varepsilon,\delta}(x) = \int_0^x w_{\varepsilon,\delta}(\xi)d\xi \varphi_{1/4}(x) + f'(x)(1 - \varphi_{1/4}(x)),$$

$$w_{\varepsilon,\delta}(x) = (-\varepsilon - \frac{x^4}{4!})\varphi_\delta(x) + f''(x)(1 - \varphi_\delta(x)),$$

where $f' = df/dx$, $f'' = d^2 f/dx^2$. Here φ_δ is a cut-off function near zero defined by

$$\varphi_\delta(x) = \varphi(|x|/\delta),$$

where φ is given for example by

$$\varphi(x) = q(2-x), \quad q(x) = \frac{p(x)}{p(x)+p(1-x)}, \quad x \in \mathbf{R} \tag{4}$$

with $p(x) = e^{-1/x}$ for $x > 0$; $p(x) = 0$ for $x \le 0$, so that $\varphi \in C_0^\infty[0,\infty)$ satisfying $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $x \le 1$ and $\varphi(x) = 0$ for $x \ge 2$. (We won't use the explicit formula of φ but we need the properties in the sequel.) The parameter ε measures the concavity of $M_{\varepsilon,\delta}f$ near zero while δ measures the size of region where the sixth derivative of $M_{\varepsilon,\delta}f$ equals -1. We shall estimate $M_{\varepsilon,\delta}f$ in L^2-Sobolev space $H^m(I)$ for f in $H^m(I)$; the norm of f in $H^m(I)$ is denoted $\|f\|_m$. By the above deformation we should be afraid that $M_{\varepsilon,\delta}f$ may lose its concavity. Fortunately, it turns out that $M_{\varepsilon,\delta}f$ is strictly concave if δ is taken sufficiently small independent of $0 < \varepsilon \le 1$. We shall state them in a precise way.

Lemma 1 . *Let f be in $H^m(I)$ with an integer $m \ge 2$. Assume that $\sup_I f'' < 0$ and that f is even, i.e., $f(x) = f(-x)$ for $x \in I$. Assume that $0 < \varepsilon \le 1$ and that $0 < \delta < 1/8$.*
(i) The function $f_{\varepsilon,\delta} = M_{\varepsilon,\delta}f$ is an even function.
(ii) $f_{\varepsilon,\delta} \in H^m(I)$ and there exists a positive constant $C_\delta^m = C(m,\delta,f,\varphi)$ such that

$$\|f_{\varepsilon,\delta}\|_m \le C_\delta^m \quad \text{for all } \varepsilon \in (0,1].$$

(iii) $f_{\varepsilon,\delta}(x) = f(x)$ for $1/2 \le |x| < 1$.
(iv) $f_{\varepsilon,\delta}^{(2)}(x) = -\varepsilon - x^4/4!$ for $|x| \le \delta$. In particular,

$$f_{\varepsilon,\delta}^{(2)}(0) = -\varepsilon, \quad f_{\varepsilon,\delta}^{(4)}(0) = 0, \quad f_{\varepsilon,\delta}^{(6)}(0) = -1,$$

where $f^{(k)}$ denotes the k-th derivative of f.
(v) There exists $\delta_0 = \delta_0(f,\varphi) > 0$ such that

$$f_{\varepsilon,\delta}^{(2)}(x) \le -\min(\varepsilon, \frac{1}{2}\inf_I |f^{(2)}|) < 0$$

for all $x \in I$, $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,1]$. In particular,

$$f_{\varepsilon,\delta}(x) \le f(\frac{1}{2}) \quad \text{for } |x| \le \tfrac{1}{2}$$

for all $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,1]$.

Proof. (i) Since φ_δ and f are even, $w_{\varepsilon,\delta}$ is even so that $v_{\varepsilon,\delta}$ is odd. Thus $f_{\varepsilon,\delta}$ is even.
(ii) By definition and the Schwarz inequality we see

$$\begin{aligned}
&\|f_{\varepsilon,\delta} - f(-\frac{1}{2})\|_0 \le \sqrt{2}\|f_{\varepsilon,\delta} - f(-\frac{1}{2})\|_{L^\infty(I)} \\
&\le \sqrt{2}\|f'_{\varepsilon,\delta}\|_{L^1(I)} \le 2\|f'_{\varepsilon,\delta}\|_0 = 2\|v_{\varepsilon,\delta}\|_0, \\
&\|v_{\varepsilon,\delta}\|_0 \le \sqrt{2}\|v_{\varepsilon,\delta}\|_1 \le \sqrt{2}(\|w_{\varepsilon,\delta}\|_{L^1(I)} + \|f^{(2)}\|_{L^1(I)}).
\end{aligned}$$

Since

$$\|w_{\varepsilon,\delta}\|_0 \le \sqrt{2}(1+\frac{1}{4!}) + \|f^{(2)}\|_0 \quad \text{for } 0 < \varepsilon \le 1$$

and

$$\|f_{\varepsilon,\delta}^{(2)}\|_0 \leq C_\varphi(\|w_{\varepsilon,\delta}\|_0 + \|f'\|_0) + \|f^{(2)}\|_0$$

for a constant depending only on φ, the Schwarz inequality yields a bound for $\|f_{\varepsilon,\delta}\|_0$, $\|f_{\varepsilon,\delta}\|_1$, $\|f_{\varepsilon,\delta}\|_2$ independent of $\varepsilon \in (0,1]$, $\delta \in (0,1/8)$.

We next estimate $f'_{\varepsilon,\delta}$ in $H^{m-1}(I)$ for $m \geq 3$. Since $H^{m-1}(I)$ is a Banach algebra for $m \geq 2$,

$$\begin{aligned}
&\|\int_0^x w_{\varepsilon,\delta}(\xi)d\xi\|_{m-1} \leq \|\int_0^x w_{\varepsilon,\delta}(\xi)d\xi\|_0 + \|w_{\varepsilon,\delta}\|_{m-2} \\
&\leq \sqrt{2}\|w_{\varepsilon,\delta}\|_0 + C^m(\|\varepsilon + \frac{x^4}{4!}\|_{m-2}\|\varphi_\delta\|_{m-2} + \|f^{(2)}\|_{m-2}\|1-\varphi_\delta\|_{m-2}) \\
&\leq C_\delta^m(1 + \|f^{(2)}\|_{m-2})
\end{aligned}$$

with some constant C_δ^m independent of $\varepsilon \in (0,1]$ but depends on δ through φ_δ while C^m depends only on m. Since

$$\|f'_{\varepsilon,\delta}\|_{m-1} \leq C^{m+1}(\|\int_0^x w_{\varepsilon,\delta}(\xi)d\xi\|_{m-1}\|\varphi_{1/4}\|_{m-1} + \|f'\|_{m-1}\|1-\varphi_{1/4}\|_{m-1}),$$

we obtain

$$\|f'_{\varepsilon,\delta}\|_{m-1} \leq C_\delta^m(1 + \|f'\|_{m-1})$$

with some constant C_δ^m independent of $\varepsilon \in (0,1]$. This completes the proof of (ii).

(iii) Since both $f_{\varepsilon,\delta}$ and f are even it suffices to prove that $f_{\varepsilon,\delta}(x) = f(x)$ for $x \in [1/2,1)$. Since $\varphi_{1/4}(\xi) = 0$ for $|\xi| \geq 1/2$,

$$\begin{aligned}
f_{\varepsilon,\delta}(x) - f(x) &= \int_{-1/2}^x (v_{\varepsilon,\delta}(\xi) - f'(\xi))d\xi \\
&= \int_{-1/2}^{1/2}(\int_0^\xi w_{\varepsilon,\delta}(\eta)d\eta - f'(\xi))\varphi_{1/4}(\xi)d\xi \quad \text{for } x \geq 1/2.
\end{aligned}$$

The integrand is an odd function so that $f_{\varepsilon,\delta}(x) = f(x)$ for $x \in [1/2,1)$.

(iv) Since $\delta < 1/8$ so that $2\delta < 1/4$, we see $\varphi_\delta(x) \equiv \varphi_{1/4}(x) \equiv 1$ for $|x| \leq \delta$. Thus

$$f_{\varepsilon,\delta}^{(2)}(x) = v'_{\varepsilon,\delta}(x) = w_{\varepsilon,\delta}(x) = -\varepsilon - \frac{x^4}{4!} \quad \text{for } |x| \leq \delta.$$

(v) Differentiate $f_{\varepsilon,\delta}$ twice to get

$$f_{\varepsilon,\delta}^{(2)}(x) = w_{\varepsilon,\delta}(x)\varphi_{1/4}(x) + \int_0^x w_{\varepsilon,\delta}(\xi)d\xi\varphi'_{1/4}(x) + f^{(2)}(x)(1-\varphi_{1/4}(x)) - f'(x)\varphi'_{1/4}(x).$$

For $|x| \leq 1/4$ we see that

$$f_{\varepsilon,\delta}^{(2)}(x) = w_{\varepsilon,\delta}(x) \leq -\min(\varepsilon, \inf_I |f^{(2)}|) < 0,$$

since $\varphi_{1/4}(x) = 1$. For $1/2 \leq |x| < 1$ we see that

$$f_{\varepsilon,\delta}^{(2)}(x) = f^{(2)}(x) \leq -\inf_I |f^{(2)}| < 0,$$

since $\varphi_{1/4}(x) = 0$. It remains to estimate $f^{(2)}_{\varepsilon,\delta}(x)$ for x satisfying $1/4 < |x| < 1/2$. We may assume that $1/4 < x < 1/2$ since $f_{\varepsilon,\delta}$ is even. For such an x since $\delta < 1/8$ we see that

$$w_{\varepsilon,\delta}(x) = f^{(2)}(x),$$
$$f^{(2)}_{\varepsilon,\delta}(x) = f^{(2)}(x) + (\int_0^x w_{\varepsilon,\delta}(\xi)d\xi - f'(x))\varphi'_{1/4}(x).$$

Since $f'(0) = 0$, we see that

$$\begin{aligned}\int_0^x w_{\varepsilon,\delta}(\xi)d\xi &= \int_0^x \{(-\varepsilon - \frac{\xi^4}{4!})\varphi_\delta(\xi) + f^{(2)}(\xi)(1 - \varphi_\delta(\xi))\}d\xi \\ &= -\int_0^x (\varepsilon + \frac{\xi^4}{4!} + f^{(2)}(\xi))\varphi_\delta(\xi)d\xi + f'(x).\end{aligned}$$

Combining these two identities we arrive at

$$\begin{aligned}f^{(2)}_{\varepsilon,\delta}(x) &= f^{(2)}(x) - \int_0^x (\varepsilon + \frac{\xi^4}{4!} + f^{(2)}(\xi))\varphi_\delta(\xi)d\xi\varphi'_{1/4}(x) \\ &\leq -\inf_I |f^{(2)}| + \int_0^{2\delta} \varphi_\delta(\xi)d\xi \sup_I |\varphi'_{1/4}|M \quad \text{for } \varepsilon \in (0,1]\end{aligned}$$

with $M = \sup_I |f^{(2)}| + 1 + 1/4!$. Since

$$\int_0^{2\delta} \varphi_\delta(\xi)d\xi = \delta \int_0^2 \varphi(\xi)d\xi \leq 2\delta,$$

we see that

$$f^{(2)}_{\varepsilon,\delta}(x) \leq -\frac{1}{2}\inf_I |f^{(2)}|$$

for all $\varepsilon \in (0,1]$ and $x \in (1/4, 1/2)$ provided that δ is sufficiently small, say

$$0 < \delta < \delta_0 := \frac{\inf_I |f^{(2)}|}{4\sup_I |\varphi'_{1/4}|M}.$$

We have thus proved (v). □

Remark 2 . From the proof we see that the constant C^m_δ in (ii) may tend to infinity as δ tends to zero for $m \geq 3$ since we have to estimate $\|\varphi_\delta\|_{m-2}$. For $m = 0, 1, 2$ the constant C^m_δ stays bounded as δ tends to zero.

3 Deformation of a strictly convex closed curve

We introduce a parametrization of a simple, closed, strictly convex curves in the plane $\mathbf{R}^2$ by using a reference curve. Let M^0 be a smooth closed convex curve embedded in $\mathbf{R}^2$. We assume that M^0 is symmetric with respect to y-axis and that M^0 is contained in $\{(x,y); y \leq 0\}$. We further assume that M^0 contains a straight line segment on x-axis. Let η be the arc-length parameter of M^0 so that it parametrizes M^0 clock-wise. By definition M^0 is of the form

$$M^0 = \{X^0(\eta) = (X^0_1(\eta), X^0_2(\eta)) \in \mathbf{R}^2; \eta \in \mathbf{T} := \mathbf{R}/2L\mathbf{Z}\},$$

where $2L$ is the length of M^0. By symmetry the arc-length parameter η is taken so that

$$\begin{aligned} &X_1^0(\eta) = -X_1^0(-\eta) \geq 0, \quad |\eta| \leq L, \\ &X_2^0(\eta) = X_2^0(-\eta) \leq 0, \quad |\eta| \leq L, \\ &X^0(\eta) = (\eta, 0) \quad \text{for } |\eta| \leq a \text{ with some } a \in (0, L/2). \end{aligned}$$

Since M^0 is convex, the outward curvature $\kappa^0(\eta)$ is always nonpositive for $\eta \in \mathbf{T}$. We fix M^0 and call M^0 a reference curve. For an integer $m \geq 2$ let $E^m(\mathbf{T})$ be

$$E^m(\mathbf{T}) = \{d_0 \in H^m(\mathbf{T}); d_0(\eta) = d_0(-\eta) > 0 \quad \text{for } \eta \text{ satisfying } |\eta| \leq L\}.$$

For $d_0 \in E^m(\mathbf{T})$ we associate (at least) a C^1 curve

$$\Gamma[d_0] = \{X^0(\eta) + d_0(\eta) n^0(\eta); \eta \in \mathbf{T}\},$$

where $n^0(\eta)$ is the outward normal of M^0. By symmetry of M^0 and $d_0 \in E^m(\mathbf{T})$, $\Gamma[d_0]$ is simple closed since M^0 is convex and $d_0(\eta) > 0$. Let Σ^m be the set of all $\Gamma[d_0]$ with $d_0 \in E^m(\mathbf{T})$ such that the outward curvature κ_0 of $\Gamma[d_0]$ is negative everywhere. Of course, for each $\Gamma_0 \in \Sigma^m$ there is a unique $d_0 \in E^m(\mathbf{T})$ such that $\Gamma_0 = \Gamma[d_0]$.

We shall deform $\Gamma[d_0] \in \Sigma^m$. Since the curvature κ_0 of $\Gamma[d_0]$ is nagative,

$$\sup_{|\eta| \leq a} d_{0\eta\eta} < 0.$$

Since the restriction $r_a d_0$ of d_0 on $(-a, a)$ is even, $r_a d_0 \in H^m(-a, a)$. We deform $r_a d_0$ by $M_{\varepsilon,\delta}$ and dilation. For a function $f \in H^m(-\alpha, \alpha)$ and $\beta > 0$ let $D_\beta f$ denote

$$(D_\beta f)(x) = f(\beta x), \quad |x| < \frac{\alpha}{\beta}.$$

Evidently, D_β gives an isomorphism from $H^m(-\alpha, \alpha)$ onto $H^m(-\alpha/\beta, \alpha/\beta)$. It also preserve uniform nonnegativity of the second derivative of function. For even $f \in H^m(-a, a)$ with $\sup_{|x| \leq a} f^{(2)} < 0$ we set

$$M^a_{\varepsilon,\delta} f = (D_{1/a} \circ M_{\varepsilon,\delta} \circ D_a) f \in H^m(-a, a),$$

where $M_{\varepsilon,\delta}$ is a deformation operator given in Section 2 with parameters $\varepsilon \in (0, 1]$, $\delta \in (0, 1/8)$. We deform d_0 by defining $d_0^{\varepsilon,\delta}$ by

$$d_0^{\varepsilon,\delta}(\eta) = \begin{cases} (M^a_{\varepsilon,\delta}(r_a d_0))(\eta), & |\eta| \leq a, \\ d_0(\eta), & a < |\eta| \leq L. \end{cases} \tag{5}$$

The next results immdiately follow from Lemma 1 and Remark 2.

Theorem 3 *Let $d_0 \in E^m(\mathbf{T})$ such that $\Gamma[d_0] \in \Sigma^m(\mathbf{T})$ with an integer $m \geq 2$. Let $0 < \varepsilon \leq 1$ and $0 < \delta < 1/8$ and let $d_0^{\varepsilon,\delta}$ be given in (5).*
(i) $\Gamma[d_0^{\varepsilon,\delta}]$ is symmetric with respect to y-axis.
(ii) $d_0^{\varepsilon,\delta} \in H^m(\mathbf{T})$ and there exists a positive constant $b^m_{a,\delta}$ independent of ε such that

$$\|d_0^{\varepsilon,\delta}\|_{H^m(\mathbf{T})} \leq b^m_{a,\delta} \quad \textit{for all } \varepsilon \in (0, 1]$$

and $b^m_{a,\delta}$ is bounded as $\delta \to 0$ for $m = 2$ but unbounded as $\delta \to 0$ for $m \geq 3$.
(iii) Outside the set $\{(x,y); |x| \leq \delta a, y > 0\}$ the set $\Gamma[d_0]$ agrees with $\Gamma[d_0^{\varepsilon,\delta}]$.
(iv) $\frac{d^2}{d\eta^2} d_0^{\varepsilon,\delta}(0) = -\varepsilon/a^2$, $\frac{d^4}{d\eta^4} d_0^{\varepsilon,\delta}(0) = 0$, $\frac{d^6}{d\eta^6} d_0^{\varepsilon,\delta}(0) = -1/a^6$.
(v) There exists $\delta_0 > 0$ such that

$$\sup_{|\eta| \leq a} \frac{d^2}{d\eta^2} d_0^{\varepsilon,\delta}(\eta) < 0,$$
$$\inf_{|\eta| \leq a} d_0^{\varepsilon,\delta}(\eta) \geq d_0(a) > 0$$

for all $\varepsilon \in (0,1]$ and $\delta \in (0, \delta_0)$. In particular, $\Gamma[d_0^{\varepsilon,\delta}] \in \Sigma^m(\mathbf{T})$ with $\inf_{\mathbf{T}} d_0^{\varepsilon,\delta} \geq \inf_{\mathbf{T}} d_0 > 0$ for all $\varepsilon \in (0,1]$ and $\delta \in (0, \delta_0)$.

For $\Gamma_0 = \Gamma[d_0] \in \Sigma^m$ we set

$$S^{\varepsilon,\delta}(\Gamma_0) := \Gamma[d_0^{\varepsilon,\delta}],$$

where $d_0^{\varepsilon,\delta}$ is given in (5). If δ is taken small ($\delta < \delta_0$) as in Theorem 3 (v) so that $\Gamma[d_0^{\varepsilon,\delta}]$ has negative outward curvature everywhere, $S^{\varepsilon,\delta}$ gives a mapping from Σ^m into itself for $\varepsilon \in (0,1]$ and $\delta \in (0, \delta_0)$. $S^{\varepsilon,\delta}(\Gamma_0)$ is a deformation of Γ_0 near the point near Γ_0 cross the positive y-axis. The outward curvature $\kappa^{\varepsilon,\delta}$ of $S^{\varepsilon,\delta}(\Gamma_0)$ and its derivatives at the intersection point A of $S^{\varepsilon,\delta}(\Gamma_0)$ and the positive y-axis can be calculated by Theorem 3 (iv). In fact,

$$\kappa^{\varepsilon,\delta} = -\varepsilon/a^2, \quad \kappa^{\varepsilon,\delta}_{\eta\eta} = 0, \quad \kappa^{\varepsilon,\delta}_{\eta\eta\eta\eta} = -1/a^6 \quad \text{at point } A.$$

This shows that the convexity of $S^{\varepsilon,\delta}(\Gamma_0)$ can be weakend compared with the minus of the fourth order derivative of curvature near A by taking ε small.

4 Existence of solutions for motion by surface diffusion

We prepare a unique local existence of solutions of (1) which is useful to prove the loss of convexity. As in [9, 13], to find a local solution $\Gamma(t)$ of (1) for $\Gamma_0 = \Gamma[d_0] \in \Sigma^m$, it is equivalent to find $d = d(t, \eta)$ solving

$$\begin{cases} \dfrac{1 - \kappa^0 d}{J} d_t = -\dfrac{1}{J}(\dfrac{1}{J}\kappa_\eta)_\eta, & (t,\eta) \in (0,T) \times \mathbf{T}, \\ d(0,\eta) = d_0(\eta), & \eta \in \mathbf{T} \end{cases} \tag{6}$$

at least for short time by setting $\Gamma(t) = \Gamma[d(t,\cdot)]$; here J and κ are the arc-length element and the outward curvature of $\Gamma(t)$, respectively. Their explicit forms are

$$J = (d_\eta^2 + (1 - \kappa^0 d)^2)^{1/2},$$
$$\kappa = \frac{1}{J^3}\{(1 - \kappa^0 d) d_{\eta\eta} + 2\kappa^0 d_\eta^2 + \kappa^0_\eta d d_\eta + \kappa^0 (1 - \kappa^0 d)^2\},$$

where κ^0 denotes the outward curvature of the reference curve M^0. The problem (6) is now written as

$$\begin{cases} d_t + J^{-4} d_{\eta\eta\eta\eta} + P d_{\eta\eta\eta} + Q = 0, & (t,\eta) \in (0,T) \times \mathbf{T}, \\ d(0,\eta) = d_0(\eta), & \eta \in \mathbf{T} \end{cases} \tag{7}$$

where $P = P(\eta, d, d_\eta, d_{\eta\eta})$ and $Q = Q(\eta, d, d_\eta, d_{\eta\eta})$ are expressed as polynomials of $(1 - \kappa^0 d)^{-1}$, J^{-1}, $d^i \kappa^0 / d\eta^i$ $(i = 0,1,2,3)$, $\partial^j d/\partial \eta^j$ $(j = 0,1,2)$. For η with $\kappa^0(\eta) = 0$ the evolution equation (6) is of the form

$$d_t = -\left\{ \frac{1}{(1+d_\eta^2)^{1/2}} \left(\frac{d_{\eta\eta}}{(1+d_\eta^2)^{3/2}} \right)_\eta \right\}_\eta . \tag{8}$$

Proposition 4 . *Let $m \geq 4$ be an integer. Let K_i be a positive constant for $i = 1,2$. Then there exists $T > 0$ such that for any $d_0 \in E^m(\mathbf{T})$ with $\|d_0\|_{H^m(\mathbf{T})} \leq K_1$ and $\min_{\mathbf{T}} d_0 \geq 1/K_2$ there exists a unique solution $d = d(t,\eta)$ of (7) that satisfies*

$$\begin{aligned} & d \in L^2(0,T; H^{m+2}(\mathbf{T})), \quad d_t \in L^2(0,T; H^{m-2}(\mathbf{T})), \\ & \sup_{0<t\leq T} \|d(t,\cdot)\|_{H^m(\mathbf{T})} \leq \mathcal{A}(K_1), \\ & \inf_{[0,T]\times\mathbf{T}} d \geq 1/(2K_2) \end{aligned}$$

with some nondecreasing function $\mathcal{A} = \mathcal{A}(K_1)$. Moreover, $d(t,\eta) = d(t,-\eta)$, $|\eta| \leq L$ so that $d(t,\cdot) \in E^m(\mathbf{T})$ for $t \in [0,T]$.

Unfortunately, the first assertion on unique existence of solutions does not directly follow from that of [13, Theorem3] since the convexity of M^0 guarantees that solution always exists without imposing smallness of d_0. The second assertion of Proposition 4 is not difficult. Indeed $d(t,-\eta)$ solves (7) with initial data $d_0(-\eta) = d_0(\eta)$. Then by uniqueness of solutions $d(t,-\eta) = d(t,\eta)$ for $(t,\eta) \in [0,T] \times \mathbf{T}$. This shows $d(t,\cdot) \in E^m(\mathbf{T})$. We shall prove the first assertion in Section 6 for reader's convenience although the basic idea for the proof is similar to that of [13, Theorem3]. Applying Proposition 4 for deformed initial data $S^{\varepsilon,\delta}(\Gamma_0) = \Gamma[d_0^{\varepsilon,\delta}]$, we have uniform local existence for (1) by virtue of Theorem 3. We state this fact explicitly.

Theorem 5 . *Let $m \geq 4$ be an integer and $\Gamma_0 = \Gamma[d_0] \in \Sigma^m$. Let δ_0 be the positive constsnt (determined by d_0) in Theorem 3 (v). Then there exists a $T_0^\delta > 0$ (independent of ε) such that for any $\varepsilon \in (0,1]$, $\delta \in (0,\delta_0)$ there exists a unique solution $d^{\varepsilon,\delta}$ of (7) in $(0,T_0^\delta) \times \mathbf{T}$ with initial data $d_0^{\varepsilon,\delta}$ (defined in Theorem 3) that satisfies*

$$\begin{aligned} & d^{\varepsilon,\delta} \in L^2(0,T_0^\delta; H^{m+2}(\mathbf{T})), \quad d_t^{\varepsilon,\delta} \in L^2(0,T_0^\delta; H^{m-2}(\mathbf{T})), \\ & \sup_{0<t\leq T_0^\delta} \|d^{\varepsilon,\delta}(t,\cdot)\|_{H^m(\mathbf{T})} \leq \mathcal{A}(b_{a,\delta}^m), \\ & \inf_{[0,T_0^\delta]\times\mathbf{T}} d^{\varepsilon,\delta} \geq \frac{1}{2} \min_{\mathbf{T}} d_0, \end{aligned}$$

where $b_{a,\delta}^m$ is given in Theorem 3 (ii) and $\mathcal{A}$ is given in Proposition 4. Moreover, $d^{\varepsilon,\delta}(t,\cdot) \in E^m(\mathbf{T})$.

We postpone the proof of Proposition 4 and Theorem 5 in Section 6.

5 Nonpreserving of convexity

For any strictly convex smooth curve Γ_0 symmetric with respect to y-axis, it is easy to take a reference curve M^0 and $d_0 \in E^m(\mathbf{T})$ so that $\Gamma_0 = \Gamma[d_0]$ up to translation in the direction of y-axis. By this interpretation the next result shows the existence of initial convex curve (even near strictly convex curve) which loses its convexity during evolution by (1).

Theorem 6 . *(Loss of convexity). Let $m \geq 11$ be an integer and $\Gamma_0 = \Gamma[d_0] \in \Sigma^m$. Let δ_0 be a positive constant defined in Theorem 3 and let δ be in $(0, \delta_0)$. Let $d_0^{\varepsilon,\delta}$ be the deforemd function of d_0 (defined in Theorem 3) for $\varepsilon \in (0,1]$. Let T_0^δ be the time (defined in Theorem 5) such that there is a unique solution $d^{\varepsilon,\delta}$ of (7) in $(0, T_0^\delta) \times \mathbf{T}$ with initial data $d_0^{\varepsilon,\delta}$. Then there is ε_0^δ such that for any $\varepsilon \in (0, \varepsilon_0^\delta)$ there are $t_0^{\varepsilon,\delta}$ and $t_1^{\varepsilon,\delta}$ with $t_0^{\varepsilon,\delta} < T_0^\delta$ and $t_0^{\varepsilon,\delta} < t_1^{\varepsilon,\delta}$ with the property that the solution $\Gamma[d^{\varepsilon,\delta}(t,\cdot)]$ of (1) starting from $S^{\varepsilon,\delta}(\Gamma_0) \in \Sigma^m$ loses its convexity at least for $t \in (t_0^{\varepsilon,\delta}, \min(T_0^\delta, t_1^{\varepsilon,\delta}))$. Moreover, $t_0^\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. First step. We shall prove

$$d_{\eta\eta t}^{\varepsilon,\delta}(0,0) = a^{-6} + 54\varepsilon^5 a^{-10} \geq a^{-6} \quad \text{for } \varepsilon \in (0,1].$$

(Smallness of $\delta < \delta_0$ is not invoked in first two steps.) Note that the equation (7) is of the form (8) for $|\eta| < a$. Differentiating (8) in η twice and use the property $d^{\varepsilon,\delta}(t,\eta) = d^{\varepsilon,\delta}(t,-\eta)$, $|\eta| < L$, we arrive at

$$d_{\eta\eta t}^{\varepsilon,\delta}(t,0) = -\left(\frac{\partial^6}{\partial\eta^6} d^{\varepsilon,\delta} - 33(d_{\eta\eta}^{\varepsilon,\delta})^2 \frac{\partial^4}{\partial\eta^4} d^{\varepsilon,\delta} + 54(d_{\eta\eta}^{\varepsilon,\delta})^5\right)(t,0). \tag{9}$$

We set $t = 0$ and use values of derivatives of $d_0^{\varepsilon,\delta}$ caluculated in Theorem 3 to get the desired value of $d_{\eta\eta t}^{\varepsilon,\delta}(0,0)$.

Second step. There is a bound K^δ depending on δ such that

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in [0, T_0^\delta]} |d_{\eta\eta tt}^{\varepsilon,\delta}(t,0)| \leq K^\delta.$$

Indeed, differentiating (9) twice in t and using (8), we observe that

$$\sup_{t \in [0,T_0^\delta]} |d_{\eta\eta tt}^{\varepsilon,\delta}(t,0)| \leq G\left(\sum_{k=0}^{10} \sup_{t \in [0,T_0^\delta]} \left|\frac{\partial^k}{\partial\eta^k} d^{\varepsilon,\delta}(t,0)\right|\right)$$

with positive nondecreasing function $G = G(\lambda)$. By the Sobolev inequality and the estimate of $d^{\varepsilon,\delta}(t,\cdot)$ in $H^m(\mathbf{T})$ in Theorem 5, we now obtain

$$\sup_{t \in [0,T_0^\delta]} |d_{\eta\eta tt}^{\varepsilon,\delta}(t,0)| \leq G(C' b_{a,\delta}^{11})$$

with a constant C' depending only on L. Setting $K^\delta = G(C' b_{a,\delta}^{11})$ yields the desired estimate.

Third step. We shall complete the proof of Theorem 6. We take $0 < \delta < \delta_0$ so that outward curvature is nagative everywhere on $\Gamma[d_0^{\varepsilon,\delta}]$. By Taylor's expansion and the second step, we see that

$$\begin{aligned} d_{\eta\eta}^{\varepsilon,\delta}(t,0) &= d_{\eta\eta}^{\varepsilon,\delta}(0,0) + d_{\eta\eta t}^{\varepsilon,\delta}(0,0)t + \int_0^t \Big(\int_0^\sigma d_{\eta\eta\tau\tau}^{\varepsilon,\delta}(\tau,0)d\tau\Big)d\sigma \\ &\geq d_{0\eta\eta}^{\varepsilon,\delta}(0) + d_{\eta\eta t}^{\varepsilon,\delta}(0,0)t - t^2K^\delta \end{aligned}$$

for $0 \leq t \leq T_0^\delta$. Since $d_{0\eta\eta}^{\varepsilon,\delta}(0) = -\varepsilon a^{-2}$ by Theorem 3 and $d_{\eta\eta}^{\varepsilon,\delta}(t,0) \geq a^{-6}$ by the first step, we now arrive at

$$d_{\eta\eta}^{\varepsilon,\delta}(t,0) \geq -\varepsilon a^{-2} + a^{-6}t - K^\delta t^2 \quad \text{for } t \in [0, T_0^\delta]. \tag{10}$$

We can take ε small so that the quadratic polynomial of the right hand side has the smallest positive zeros less than T_0^δ. In fact, we can take $\varepsilon_0^\delta > 0$ small so that

$$0 < \frac{a^{-6} - (a^{-12} - 4K^\delta\varepsilon_0^\delta a^{-2})^{1/2}}{2K^\delta} < T_0^\delta.$$

Then for each $\varepsilon \in (0, \varepsilon_0^\delta)$ the polynomial $-\varepsilon a^{-2} + a^{-6}t - K^\delta t^2$ has two positive zeros $t_0^{\varepsilon,\delta} < t_1^{\varepsilon,\delta}$ such that $t_0^{\varepsilon,\delta} < T_0^\delta$. By (10) for $\varepsilon \in (0, \varepsilon_0^\delta)$

$$d^{\varepsilon,\delta}(t,0) \geq -(t - t_0^{\varepsilon,\delta})(t - t_1^{\varepsilon,\delta}) \quad \text{for } t \in [0, T_0^\delta].$$

This implies that

$$d^{\varepsilon,\delta}(t,0) > 0 \quad \text{for } t_0^{\varepsilon,\delta} < t < \min(t_1^{\varepsilon,\delta}, T_0^\delta).$$

This shows that $\Gamma[d^{\varepsilon,\delta}(t,\cdot)]$ loses its convexity at least for $t_0^{\varepsilon,\delta} < t < \min(t_1^{\varepsilon,\delta}, T_0^\delta)$. The assertion $t_0^{\varepsilon,\delta} \to 0$ as $\varepsilon \to 0$ follows from its definition. □

6 Proof of Proposition 4 and Theorem 5

Proposition 4 is based on the convexity $\kappa^0 \leq 0$ of the reference curve M^0. It guarantees that $1 - \kappa^0 d$ does not take zero for $d \geq 0$. Thus J^{-4}, P and Q in (7) are always regular for $d \geq 0$. To prove Proposition 4 we begin with a general result guaranteeing the lower bound of solutions.

General framework. Let $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$, where ω is a positive constant. We consider a general equation of the form:

$$\begin{cases} u_t + a(x,u,u_x)u_{xxxx} + b(x,u,u_x,u_{xx})u_{xxx} + c(x,u,u_x,u_{xx}) = 0, \\ u(0,x) = u_0(x) \end{cases} \tag{11}$$

for $t \geq 0$ and $x \in \mathbf{T}$. We assume the followings:
(H1) There are positive constants Λ_*, Λ and a_1 with $\Lambda_* < \Lambda$ such that $a(x,\alpha_0,\alpha_1) \geq a_1$ for $x \in \mathbf{T}$, $\Lambda_* \leq \alpha_0 \leq \Lambda$ and $|\alpha_1| \leq \Lambda$.
(H2) Functions $a(x,\alpha_0,\alpha_1)$, $b(x,\alpha_0,\alpha_1,\alpha_2)$ and $c(x,\alpha_0,\alpha_1,\alpha_2)$ are regular in $\mathbf{T} \times \mathbf{R}^3$.
Then we have

Lemma 7 . *Let $m \geq 4$ be integers. Assume (H1) and (H2). Then, for any $u_0 \in H^m(\mathbf{T})$ with $\min_{x\in\mathbf{T}} u_0(x) \geq 2\Lambda_*$, there are a $T_0 = T_0(\|u_0\|_{H^m(\mathbf{T})}, \min_{x\in\mathbf{T}} u_0(x)) > 0$ and a unique solution $u(t,x)$ of (11) satisfying*

$$u \in L^2(0,T_0;H^{m+2}(\mathbf{T})), \quad u_t \in L^2(0,T_0;H^{m-2}(\mathbf{T})),$$

$$\sup_{t\in[0,T_0]} \|u(t)\|_{H^m(\mathbf{T})} \leq \mathcal{A}(\|u_0\|_{H^m(\mathbf{T})}),$$

$$\min_{(t,x)\in[0,T_0]\times\mathbf{T}} u(t,x) \geq \frac{1}{2}\min_{x\in\mathbf{T}} u_0(x) \tag{12}$$

where $\mathcal{A}$ is some nondecreasing function.

Remark 8 . (i) $T_0(\mu_0,\mu_1)$ is nonincreasing in $\mu_0 \geq 0$ and nondecreasing in $\mu_1 \geq 0$.
(ii) Proposition 4 follows directly from Lemma 7.

Proof. Set

$$Z_T^m = \{u \in L^2(0,T;H^{m+2}(\mathbf{T})); u_t \in L^2(0,T;H^{m-2}(\mathbf{T})), u(0,x) = u_0(x)\}.$$

As in the proof of Theorem 1 of [13], we can show by virtue of (H1) and (H2) that for any $T > 0$ and given $v \in Z_T^m$ the linear inhomogeneous equation for $w(t,x)$

$$\begin{cases} w_t + a(x,u_0,u_{0x})w_{xxxx} = & \{a(x,u_0,u_{0x}) - a(x,v,v_x)\}v_{xxxx} \\ & -b(x,v,v_x,v_{xx})v_{xxx} - c(x,v,v_x,v_{xx}), \\ w(0,x) = u_0(x) & \end{cases}$$

admits a unique solution $w \in Z_T^m$. It can be also shown that there are positive constants T_1 nonincreasing in $\|u_0\|_{H^m(\mathbf{T})}$ and R nondecreasing in $\|u_0\|_{H^m(\mathbf{T})}$ such that the mapping

$$B \to B; \quad v \mapsto w$$

admits a unique fixed point u satisfying

$$\sup_{t\in[0,T_1]} \|u(t)\|_{H^m(\mathbf{T})} \leq \mathcal{A}(\|u_0\|_{H^m(\mathbf{T})})$$

with some nondecreasing function $\mathcal{A}$, where

$$B = \{u \in Z_{T_1}^m; \|u\|_{Z_{T_1}^m} \leq R\},$$

$$\|u\|_{Z_{T_1}^m} = \|u\|_{L^2(0,T_1;H^{m+2}(\mathbf{T}))} + \|u_t\|_{L^2(0,T_1;H^{m-2}(\mathbf{T}))}.$$

Furthermore, since

$$\|u(t) - u_0\|_{L^\infty(\mathbf{T})} \leq C_L\|u(t) - u_0\|_{H^1(\mathbf{T})} \leq C_L \int_0^t \|u_\tau(\tau)\|_{H^1(\mathbf{T})}d\tau$$
$$\leq C_L\left(\int_0^{T_1} \|u_t(t)\|_{H^{m-2}(\mathbf{T})}^2 dt\right)^{1/2} t^{1/2} \leq C_L R t^{1/2}$$

for some constant $C_L > 0$, we have

$$u(t,x) \geq \min_{x\in\mathbf{T}} u_0(x) - C_L R t^{1/2}$$

for $t \in [0, T_1]$ and $x \in \mathbf{T}$. Thus, if we put

$$T_0 = \left\{ \min \left(\frac{\min_{x\in\mathbf{T}} u_0(x)}{2C_L R}, T_1 \right) \right\}^2,$$

then we obtain (12). $\square$

Proof of Theorem 5. We put

$$\alpha_* := \frac{1}{2} \min_{\eta\in\mathbf{T}} d_0(\eta) > 0,$$

$$A(\eta, \alpha_0, \alpha_1) := q(\frac{2}{\alpha_*}\alpha_0)(\alpha_1^2 + (1 - \kappa^0(\eta)\alpha_0)^2)^{-2},$$

$$B(\eta, \alpha_0, \alpha_1, \alpha_2) := q(\frac{2}{\alpha_*}\alpha_0)P(\eta, \alpha_0, \alpha_1, \alpha_2),$$

$$C(\eta, \alpha_0, \alpha_1, \alpha_2) := q(\frac{2}{\alpha_*}\alpha_0)Q(\eta, \alpha_0, \alpha_1, \alpha_2)$$

for $\eta \in \mathbf{T}$ and $(\alpha_0, \alpha_1, \alpha_2) \in \mathbf{R}^3$, where q is in (4). We note that

$$A(\eta, \alpha_0, \alpha_1) = (\alpha_1^2 + (1 - \kappa^0(\eta)\alpha_0)^2)^{-2}, \tag{13}$$

$$B(\eta, \alpha_0, \alpha_1, \alpha_2) = P(\eta, \alpha_0, \alpha_1, \alpha_2), \tag{14}$$

$$C(\eta, \alpha_0, \alpha_1, \alpha_2) = Q(\eta, \alpha_0, \alpha_1, \alpha_2) \tag{15}$$

for $\eta \in \mathbf{T}$, $\alpha_0 \geq \alpha_*/2$ and $(\alpha_1, \alpha_2) \in \mathbf{R}^2$, since $q(2\alpha_0/\alpha_*) = 1$ for $\alpha_0 \geq \alpha_*/2$.

We consider the following equation:

$$\begin{cases} d_t + A(\eta, d, d_\eta)d_{\eta\eta\eta\eta} + B(\eta, d, d_\eta, d_{\eta\eta})d_{\eta\eta\eta} + C(\eta, d, d_\eta, d_{\eta\eta}) = 0, \\ d(0, \eta) = d_0^{\varepsilon,\delta}(\eta) \end{cases} \tag{16}$$

for $t \geq 0$ and $\eta \in \mathbf{T}$.

We check (H1) and (H2) for (16). $A(\eta, \alpha_0, \alpha_1)$ satisfies (H1) with $\Lambda_* = \alpha_*$, $\Lambda = 2C_L\|d_0^{\varepsilon,\delta}\|_{H^m(\mathbf{T})}$ (where C_L is a constant in Sobolev inequality $\|f\|_{L^\infty} \leq C_L\|f\|_{H^1(\mathbf{T})}$) and

$$a_1 = (\Lambda^2 + (1 + \|\kappa^0\|_{L^\infty(\mathbf{T})}\Lambda)^2)^{-2}.$$

Since $\kappa^0(\eta) \leq 0$, $A(\eta, \alpha_0, \alpha_1)$, $B(\eta, \alpha_0, \alpha_1, \alpha_2)$ and $C(\eta, \alpha_0, \alpha_1, \alpha_2)$ also satisfy (H2).

Therefore, it follows from Lemma 7 that there are

$$T_0^{\varepsilon,\delta} := T_0(\|d_0^{\varepsilon,\delta}\|_{H^m(\mathbf{T})}, \min_{\eta\in\mathbf{T}} d_0^{\varepsilon,\delta}(\eta)) > 0$$

and a unique solution $d^{\varepsilon,\delta}(t, \eta)$ of (16) satisfying

$$d^{\varepsilon,\delta} \in L^2(0, T_0^{\varepsilon,\delta}; H^{m+2}(\mathbf{T})), \quad d_t^{\varepsilon,\delta} \in L^2(0, T_0^{\varepsilon,\delta}; H^{m-2}(\mathbf{T}))$$

$$\sup_{t\in[0,T_0^{\varepsilon,\delta}]} \|d^{\varepsilon,\delta}(t)\|_{H^m(\mathbf{T})} \leq \mathcal{A}(\|d_0^{\varepsilon,\delta}\|_{H^m(\mathbf{T})}),$$

$$\min_{(t,\eta)\in[0,T_0^{\varepsilon,\delta}]\times\mathbf{T}} d^{\varepsilon,\delta}(t,\eta) \geq \frac{1}{2}\min_{\eta\in\mathbf{T}} d_0^{\varepsilon,\delta}(\eta).$$

Put

$$T_0^\delta := T_0(b_{a,\delta}^m, \min_{\eta\in\mathbf{T}} d_0(\eta)) > 0,$$

where $b_{a,\delta}^m$ is in Theorem 3 (ii). Then, it follows from Remark 8 and Theorem 3 (ii) and (v) that

$$T_0^{\varepsilon,\delta} \geq T_0^\delta,$$

$$\sup_{t\in[0,T_0^\delta]} \|d^{\varepsilon,\delta}(t)\|_{H^m(\mathbf{T})} \leq \mathcal{A}(b_{a,\delta}^m),$$

$$\min_{(t,\eta)\in[0,T_0^\delta]\times\mathbf{T}} d^{\varepsilon,\delta}(t,\eta) \geq \frac{1}{2}\min_{\eta\in\mathbf{T}} d_0(\eta) = \alpha_*$$

for $\varepsilon \in (0,1]$. Furthermore, the last inequality, (13)-(15) yield

$$A(\eta, d^{\varepsilon,\delta}, d_\eta^{\varepsilon,\delta}) = ((d_\eta^{\varepsilon,\delta})^2 + (1-\kappa^0 d^{\varepsilon,\delta})^2)^{-2},$$

$$B(\eta, d^{\varepsilon,\delta}, d_\eta^{\varepsilon,\delta}, d_{\eta\eta}^{\varepsilon,\delta}) = P(\eta, d^{\varepsilon,\delta}, d_\eta^{\varepsilon,\delta}, d_{\eta\eta}^{\varepsilon,\delta}),$$

$$C(\eta, d^{\varepsilon,\delta}, d_\eta^{\varepsilon,\delta}, d_{\eta\eta}^{\varepsilon,\delta}) = Q(\eta, d^{\varepsilon,\delta}, d_\eta^{\varepsilon,\delta}, d_{\eta\eta}^{\varepsilon,\delta}).$$

Thus, we conclude that $d^{\varepsilon,\delta}(t,\eta)$ for $t \in [0, T_0^\delta]$ and $\eta \in \mathbf{T}$ is the desired solution of (7). The proof of Theorem 5 is complete. □

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