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Series #411. May 1998

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THE LIGHTCONE GAUSS MAP AND THE LIGHTCONE DEVELOPABLE OF A SPACELIKE CURVE IN MINKOWSKI 3-SPACE

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ABSTRACT. We define the notion of lightcone Gauss maps, lightcone pedal curves and lightcone developables of spacelike curves in Minkowski 3-space and establish the relationships between singularities of these subjects and geometric invariants of curves under the action of Lorentz group.

1. INTRODUCTION

There are several articles concerning 'generic differential geometry' in Euclidean space [1–5,8,10, etc.]. The main tools in these articles are the distance-squared functions and the height functions on submanifolds. In this paper we introduce the notion of lightcone height functions and Lorenzian distance-squared functions on spacelike curves in Minkowski 3-space. We also define the notion of lightcone Gauss maps, lightcone pedal curves and lightcone developables of spacelike curves and establish the relationships between singularities of these subjects and geometric invariants of curves under the action of Lorentz group as applications of standard techniques of singularity theory for the above functions. For the basic notions in Lorentzian geometry, see [9].

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $< \boldsymbol{x}, \boldsymbol{y} >= -x_1y_1 + x_2y_2 + x_3y_3$. We call $(\mathbb{R}^3, <,>)$ a 3-dimensional *pseudo Euclidean space*, or *Minkowski 3-space*. We denote $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, <,>)$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $< \boldsymbol{x}, \boldsymbol{x} > > 0$, $< \boldsymbol{x}, \boldsymbol{x} > = 0$ or $< \boldsymbol{x}, \boldsymbol{x} > < 0$ respectively.

Let $\gamma : I \longrightarrow \mathbb{R}^3_1;\ \gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a smooth regular curve in $\mathbb{R}^3_1$ (i.e.,$\dot\gamma(t) \neq 0$ for any $t \in I$), where I is an open interval. The curve γ is called a *spacelike curve* if $< \dot\gamma(t), \dot\gamma(t) > > 0$ for any $t \in I$. The *norm* of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{| < \boldsymbol{x}, \boldsymbol{x} > |}$. The *arc-length* of a spacelike curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot\gamma(t)\|\, dt.$$

1991 *Mathematics Subject Classification.* 53B30, 58C28, 57R70.

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Then the parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. So we say that a spacelike curve γ is *parameterized by arc-length* if it satisfies that $\|\gamma'(s)\| = 1$. Throughout the reminder in this paper we denote the parameter s of γ as the arc-length parameter. Let us denote $\boldsymbol{t}(s) = \gamma'(s)$, and we call $\boldsymbol{t}(s)$ a *unit tangent vector* of γ at s. We define the *curvature* by $k(s) = \sqrt{|<\gamma''(s), \gamma''(s)>|}$. If $k(s) \neq 0$ then the *unit principal normal vector* $\mathbf{n}$ of the curve γ at s is given by $\gamma''(s) = k(s) \cdot \boldsymbol{n}(s)$. The *signature* of $\boldsymbol{x}$ is defined to be

$$\operatorname{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} : \text{spacelike} \\ 0 & \boldsymbol{x} : \text{lightlike} \\ -1 & \boldsymbol{x} : \text{timelike}\,. \end{cases}$$

We denote that $\delta(\gamma(s)) = \operatorname{sign}(\boldsymbol{n}(s))$.

For any $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the *pseudo vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The unit vector $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$ is called a *unit binormal vector* of the curve γ at s. Since $\boldsymbol{t}(s)$ is spacelike, we have $<\boldsymbol{b}(s), \boldsymbol{b}(s)> = -\delta(\gamma(s))$ and $\operatorname{sign}(\gamma'(s)) = 1$ Then the following Frenet-Serret type formula holds:

$$\begin{cases} \boldsymbol{t}'(s) = k(s) \cdot \boldsymbol{n}(s) \\ \boldsymbol{n}'(s) = -\delta(\gamma(s)) \cdot k(s) \cdot \boldsymbol{t}(s) + \tau(s) \cdot \boldsymbol{b}(s) \\ \boldsymbol{b}'(s) = \tau(s) \cdot \boldsymbol{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve γ at s (cf.,[6]). This is the fundamental formula for the study of spacelike curve in $\mathbb{R}^3_1$, it is, however, useless at the point $\gamma(s)$ with $k(s) = 0$. We now denote that $N(s) = \gamma''(s)$ and $B(s) = \boldsymbol{t}(s) \wedge N(s)$. We simply call $N(s)$ a *principal normal vector* and $B(s)$ a *binormal vector*. If $k(s) \neq 0$, then we have $N(s) = k(s)\boldsymbol{n}(s)$ and $B(s) = k(s)\boldsymbol{b}(s)$. It follows that

$$< N(s) \pm B(s), N(s) \pm B(s) > = k(s)(\delta(\gamma(s)) - \delta(\gamma(s))) = 0.$$

If $k(s) = 0$, then $N(s)$ is a lightlike vector, so that any pseudo perpendicular vector in the normal plane of $\gamma(s)$ is parallel to $N(s)$. We can prove that $N(s) \pm B(s) \neq \mathbf{0}$. This means that $N(s) \pm B(s)$ is a lightlike vector which is parallel to the vector $N(s)$ for $s \in I$ with $k(s) = 0$. Define

$$S^1_+ = \{x \in \mathbb{R}^3_1 | x = (1, x_2, x_3), x_2^2 + x_3^2 = 1\},$$

$$C_p = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3_1 | -(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = 0\},$$

where $p = (p_1, p_2, p_3)$. We call S^1_+ a *lightlike unit circle* and $C^*_p = C_p - \{p\}$ a *lightcone at the vertex* p. For any lightlike vector $\boldsymbol{x} = (x_1, x_2, x_3)$, we denote that $\tilde{\boldsymbol{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}) \in S^1_+$. Under this notation, we have $\widetilde{N(s) \pm B(s)} = \widetilde{N(s)}$ if $k(s) = 0$.

We now define a map $LG_\gamma^+ : I \longrightarrow S_+^1$ by

$$LG_\gamma^+(s) = \widetilde{N(s) + B(s)}$$

and a curve $LP_\gamma^+ : I \longrightarrow C^*$ by

$$LP_\gamma^+(s) = < \gamma(s), \widetilde{N(s) + B(s)} > \cdot(\widetilde{N(s) + B(s)}),$$

where we may assume that $\gamma(s) \neq 0$. Under the assumption that $k(s) \neq 0$, we also define a ruled surface $LD_\gamma^+ : I \times \mathbb{R} \longrightarrow \mathbb{R}_1^3$ by

$$LD_\gamma^+(s, u) = \gamma(s) + u(\boldsymbol{n}(s) + \boldsymbol{b}(s)).$$

We call LG_γ^+ the *lightcone Gauss map* and LP_γ^+ the *lightcone pedal curve (or, lightcone dual curve)* of γ. We also call LD_γ^+ the *lightcone developable* of γ. The geometric properties of these subjects will be discussed in §3 We can also define LG_γ^-, LP_γ^- and LD_γ^- exactly the same way as the above. Since these have same properties as those of the above, so that we do not consider these.

Let $\gamma : S^1 \longrightarrow \mathbb{R}_1^3$ be a spacelike curve with $\gamma''(s) \neq 0$. We consider the following properties of γ.

(A 1) The number of points p of $\gamma(S^1)$ where the lightcone at p having two-point contact with the principal normal curve γ'' is finite.

(A 2) There is no point p of $\gamma(S^1)$ where the lightcone at p having greater than three-point contact with the principal normal curve γ''.

In §3, under the assumption that $k(s_0) \neq 0$, we shall show that there exist just two lightcones at points $v_\pm = \gamma(s_0) + \frac{1}{k(s_0)\delta(\gamma(s_0))}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ such that γ has three-point contact with the lightcones at $v_\pm$. We call each lightcone *an osculating lightcone* of γ.

(A 3) The number of points at where γ has at least four-point contact with the osculating lightcone is finite.

(A 4) There are no osculating lightcone with which γ has five-point contact at a point.

Our main results are formulated as follows:

Theorem A. (1) *Let $Imm_s(S^1, \mathbb{R}_1^3)$ be a space of spacelike curves equipped with Whitney C^∞-topology. Then the set of curves which satisfy* (A 1) *and* (A 2) *is a residual set in $Imm_s(S^1, \mathbb{R}_1^3)$.*
(2) *Let $Imm_s^+(S^1, \mathbb{R}_1^3)$ be a space of spacelike curves with $k(s) \neq 0$ equipped with Whitney C^∞-topology. Then the set of curves which satisfy* (A 3) *and* (A 4) *is a residual set in $Imm_s^+(S^1, \mathbb{R}_1^3)$.*

Theorem B. (1) *Under the assumption of* (A 1) *and* (A 2),

a) *The lightcone Gauss map LG_γ^+ has a fold point at s_0 if and only if $k(s_0) = 0$.*

b) *The lightcone pedal curve LP_γ^+ has a cusp point at $LP_\gamma^+(s_0)$ if and only if $k(_0) = 0$.*

(2) *Under the assumption of* (A 3) *and* (A 4),

a) *The lightcone developable LD_γ^+ is nonsingular along the curve $\gamma(s)$. Moreover, if we consider another lightcone developable LD_γ^-, these surfaces intersect transversally along the curve $\gamma(s)$.*

b) *The lightcone developable* LD^{+}_{γ} *is locally diffeomorphic to the cuspidal edge at* $\gamma(s_0) + u_0(\boldsymbol{n}(s_0) + \boldsymbol{b}(s_0)))$ *if and only if* $u_0 = \frac{1}{k(s_0)\delta(\gamma(s_0))}$. *Moreover, the locus of the vertices of the osculating lightcones* $\gamma(s) + \frac{1}{k(s)\delta(\gamma(s))}(\boldsymbol{n}(s) + \boldsymbol{b}(s)))$ *is the cuspidal edge.*

c) *The lightcone developable* LD^{+}_{γ} *is locally diffeomorphic to the swallow tail at* $\gamma(s_0)+ u_0(\boldsymbol{n}(s_0) + \boldsymbol{b}(s_0)))$ *if and only if* $u_0 = \frac{1}{k(s_0)\delta(\gamma(s_0))}$ *and* $(k' - \tau \cdot k)(s) = 0$.

In [7] M. Kossowski introduced the notion of $S^1 \times S^1$-valued Gauss maps associated with spacelike curves. The authors are much inspired by his paper. Especially, the author learned the fact that $\boldsymbol{n}(s) \pm \boldsymbol{b}(s)$ is lightlike in his paper. As a matter of fact, the notion of lightcone Gauss maps in this paper is almost the same as that of $S^1 \times S^1$-valued Gauss maps. The basic techniques in this paper depend heavily on those in the attractive book of Bruce and Giblin [4]. In §2 we introduce the notion of lightcone height functions and Lorentzian distance-squared functions on spacelike curves and study these properties. The Lorentzian distance-squared function is just a direct analogy of the distance squared function in Euclidean 3-space. We can consider the Lorentzian focal surfaces and the Lorentzian canal surface of a spacelike curve by using Lorentzian distance squared functions. It is, however, a simple analogy of Euclidean case. Here, we consider the notion of lightcone developables which is a special subject in Lorentzian geometry. We study some Lorentzian invariants in §3. These invariants are squeezed out by the study of lightcone height functions and Lorentzian distance squared functions in §2. The proof of Theorem B is given in §4. In §5 we consider the generic properties by using the Lorentzian analogy of the notion of Monge-Taylor maps of curves in [4].

2. Lorentzian invariant functions on spacelike curves

In this section we introduce two different families of functions on a spacelike curve which are useful to the study of Lorentzian invariants of spacelike curves.

Lightcone height functions

For a spacelike curve $\gamma : I \longrightarrow \mathbb{R}^3_1$, we now define a function

$$H : I \times S^1_+ \longrightarrow \mathbb{R}$$

by $H(s, v) =< \gamma, v >$. We call H a *lightcone height function* on the spacelike curves γ. We denote that $h(s) = H_{v_0}(s) = H(s, v_0)$ for any fixed $v_0 \in S^1_+$. We have the following proposition.

Proposition 2.1. *Let* $\gamma : I \longrightarrow \mathbb{R}^3$ *be a unit speed spacelike curve with* $\gamma''(s) \neq 0$. *Then*

(1) $h'(s_0) = 0$ *if and only if* v *is in the normal plane at* $\gamma(s_0)$. *Especially, if* $k(s_0) \neq 0$, *then* $v = N(s_0) \pm B(s_0)$.

(2) $h'(s_0) = h''(s_0) = 0$ *if and only if* $v = N(s_0) \pm B(s_0) = N(s_0)$ *and* $k(s_0) = 0$.

(3) $h'(s_0) = h''(s_0) = h^{(3)}(s_0) = 0$ *if and only if* $v(s_0) = N(s) \pm B(s_0) = N(s_0)$, *and* $k(s_0) =< N'(s_0), N(s_0) >= 0$.

(4) $h'(s_0) = h''(s_0) = h^{(3)}(s_0) = h^{(4)}(s_0) = 0$ *if and only if* $v = N(s_0) \pm B(s_0) = N(s_0)$, *and* $k(s_0) =< N'(s_0), N(s_0) >=< N''(s_0), N(s_0) >= 0$.

Proof. Let $H : I \times S^1_+ \longrightarrow \mathbb{R}$ be the lightcone height function on the spacelike curve $\gamma : I \longrightarrow \mathbb{R}^3_1$. Then we have $\frac{\partial H}{\partial s} =< \gamma'(s), v >=< \boldsymbol{t}(s), v >$, where $v = (1, x_2, x_3) \in {S_+}^1$. It follows that $\frac{\partial H}{\partial s} = 0$ if and only if $< v, \boldsymbol{t} >= 0$. Especially, if $k(s) \neq 0$, then there exist λ, μ such that $v = \lambda(\boldsymbol{n} + \mu \boldsymbol{b})$. Since $< v, v >= 0$, we have $\delta(\gamma) \cdot \lambda^2 - \delta(\gamma) \cdot \mu^2 = 0$, so that we have $\lambda^2 = \mu^2$. It follows that we have $v = \boldsymbol{n} \pm \boldsymbol{b}$.

On the other hand, we have $\frac{\partial^2 H}{\partial s^2} =< N(s), v >$. If $k(s) \neq 0$, then $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = 0$ if and only if $v = \boldsymbol{n} \pm \boldsymbol{b} = N(s) \pm B(s)$ and $< k \cdot \boldsymbol{n}, v >= 0$ this is equivalent to the condition that $v = \boldsymbol{n} \pm \boldsymbol{b}$ and $k < \boldsymbol{n}, \boldsymbol{n} \pm \boldsymbol{b} >= k\delta(\gamma) = 0$. Since $\delta(\gamma) = \pm 1$, the above condition means that $v = \boldsymbol{n} \pm \boldsymbol{b}$ and $k = 0$. This is a contradiction, so that we have $k(s) = 0$, then $N(s)$ is a lightlike vector. Since $< v, N(s) >= 0$, v is parallel to $N(s)$. It is equivalent to the fact that $v = N(s) = N(s) \pm B(s)$.

Since $\frac{\partial^3 H}{\partial s^3} =< N'(s), v >$, $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 F}{\partial s^3} = 0$ if and only if $v = N(s) = N(s) \pm B(s)$, $k(s) = 0$ and $< N'(s), N(s)) >= 0$.

Moreover, we have $\frac{\partial^4 H}{\partial s^4} =< N''(s), v >$. Then $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 H}{\partial s^3} = \frac{\partial^4 H}{\partial s^4} = 0$ if and only if $v = N(s_0) \pm B(s_0) = N(s_0)$, $k =< N', N >= 0$ and $< N'', v >= 0$. Since v is parallel to N, we have $< N'', N >= 0$. □

Lorentzian distance-squared functions

We now define a function

$$G : I \times \mathbb{R}^3_1 \longrightarrow \mathbb{R}$$

by

$$G(s, v) =< \gamma - v, \gamma - v > .$$

We call G the *Lorentzian distance-squared function* on a spacelike curve γ. We denote that $g(s) = G_{v_0}(s) = G(s, v_0)$ for any fixed $v_0 \in {\mathbb{R}_1}^3$. We also have the following proposition.

Proposition 2.2. *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $k(s) \neq 0$. Then*

(1) $g(s_0) = g'(s_0) = 0$ *if and only if* $\gamma(s_0) - v(s_0) = \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ $(\lambda \in \mathbb{R} - \{0\})$.

(2) $g(s_0) = g'(s_0) = g''(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \frac{1}{k(s_0) \cdot \delta(\gamma(s_0))}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ *and* $k(s_0) \neq 0$.

(3) $g(s_0) = g'(s_0) = g''(s_0) = g^{(3)}(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \frac{1}{k(s_0) \cdot \delta(\gamma(s_0))}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$, $k(s_0) \neq 0$ *and* $(k' \mp \cdot k \cdot \tau)(s_0) = 0$.

(4) $g(s_0) = g'(s_0) = g''(s_0) = g^{(3)}(s_0) = g^{(4)}(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \frac{(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))}{k(s_0) \cdot \delta(\gamma(s_0))}$, $k(s_0) \neq 0$ *and* $(k' \mp \cdot k \cdot \tau)(s_0) = (k' \mp \cdot k \cdot \tau)'(s_0) = 0$.

Proof. Let $G : I \times \mathbb{R}^3_1 \longrightarrow \mathbb{R}$ be a Lorentzian distance-squared function on the spacelike curve $\gamma : I \longrightarrow \mathbb{R}^3_1$. Then we have $\frac{\partial G}{\partial s} = 2 < \gamma', \gamma - v >$. It follows that $\frac{\partial G}{\partial s} = 0$ if and only if $\gamma - v = \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b}$. Thus $G(s, v) = \frac{\partial G}{\partial s} = 0$ if and only if $\gamma - v = \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b}$ and $< \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b}, \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b} >= 0$. This is equivalent to the condition that $\gamma - v = \lambda(\boldsymbol{n} \pm \boldsymbol{b})$.

On the other hand, since $\frac{\partial^2 G}{\partial s^2} = 2 < \boldsymbol{t}', \gamma - v > +2 < \boldsymbol{t}, \boldsymbol{t} >= 2\{< k \cdot \boldsymbol{n}, \gamma - v > +1\}$, $G = \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = 0$ if and only if $\gamma - v = \lambda(\boldsymbol{n} \pm \boldsymbol{b})$ and $< k \cdot \boldsymbol{n}, \lambda(\boldsymbol{n} \pm \boldsymbol{b}) >= -1$. The last equality is equivalent to the relation that $k \cdot \lambda \cdot \delta(\gamma) = k \cdot \lambda < \boldsymbol{n}, \boldsymbol{n} >= -1$ This means that $v = \gamma + \frac{1}{k \cdot \delta(\gamma)}(\boldsymbol{n} \pm \boldsymbol{b})$ and $k \neq 0$.

Since $\frac{\partial^3 G}{\partial s^3} = 2 < (k \cdot \boldsymbol{n})', \gamma - v >$, $G = \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = 0$ if and only if $k \neq 0$ and $< k' \cdot \boldsymbol{n} + k \cdot \boldsymbol{n}', -\frac{1}{k \cdot \delta(\gamma)}(\boldsymbol{n} \pm \boldsymbol{b}) >= 0$, which is equivalent to the condition that $k \neq 0$ and $k' \cdot \delta(\gamma) \pm k \cdot \tau < \boldsymbol{b}, \boldsymbol{b} > = 0$. This can be reduced to the condition that $k \neq 0$ and $k' - \pm \cdot k \cdot \tau = 0$.

Finally, we have $\frac{\partial^4 G}{\partial s^4} = 2 < (k \cdot \boldsymbol{n})'', \gamma - v > +2 < (k \cdot \boldsymbol{n})', \boldsymbol{t} >$. Then $\frac{\partial^4 G}{\partial s^4} = 0$ if and only if

$$< k'' \cdot \boldsymbol{n} + k' \cdot \boldsymbol{n}' + \tau \boldsymbol{b}' k + \tau \boldsymbol{b} k' - 2\delta k k' \boldsymbol{t} - \delta k^2 \boldsymbol{t}' + k \tau' \boldsymbol{b}, \gamma - v >$$
$$+ < k' \boldsymbol{n} + \tau \boldsymbol{b} k - \delta k^2 \boldsymbol{t}, \boldsymbol{t} >= 0.$$

This is equivalent to the condition that

$$< k'' \boldsymbol{n} + k' \boldsymbol{n}' + k \tau \boldsymbol{b}' + \tau \boldsymbol{b} k' - 2\delta k k' \boldsymbol{t} - \delta k^3 \boldsymbol{n} + k \tau' \boldsymbol{b}, \gamma - v > -\delta k^2 = 0.$$

Therefore $G = \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = \frac{\partial^4 G}{\partial s^4} = 0$ if and only if

$$< k'' \boldsymbol{n} + k' \boldsymbol{n}' + \tau \boldsymbol{b}' k + \tau \boldsymbol{b} k' - \delta k^3 \boldsymbol{n} + k \tau' \boldsymbol{b}, -\frac{(\boldsymbol{n} \pm \boldsymbol{b})}{k \cdot \delta} > -\delta k^2 = 0.$$

By the Frenet-Serret type formula, we can translate this condition into the condition that

$$< k'' \boldsymbol{n} + k' \tau \boldsymbol{b} + k \tau^2 \boldsymbol{n} + k' \tau \boldsymbol{b} - \delta k^3 \boldsymbol{n} + k \tau' \boldsymbol{b}, -\frac{(\boldsymbol{n} \pm \boldsymbol{b})}{k \cdot \delta} > -\delta k^2 = 0.$$

So we have the condition that $k'' \mp 2k'\tau \mp k\tau + k\tau^2 = 0$. Under the condition that $k \neq 0, k' = \pm k \cdot \tau$, the above condition is equivalent to $(k' \mp k \cdot \tau)' = 0$. □

3. Lorentzian invariants of spacelike curves

In this section we study the geometric properties of the lightcone Gauss maps, the lightcone pedal curve and the lightcone developables of spacelike curves. By the propositions in the last section, we can recognize the functions $k(s)$ and $(k' \mp \tau k)(s)$ have special meanings. Firstly we have the following proposition.

Proposition 3.1. (1) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve with $\gamma''(s) \neq 0$. Then $k(s) \equiv 0$ if and only if $\gamma''(s) \in C^*_0$.*
(2) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve with $k(s) \neq 0$. Then $(k' \mp \tau k)(s) \equiv 0$ if and only if $p_\pm = \gamma(s) + \frac{1}{k(s)\delta(\gamma)}(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))$ are constant vectors. Under this condition, $\gamma(s) \in C^*_{p_\pm}$.*

Proof. By definition, the assertion (1) holds. In order to prove the assertion (2), we put

$$P_\pm(s) = \gamma(s) + \frac{1}{k(s)\delta(\gamma)}(\boldsymbol{n}(s) \pm \boldsymbol{b}(s)),$$

then we have

$$P'_\pm(s) = \frac{-(k' \mp \tau k)(s)}{k(s)\delta(s)}(\boldsymbol{n}(s) \pm \boldsymbol{b}(s)).$$

It follows that $P'_\pm(s) \equiv 0$ if and only if $(k' \mp \tau k)(s) \equiv 0$. Since $(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))$ is lightlike, $\gamma(s) \in C_{p_\pm}$, where $p_\pm = P_\pm(s)$. □

Corollary 3.2. (1) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve with $\gamma''(s) \neq 0$. If $k(s) \equiv 0$, then the lightcone Gauss map of γ is constant and the lightcone pedal curve of γ is a lightlike line.*
(2) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve with $k(s) \neq 0$. If $(k' - \tau k)(s) \equiv 0$, then the lightcone developable is the lightcone C_{p_+} with vertices p_+, where $p_+ = \gamma(s) + \frac{1}{k(s)\delta(\gamma)}(\boldsymbol{n}(s) + \boldsymbol{b}(s))$ are the constant points.*

Proof. (1) Since $N(s) = \gamma''(s)$ is lightlike for any $s \in I$, the locus of $\boldsymbol{t}(s) = \gamma'(s)$ is a lightlike line on *the pseudo sphere* S^2_1, where

$$S^2_1 = \{x \in \mathbb{R}^3_1 | < x, x >= 1\}.$$

It follows that $\widetilde{N}(s)$ is constant and the locus of $< \gamma(s), \widetilde{N(s)} > \widetilde{N(s)}$ is a lightlike line.
(2) By definition, we have

$$LD^+_\gamma(s,u) = \gamma(s) + u(\boldsymbol{n}(s) + \boldsymbol{b}(s)) = p_+ + (u - \frac{1}{k(s)\delta(\gamma)})(\boldsymbol{n}(s) + \boldsymbol{b}(s))$$

so that the image of $LD^+_\gamma(s,u)$ is in the pair of lightcones with vertices C_{p_+}. Since $\frac{\partial LD^+_\gamma}{\partial s}(s,u) = (1 - u\delta(\gamma)k(s))\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s)$ and $\frac{\partial LD^+_\gamma}{\partial u}(s,u) = (\boldsymbol{n}(s) + \boldsymbol{b}(s))$, we have

$$\frac{\partial LD^+_\gamma}{\partial s}(s,u) \wedge \frac{\partial LD^+_\gamma}{\partial u}(s,u) = (u\delta(\gamma)k(s) - 1)(\boldsymbol{n}(s) + \boldsymbol{b}(s)) + \tau(s)\boldsymbol{t}(s).$$

It follows that $LD^+_\gamma(s,u)$ is an immersed surface at (s,u) with $u \neq \frac{1}{k(s)\delta(\gamma)}$. The image of the set $u = \frac{1}{k(s)\delta(\gamma)}$ are the vertices p_+ of the lightcones. This completes the proof. $\square$

Let $F : \mathbb{R}^3_1 \longrightarrow \mathbb{R}$ be a submersion and $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve. We say that γ and $F^{-1}(0)$ have *k-point contact for $t = t_0$* if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0,\ g^{(k)}(t_0) \neq 0$. By Propositions 2.1, 2.2 and 3.1, we have the following proposition.

Proposition 3.3. 1) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $\gamma''(s) \neq 0$. Then γ'' and the lightcone C^*_0 have 2-point contact for $s = s_0$ if and only if $k(s_0) = 0$ and $< N'(s_0), N(s_0) >\neq 0$.*
2) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $k(s) \neq 0$. Then γ and the lightcone C^*_{p+} have 4-point contact for $s = s_0$ if and only if $(k' - k\tau)(s_0) = 0$ and $(k' - k\tau)'(s_0) \neq 0$.*

4. Unfoldings of functions by one-variable

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [4]. Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, x_0)$. We say that f has *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity ($k \geqq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$ for $i = 1, \ldots, r$. Then

F is called a *(p) versal unfolding* if the $(k-1)\times r$ matrix of coefficients (α_{ji}) has rank $k-1$ $(k-1 \leqq r)$. Under the same condition as the above, F is called a *versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k $(k \leqq r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *singular set* of F is the set

$$S_F = \{(s,x) | \frac{\partial F}{\partial s}(s,x) = 0\}.$$

The *bifurcation set* $\mathfrak{B}_F$ of F is the critical value set of the restriction to S_F of the canonical projection $\pi : \mathbb{R} \times \mathbb{R}^r \longrightarrow \mathbb{R}$:

$$\mathfrak{B}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s,x)\}$$

The *discriminant set* of F is the set

$$\mathfrak{D}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s,x)\}.$$

Then we have the following well-known result (cf., [4]).

Theorem 4.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0.*
(1) Suppose that F is an (p) versal unfolding.
- (a) *If $k=2$, then (s_0, x_0) is the fold point of $\pi | S_F$ and $\mathfrak{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
- (b) *If $k=3$, then $\mathfrak{B}_F$ is diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(2) Suppose that F is an versal unfolding.
- (a) *If $k=1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
- (b) *If $k=2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
- (c) *If $k=3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

Here, $C = \{(x_1, x_2) | {x_1}^2 = {x_2}^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallow tail. We also say that a point $x_0 \in \mathbb{R}^r$ is a fold point of a map germ $f : (\mathbb{R}^r, x_0) \longrightarrow (\mathbb{R}^r, f(x_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, x_0) \longrightarrow (\mathbb{R}^r, 0)$ and $\psi : (\mathbb{R}^r, f(x_0)) \longrightarrow (\mathbb{R}^r, 0)$ such that $\psi \circ \phi(x_1, \ldots, x_r) = (x_1, \ldots, x_{r-1}, x_r^2)$.

For a unit speed spacelike curve $\gamma : I \longrightarrow \mathbb{R}^3_1$, we now define a function

$$\widetilde{H} : I \times S^1_+ \times \mathbb{R} \to \mathbb{R}$$

by $\widetilde{H}(s,v,u) = H(s,v) - u = <\gamma(s), v> - u$, where H is the lightcone height function. Then we have the following fundamental theorem in this paper.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $\gamma''(s) \neq 0$ and $H : I \times S^1_+ \longrightarrow \mathbb{R}$ be the lightcone height function on γ.*
- (1) *If $h(s) = H_{v_0}(s)$ has the A_2-singularity at s_0, then H is the (p)versal unfolding of h.*
- (2) *If $h(s) = \widetilde{H}_{(v_0,u_0)}(s)$ has the A_k-singularity $(k = 1, 2)$ at s_0, then $\widetilde{H}$ is the versal unfolding of h.*

Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $k(s) \neq 0$ and $G : I \times \mathbb{R}^3_1 \longrightarrow \mathbb{R}$ be the Lorenzian distance-squared function. We consider the point $(s_0, v_0) \in I \times \mathbb{R}^3_1$ with $G(s_0, v_0) = 0$ and $v_0 \neq \gamma(s_0)$.

(3) *If $g(s)$ has the A_k-singularity $(k = 1, 2, 3)$ at s_0, then G is the versal unfolding of $g := G_{v_0}$.*

proof. (1) We have shown that H_{v_0} has the A_2-singularity at s_0 if and only if there exists a non-zero real number λ such that $v = \lambda N(s_0)$ and $k(s_0) = 0$. We denote by $\gamma(s) = (x_1(s), x_2(s), x_3(s))$, $v = (1, \cos\theta, \sin\theta)$. By definition,we have $H(s,\theta) = -x_1(s) + x_2(s) \cdot \cos\theta + x_3(s) \cdot \sin\theta$. It follows that $\frac{\partial H}{\partial \theta} = -x_2(s) \cdot \sin\theta + x_3(s) \cdot \cos\theta$ and the 1-jet of $\frac{\partial H}{\partial \theta}$ at s_0 is given by $s \cdot -x_2'(s_0) \cdot \sin\theta + s \cdot x_3'(s_0) \cdot \cos\theta$. So we require the 1×2 matrix $(-x_2(s) \cdot \sin\theta + x_3(s) \cdot \cos\theta, -x_2'(s_0) \cdot \sin\theta + x_3'(s_0) \cdot \cos\theta)$ to have rank 1 which it always does since $-x_2'(s_0) \cdot \sin\theta + x_3'(s_0) \cdot \cos\theta \neq 0$. In fact, $-x_2'(s_0) \cdot \sin\theta + x_3'(s_0) \cdot \cos\theta$ is equal to the first component of $\gamma' \wedge v$. Suppose that $-x_2'(s_0) \cdot \sin\theta + x_3'(s_0) \cdot \cos\theta = 0$. Since $< \gamma' \wedge v, \gamma' \wedge v >=< \boldsymbol{t} \wedge \lambda_1 N, \boldsymbol{t} \wedge \lambda_1 N >= \lambda^2 < B, B >= \lambda^2 < N, N >= 0$, we have

$$< (0, x_3' - x_1' \sin\theta, x_1' \cos\theta - x_2'), (0, x_3' - x_1' \sin\theta, x_1' \cos\theta - x_2') >= 0$$

This is equivalent to the condition that

$$x_3'^2 - 2x_1'x_3' \sin\theta + x_1'^2 \sin^2\theta + x_1'^2 \cos^2\theta - 2x_1'x_2' \cos\theta + x_2'^2 = 0$$

It follows that

$$x_3'^2 + x_2'^2 + x_1'^2 - 2x_1'(x_2' \sin\theta + x_3' \cos\theta) = 0.$$

Since $-x_1' + x_2' \sin\theta + x_3' \cos\theta =< \gamma', v >= 0$, we have $-x_1'^2 + x_2'^2 + x_3'^2 = 0$. On the other hand, $-x_1'^2 + x_2'^2 + x_3'^2 =< \gamma', \gamma' >= 1$. This is a contradiction. Hence H is (p) versal.

(2) In this case we have

$$\widetilde{H}(s, \theta, u) = H(s, \theta) - u = -x_1(s) + x_2(s) \cdot \cos\theta + x_3(s) \cdot \sin\theta - u.$$

By Proposition 2.1, h has A_1 at s_0 if and only if v is in the normal plane at $\gamma(s_0)$ and A_2 at s_0 if and only if there exists a non-zero real number λ_1 such that $v = \lambda_1(s_0)N(s_0)$ and $k(s_0) = 0$.

Since $\widetilde{H}(s, v, u) = H(s, v) - u =< \gamma(s), v > -u = -x_1(s) + x_2(s) \cdot \cos\theta + x_3(s) \cdot \sin\theta - u$. we have $\frac{\partial H}{\partial \theta} = -x_2 \cdot \sin\theta + x_3 \cdot \cos\theta$ and $\frac{\partial H}{\partial u} = 1$ at s_0. So the rank of the 1×2 matrix $(-x_2 \cdot \sin\theta + x_3 \cdot \cos\theta, 1)$ is 1. By the same reason as the case 1), we have $-x_2'(s_0) \cdot \sin\theta + x_3'(s_0) \cdot \cos\theta \neq 0$. It follows that the rank of the 2×2 matrix

$$\begin{pmatrix} -x_2 \cdot \sin\theta + x_3 \cdot \cos\theta & 1 \\ -x_2' \cdot \sin\theta + x_3' \cdot \cos\theta & 0 \end{pmatrix}$$

is 2. This means that $\widetilde{H}$ is versal when $h(s)$ has the A_k-singularity $(k = 1, 2)$ at s_0.

(3) In this case we have

$$G(s, v) = -(x_1(s) - v_1)^2 + (x_2(s) - v_2)^2 + (x_3(s) - v_3)^2,$$

where $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ and $v = (v_1, v_2, v_3)$.

Thus we have $\frac{\partial G}{\partial v_1}(s) = 2(x_1(s) - v_1)$, so the 2-jet at s_0 is $2(sx_1'(s_0) + \frac{1}{2}s^2 x_1''(s_0))$. We also have $\frac{\partial G}{\partial v_i} = -2(x_i(s) - v_i)$ $(i = 2, 3)$, so the 2-jet at s_0 is $-2(sx_i'(s_0) + \frac{1}{2}s^2 x_i''(s_0))$. The condition for versality can be checked as follows:

(i) By Proposition 2.2, g has the A_1-singularity at s_0 if and only if there exists a non-zero real number λ such that $v = \gamma(s_0) - \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ and $k(s) \cdot \lambda \cdot \delta \neq -1$. When g has the A_1-singularity at s_0, we require the 1×3 matrix $(2(x_1(s_0) - v_{0,1}), -2(x_2(s_0) - v_{0,2}), -2(x_3(s_0) - v_{0,3}))$ to have rank 1, which it always does since $v_0 \neq \gamma(s_0)$.

(ii) It also follows from Proposition 2.2 that g has the A_2-singularity at s_0 if and only if $v = \gamma + \frac{1}{k(s_0)\cdot\delta(\gamma)}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ and $k(s) \neq 0, k'(s) \neq \pm k(s) \cdot \tau$. When g has A_2 at s, we require the 2×3 matrix

$$\begin{pmatrix} 2(x_1(s_0) - v_1) & -2(x_2(s_0) - v_2) & -2(x_3(s_0) - v_3) \\ 2x_1'(s_0) & -2x_2'(s_0) & -2x_3'(s_0) \end{pmatrix}$$

to have rank 2, which follows from the proof of the case (iii).

(iii) By Proposition 2.2, g has the A_3-singularity at s_0 if and only if $v = \gamma + \frac{(\boldsymbol{n}\pm\boldsymbol{b})}{k\cdot\delta})$ and $k(s_0) \neq 0, (k' \mp k \cdot \tau)(s_0) = 0$ and $(k' \mp k \cdot \tau)'(s_0) = 0$. When g has A_3 at s, we require the 3×3 matrix

$$\begin{pmatrix} 2(x_1(s_0) - v_1) & -2(x_2(s_0) - v_2) & -2(x_3(s_0) - v_3) \\ 2x_1'(s_0) & -2x_2'(s_0) & -2x_3'(s_0) \\ x_1''(s_0) & -x_2''(s_0) & -x_3''(s_0) \end{pmatrix}$$

to be nonsingular. The determinant of this matrix is $4 \det((v - \gamma(s_0))\ \gamma'(s_0)\ \gamma''(s_0)) = 4 < (v - \gamma(s_0)) \wedge \boldsymbol{t}(s_0), k(s_0) \cdot \boldsymbol{n}(s_0) >= 4 < \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0)) \wedge \boldsymbol{t}(s_0), k(s_0) \cdot \boldsymbol{n}(s_0) >= 4 < -\lambda \boldsymbol{b}(s_0) \mp \lambda \boldsymbol{n}(s_0), k(s_0)\boldsymbol{n}(s_0) >= \pm 4\lambda k(s_0)\delta(\gamma) \neq 0$. But this just says $k(s_0) \neq 0$, This completes the proof. □

We now give the proof of Theorem B.

Proof of Theorem B. For the proof of the assertion (1) a), we consider the set S_H associated with the lightcone height function H given by $S_H = \{(s, v) \in I \times S^1_+ | h'(s) = 0\}$. By Proposition 2.1, we have $S_H = \{(s, v) | v = N(s) \pm B(s)\}$. We also consider the canonical projection $\pi : I \times S^1_+ \longrightarrow S^1_+$ and we can identify $\pi | S_H$ and the lightcone Gauss map $LG^{\pm}_{\gamma}$. By the assumption and Propositions 2.1 and 3.3, h has the A_2-singularity at s_0 if and only if $k(s_0) = 0$. It follows from Theorem 4.2 that H is the (p)versal unfolding of h at s_0.

Therefore Lemma 4.1, (1) (a) asserts that $\pi | S_H$ has a fold point at s_0.

In order to prove the assertion (1) b), we define a map

$$\widetilde{H} : I \times S^1_+ \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$\widetilde{H}(s, v, u) = H(s, v) - u =< \gamma(s), v > -u.$$

The discriminant set of $\widetilde{H}$ is

$$\mathfrak{D}_H = \{(v,u)|u =< \gamma, v >,\ v = \widetilde{N(s) \pm B(s)}\}.$$

Applying a Lorentzian motion to the curve γ, we may assume that $< \gamma(s), \gamma(s) > \neq 0$. This means that $u =< \gamma(s), \widetilde{N(s) \pm B(s)} > \neq 0$.

We now define a map $\Phi : S^1_+ \times \mathbb{R}_+ \longrightarrow C_+$ by $\Phi((1, x_2, x_3), \mu) = (\mu, \mu \cdot x_2, \mu \cdot x_3)$, where $\mathbb{R}_+ = \{x \in \mathbb{R}|x > 0\}$ and $C_+ = \{(x_1, x_2, x_3)|{x_1}^2 = {x_2}^2 + {x_3}^2,\ x_1 > 0\}$. Since $\Phi^{-1}(x_1, x_2, x_3) = ((1, \frac{x_2}{x_1}, \frac{x_3}{x_1}), x_1)$, Φ is a diffeomorphism. By the above arguments we may assume that $\mathfrak{D}_H \subset S^1_+ \times \mathbb{R}_+$. It is clear that

$$\Phi(\mathfrak{D}_H) = \{(u \cdot v)|v =< \gamma, \widetilde{N(s) \pm B(s)} > \cdot(\widetilde{N(s) \pm B(s)}),\ s \in I\}.$$

By Lemma 4.1 and Theorem 4.2 (2), the discriminant set $\mathfrak{D}_H$ of $\widetilde{H}$ is locally diffeomorphic to a line or the cusp. It follows from Proposition 2.1 that the proof of Theorem B (1) b is completed.
The assertions (2) a) is trivial by definition of the lightcone developable.

For the proof of the assertion (2) b), c), we consider the discriminant set of G. By Proposition 2.2, the discriminant set of G is

$$\mathfrak{D}_G = \{v \in {\mathbb{R}_1}^3|\text{ there exists } s \text{and non-zero real number } \lambda \text{ with } v = \gamma(s) - \lambda(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))\}.$$

So the assertions follow from Proposition 2.2 and Theorem 4.2. □

5. Generic properties of spacelike curves

In this section we consider the notion of Lorentzian Monge-Taylor maps for spacelike curves analogous to the ordinary notion of Monge-Taylor maps for space curves in Euclidean space (cf.,[4]). Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a (regular) spacelike curve, with I an open connected subset of the unit circle S^1_+, increasing t corresponding the anticlockwise orientation of S^1_+. We now choose a smooth family of unit vectors $\boldsymbol{n}(t)$, with $\boldsymbol{n}(t)$ pseudonormal to γ at t, so $\|\boldsymbol{n}(t)\| = 1$ and $< \boldsymbol{n}(t), \boldsymbol{t}(t) >= 0$ for all $t \in I$. Such $\boldsymbol{n}(t)$ can be obtained as follows: consider the smooth map $\boldsymbol{t} : I \to {S_1}^2$ which takes t to the unit tangent vector $\boldsymbol{t}(t)$. If V is any vector in ${H_1}^2$ we can obtain the vector field $\boldsymbol{n}(t)$ by pseudo-orthogonally projecting V onto each of the normal planes and normalizing. Thus $\boldsymbol{n}(t) = \frac{V - < V, \boldsymbol{t}(t) > \boldsymbol{t}(t)}{\|V - < V, \boldsymbol{t}(t) > \boldsymbol{t}(t)\|}$, then we have $< \boldsymbol{n}, \boldsymbol{n} >= -1$ and $< \boldsymbol{n}, \boldsymbol{t} >= 0$. We can obtain a second smooth family of unit vectors $\boldsymbol{b}(t) = \boldsymbol{t}(t) \wedge \boldsymbol{n}(t)$ normal to γ at t. We remark that the triple $\boldsymbol{t}(t)$, $\boldsymbol{n}(t)$, $\boldsymbol{b}(t)$ is the Lorentzian frame along γ. We now use the psuedoperpendicular lines spanned by $\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)$ as axes at $\gamma(t)$ with the unit points on the axes corresponding to the three given vectors. We remark that the Minkowski metric is invariant under the Lorentz transformation. Note the curve $\gamma(t)$ not necessarily unit speed, with $\gamma(t_0) = 0$. Then the coordinates η, ζ and χ of $\gamma(t)$ relative to axes $\boldsymbol{t}, \boldsymbol{n}$ and $\boldsymbol{b}$ are functions of $t : \zeta(t) = \gamma(t) \cdot \boldsymbol{t}(t_0)$, $\eta(t) = \gamma(t) \cdot \boldsymbol{n}(t_0)$, $\chi(t) = \gamma(t) \cdot \boldsymbol{b}(t_0)$, $\eta(t) = f(\zeta(t))$, $\chi(t) = g((\zeta(t))$ where $f = f_0$, $g = g_0$. So $j^k f_t(0) = a_2(t)\zeta^2 + a_3(t)\zeta^3 + \cdots + a_k(t)\zeta^k$, $j^k g_t(0) = b_2(t)\zeta^2 + b_3(t)\zeta^3 + \cdots + b_k(t)\zeta^k$ in the neighbourhood $(0,0,0)$. Locally then $\gamma(I)$ can be written in the form $\{(f_t(\zeta), \zeta, g_t(\zeta))\}$, with $f(0) = g(0) = j^1 f_t(0) = j^1 g_t(0) = 0$. If V_k denotes the space of polynomials in ζ of degree $\geqq 2$ and $\leqq k$ we have a map, the *Lorentzian Monge-Taylor map* for the space curve γ, $\mu_\gamma : I \longrightarrow V_k \times V_k$ given by $\mu_\gamma(t) = (j^k f_t(0), j^k g_t(0))$. ($V_k \times V_k$ can be identified with $\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} = \mathbb{R}^{2(k-1)}$ via the coordinates $(a_2, \ldots, a_k, b_2, \ldots, b_k)$.) Of course μ_γ depends rather heavily on our choice of unit normals $\boldsymbol{n}(t)$. Where, $a_i(t) = \frac{f_t(0)^{(i)}}{i!}$, $b_i(t) = \frac{g_t(0)^{(i)}}{i!}$ $(2 \leqq i \leqq k)$, that is $V_k \times V_k =$

$$\{(a_2\zeta^2 + b_3\zeta^3 + \cdots + b_k\zeta^k), (b_2\zeta^2 + b_3\zeta^3 + \cdots + a_k\zeta^k)\}.$$

Let P_k denote the set of maps $\psi : {\mathbb{R}_1}^3 \to {\mathbb{R}_1}^3$ of the form $\psi(x, y, z) = (\psi_1(x, y, z), \psi_2(x, y, z), \psi_3(x, y, z))$ where $\psi_i(x, y, z)$ is a polynomial in x, y and z of degree $\leqq k$. So an element $\psi \in P_k$ is determines by the coefficients of the various monomials $x^l y^i z^j$ occurring in ψ_1, ψ_2 and ψ_3. There are altogether $1 + 3 + \cdots + (2k-1) + 2k + 1 = (k+1)^2$ monomials of degree $\leqq k$, so P_k can be thought of as a Euclidean space $\mathbb{R}^{(k+1)^2}$. It is this space which will provide the required deformations of the curve.

To simplify matters we now assume that the curve $\gamma(I)$ is compact, that is $I = {S_+}^1$. The identity map $1_{\mathbb{R}^3_1} : {\mathbb{R}_1}^3 \to {\mathbb{R}_1}^3$,, is of course an element of P_k (provided $k \geqq 1$), and using the compactness of $\gamma(S^1)$ it easily follows that there is a open neighbourhood U of $1_{\mathbb{R}^3_1}$ in P_k with the property that if $\psi \in U$ then the linear map $T\psi(\gamma(t)) : \mathbb{R}^2_1 \to \mathbb{R}^2_1$; $v \mapsto D\psi(\gamma(t)) \cdot v$ satisfy $D\psi(\gamma(t)) \cdot \boldsymbol{n}(t)$ is timelike vector and $D\psi(\gamma(t)) \cdot \boldsymbol{t}(t)$ is spacelike vector. Where, $D\psi(\gamma(t))$ denotes the derivative of ψ at $\gamma(t)$. (In fact, in this case the two conditions can be written two conditions of algebraic inequality in the open neighbourhood of $\gamma(t)$, so by the compactness of $\gamma({S_+}^1)$ the set which satisfy the two conditions is intersection of finite sets. Hence, it is open.) If we deform the original curve by the map ψ, then we can also obtain the required new smooth family of normal vectors $\boldsymbol{n}_\psi(t)$ as follows: since the map $\psi : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ is a diffeomorphism on some open set containing $\gamma(I)$, the vector $\boldsymbol{n}(t)$ will be sent to some new timelike vector $D\psi(\gamma(t))\boldsymbol{n}(t)$ which will be neither zero nor tangent to $\psi \circ \gamma$ at t. Orthogonally projecting this vector onto the normal plane to $\psi \circ \gamma$ at t and normalizing, that is $\boldsymbol{n}_\psi(t) = \frac{D\psi(\gamma(t))\boldsymbol{n}(t) - <D\psi(\gamma(t))\boldsymbol{n}(t), \boldsymbol{t}_\psi> \boldsymbol{t}_\psi}{\|D\psi(\gamma(t))\boldsymbol{n}(t) - <D\psi(\gamma(t))\boldsymbol{n}(t), \boldsymbol{t}_\psi> \boldsymbol{t}_\psi\|}$, $< \boldsymbol{n}_\psi(t), \boldsymbol{n}_\psi(t) >= -1$. Where, $\boldsymbol{t}_\psi$ denotes the tangent vector of the curve $\psi \circ \gamma$ at t. So assuming as before that $I = S^1_+$, we choose an open neighbourhood U of $1_{S^1_+} \in P_k$ consisting of polynomial maps which map an open set containing $\gamma(S^1_+)$ diffeomorphically to its image. We have now shown that there is a smooth map

$$\mu : S^1_+ \times U \longrightarrow V_k \times V_k$$

defined by $\mu(-, \psi) =$ Monge-Taylor map for the curve $\psi \circ \gamma$ using the family of normal vectors $\boldsymbol{n}_\psi(t)$. By exactly the same arguments as in the proof of Theorem 9.9 in [4], we have the following theorem.

Theorem 5.1. *Let Q be a manifold in $V_k \times V_k = {\mathbb{R}_1}^{2k-2}$. For some open set $U_1 \subset U$ containing the identity map, the map $\mu : S^1 \times U_1 \to V_k$ defined by $\mu(t, \psi) = \mu_{\psi\circ\gamma}(t)$ is transverse to Q. (In fact we can prove that μ is a submersion so Q does not enter the argument at all.)*

In order to give a proof of Theorem A, we prepare some lemmas.

Lemma 5.2. *Let $\gamma : S^1 \to \mathbb{R}_1{}^3$ be a spacelike curve defined by $\gamma(t) = (f_t(\zeta), \zeta, g_t(\zeta))$ $= (a_2\zeta^2 + a_3\zeta^3 + \cdots, \zeta, b_2\zeta^2 + b_3\zeta^3 + \cdots)$ with $\zeta(t_0) = 0$. Then:*

(1) *$k = 0$ and $< N', N >= 0$ at t_0 if and only if $a_2{}^2 - b_2{}^2 = 0$ and $a_2a_3 - b_2b_3 = 0$.*
(2) *$k'(s_0) - k(s_0) \cdot \tau(s_0) = 0$ and $(k' - k \cdot \tau)' = 0$ at t_0 if and only if*

$$\begin{cases} (a_2a_3 - b_2b_3) - |a_2{}^2 - b_2{}^2|^{\frac{1}{2}} \cdot |a_3b_2 - a_2b_3| = 0 \\ 3(a_2a_3 - b_2b_3)^2 \cdot |a_2{}^2 - b_2{}^2|^{\frac{1}{2}} + \{4(a_2a_4 - b_2b_4) + 3|a_3{}^2 - b_3{}^2|\} \cdot |a_2{}^2 - b_2{}^2|^{\frac{3}{2}} \\ +4|a_2{}^2 - b_2{}^2|^{\frac{7}{2}} - 3(a_2a_3 - b_2b_3) \cdot |a_3b_2 - a_2b_3| - 4|a_2{}^2 - b_2{}^2|^{\frac{3}{2}} \cdot |a_4b_2 - a_2b_4| \\ +6|a_2a_3 - b_2b_3| \cdot (a_3b_2 - a_2b_3) \cdot |a_2{}^2 - b_2{}^2|^{\frac{1}{2}} = 0, \end{cases}$$

where ζ is the coordinate along the $\mathbf{t}$-direction, $f_t(\zeta)$ along the $\mathbf{n}$-direction and $g_t(\zeta)$ along the $\mathbf{b}$-direction. Moreover, $(a_2, a_3, \cdots, a_k, b_2, b_3, \cdots, b_k) \in \mathbb{R}_1^{2k}$, k, k' as in §2.

Proof. Since $\gamma(t) = (a_2\zeta^2 + a_3\zeta^3 + \cdots, \zeta, b_2\zeta^2 + b_3\zeta^3 + \cdots)$. Then

$$\begin{aligned} \gamma'(t) &= (2a_2\zeta + 3a_3\zeta^2 + \cdots, 1, 2b_2\zeta + 3b_3\zeta^2 + \cdots), \\ \gamma''(t) &= (2a_2 + 6a_3\zeta + \cdots, 0, 2b_2 + 6b_3\zeta + \cdots), \\ \gamma^{(3)}(t) &= (6a_3 + 24a_4\zeta + \cdots, 0, 6b_3 + 24b_4\zeta + \cdots), \\ \gamma^{(4)}(t) &= (24a_4 + \cdots, 0, 24b_4 + \cdots), \end{aligned}$$

where $' = d/d\zeta$.

By the direct calculation, we have $k = \frac{\|\gamma' \wedge \gamma''\|}{\|\gamma'\|^3}$. We also have

$$\begin{aligned} < N, N' >= &\frac{< \gamma'', \gamma^{(3)} >}{\|\gamma'\|^5} + < \gamma''(\frac{d\zeta}{ds})^2, 3\gamma''\frac{d\zeta}{ds}\frac{d^2\zeta}{ds^2} > + < \gamma''(\frac{d\zeta}{ds})^2, \gamma'\frac{d^3\zeta}{ds^3} > \\ &+ < \gamma'\frac{d^2\zeta}{ds^2}, \gamma^{(3)}(\frac{d\zeta}{ds})^3 > + < \gamma'\frac{d^2\zeta}{ds^2}, 3\gamma''\frac{d\zeta}{ds}\frac{d^2\zeta}{ds^2} > + < \gamma'\frac{d^2\zeta}{ds^2}, \gamma'\frac{d^3\zeta}{ds^3} > . \end{aligned}$$

Therefore, at the point t_0 with $\zeta(t_0) = 0$ we have $k = \frac{\|(0,1,0) \wedge (2a_2, 0, 2b_2)\|}{\|(0,1,0)\|^3} = 2|a_2{}^2 - b_2{}^2|^{\frac{1}{2}}$ We can show that $< \gamma''(t_0), \gamma'(t_0) >=< \gamma'(t_0), \gamma^{(3)}(t_0) >= \frac{d^2\zeta}{ds^2}(t_0) = 0$. Under the assumption that $k(t_0) = a_2{}^2 - b_2{}^2 = 0$, we have

$$< N, N' >= \frac{< \gamma'', \gamma^{(3)} >}{\|\gamma'\|^5} = 12(-a_2a_3 + b_2b_3).$$

Hence, $k = 0$ and $< N', N >= 0$ at t_0 if and only if $a_2{}^2 - b_2{}^2 = 0$ and $a_2a_3 - b_2b_3 = 0$.

(2) By the similar calculation as the case 1), we have the following:

$$\tau = \frac{\|\gamma'\ \gamma''\ \gamma^{(3)}\|}{\|\gamma' \wedge \gamma''\|^2},$$

$$k' = \frac{< \gamma' \wedge \gamma^{(3)}, \gamma' \wedge \gamma'' >}{\|\gamma' \wedge \gamma''\| \cdot \|\gamma'\|^3} - \frac{3\cdot < \gamma', \gamma'' > \cdot \|\gamma' \wedge \gamma''\|}{\|\gamma'\|^5},$$

$$\begin{aligned} k'' = & -\frac{< \gamma' \wedge \gamma^{(3)}, \gamma' \wedge \gamma'' >^2}{\|\gamma'\|^3 \cdot \|\gamma' \wedge \gamma''\|^3} \\ & + \frac{< \gamma'' \wedge \gamma^{(3)}, \gamma' \wedge \gamma'' > + < \gamma' \wedge \gamma^{(4)}, \gamma' \wedge \gamma'' > + \|\gamma' \wedge \gamma^{(3)}\|^2}{\|\gamma'\|^3 \cdot \|\gamma' \wedge \gamma''\|} \\ & - \frac{3 < \gamma' \wedge \gamma^{(3)}, \gamma' \wedge \gamma'' > \cdot < \gamma', \gamma'' >}{\|\gamma'\|^5 \cdot \|\gamma' \wedge \gamma''\|} + \frac{15 < \gamma', \gamma'' >^2 \cdot \|\gamma' \wedge \gamma''\|}{\|\gamma'\|^7} \\ & - \frac{3\|\gamma' \wedge \gamma''\|\{< \gamma', \gamma^{(3)} > + \|\gamma''\|^2\}}{\|\gamma'\|^5} - \frac{3 < \gamma' \wedge \gamma^{(3)}, \gamma' \wedge \gamma'' > \cdot < \gamma', \gamma'' >}{\|\gamma' \wedge \gamma''\| \cdot \|\gamma'\|^5} \end{aligned}$$

and

$$\tau' = \frac{\|\gamma'\ \gamma''\ \gamma^{(4)}\|}{\|\gamma' \wedge \gamma''\|^2} - \frac{2\|\gamma'\ \gamma''\ \gamma^{(3)}\| \cdot < \gamma' \wedge \gamma^{(3)}, \gamma' \wedge \gamma'' >}{\|\gamma' \wedge \gamma''\|^4}.$$

So at the origin we have

$$k' = \frac{6(-b_2 b_3 + a_2 a_3)}{|{a_2}^2 - {b_2}^2|},$$

$$k'' = \frac{18(-b_2 b_3 + a_2 a_3)^2}{|{a_2}^2 - {b_2}^2|^{\frac{3}{2}}} + \frac{24(-b_2 b_4 + a_2 a_4) + 18| - {b_3}^2 + {a_3}^2|}{|{a_2}^2 - {b_2}^2|^{\frac{1}{2}}} + 24|{a_2}^2 - {b_2}^2|^{\frac{3}{2}},$$

$$\tau = \frac{| - 3a_2 b_3 + 3a_3 b_2|}{|{a_2}^2 - {b_2}^2|}$$

and

$$\tau' = \frac{| - 12a_2 b_4 + 12a_4 b_2|}{|{a_2}^2 - {b_2}^2|} - \frac{18| - b_2 b_3 + a_2 a_3| \cdot (-a_2 b_3 + a_3 b_2)}{|{a_2}^2 - {b_2}^2|^2}.$$

Hence under the assumption that $k \neq 0$, $k' - k \cdot \tau = 0$ and $(k' - k \cdot \tau)' = 0$ if and only if

$$\left\{ \begin{aligned} & \frac{6(-b_2 b_3 + a_2 a_3)}{|{a_2}^2 - {b_2}^2|} - \frac{2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} \cdot | - 3a_2 b_3 + 3a_3 b_2|}{|{a_2}^2 - {b_2}^2|} = 0 \\ & \frac{18(-b_2 b_3 + a_2 a_3)^2}{|{a_2}^2 - {b_2}^2|^{\frac{3}{2}}} + \frac{24(-b_2 b_4 + a_2 a_4) + 18| - {b_3}^2 + {a_3}^2|}{|{a_2}^2 - {b_2}^2|^{\frac{1}{2}}} + 24|{a_2}^2 - {b_2}^2|^{\frac{3}{2}} \\ & - \frac{6a_2 a_3 - 6b_2 b_3}{|{a_2}^2 - {b_2}^2|} \cdot \frac{|3a_3 b_2 - 3a_2 b_3|}{|{a_2}^2 - {b_2}^2|} - 2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} \cdot \frac{|12a_4 b_2 - 12a_2 b_4|}{|{a_2}^2 - {b_2}^2|} \\ & + 2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} \cdot \frac{18| - b_2 b_3 + a_2 a_3| \cdot (-a_2 b_3 + a_3 b_2)}{({a_2}^2 - {b_2}^2)^2} = 0. \end{aligned} \right.$$

This condition is equivalent to the following condition:

$$\begin{cases} (a_2a_3 - b_2b_3) - |{a_2}^2 - {b_2}^2|^{\frac{1}{2}} \cdot |a_3b_2 - a_2b_3| = 0 \\ 3(a_2a_3 - b_2b_3)^2 \cdot |{a_2}^2 - {b_2}^2|^{\frac{1}{2}} + \{4(a_2a_4 - b_2b_4) + 3|{a_3}^2 - {b_3}^2|\} \cdot |{a_2}^2 - {b_2}^2|^{\frac{3}{2}} \\ +4|{a_2}^2 - {b_2}^2|^{\frac{7}{2}} - 3(a_2a_3 - b_2b_3) \cdot |a_3b_2 - a_2b_3| - 4|{a_2}^2 - {b_2}^2|^{\frac{3}{2}} \cdot |a_4b_2 - a_2b_4| \\ +6|a_2a_3 - b_2b_3| \cdot (a_3b_2 - a_2b_3) \cdot |{a_2}^2 - {b_2}^2|^{\frac{1}{2}} = 0 \end{cases}$$

□

Lemma 5.3. *We consider smooth maps* $\rho_i : V_3 \times V_3 = \mathbb{R}^4 \longrightarrow \mathbb{R}$ $(i = 1, 2)$ *given by*

$$\begin{cases} \rho_1 = {a_2}^2 - {b_2}^2 \\ \rho_2 = -b_2b_3 + a_2a_3 \end{cases}$$

and $\phi_i : V_4 \times V_4 = \mathbb{R}^6 \longrightarrow \mathbb{R}$ $(i = 1, 2)$ *given by*

$$\begin{cases} \phi_1 = (a_2a_3 - b_2b_3) - |{a_2}^2 - {b_2}^2|^{\frac{1}{2}} \cdot |a_3b_2 - a_2b_3| \\ \phi_2 = 3(a_2a_3 - b_2b_3)^2 \cdot |{a_2}^2 - {b_2}^2|^{\frac{1}{2}} + \{4(a_2a_4 - b_2b_4) + 3|{a_3}^2 - {b_3}^2|\} \cdot |{a_2}^2 - {b_2}^2|^{\frac{3}{2}}, \\ +4|{a_2}^2 - {b_2}^2|^{\frac{7}{2}} - 3(a_2a_3 - b_2b_3) \cdot |a_3b_2 - a_2b_3| - 4|{a_2}^2 - {b_2}^2|^{\frac{3}{2}} \cdot |a_4b_2 - a_2b_4| \\ +6|a_2a_3 - b_2b_3| \cdot (a_3b_2 - a_2b_3) \cdot |{a_2}^2 - {b_2}^2|^{\frac{1}{2}}. \end{cases}$$

Then
(1) The set $Q_1 = \{(a_2, a_3, b_2, b_3) \in \mathbb{R}^4 | \rho_1 = \rho_2 = 0\}$ *is a codimension two submanifold in* $\mathbb{R}^4$.
(2) The set $Q_2 = \{(a_2, a_3, a_4, b_2, b_3, b_4) \in \mathbb{R}^6 | \phi_1 = \phi_2 = 0\}$ *is a codimension two submanifold in* $\mathbb{R}^6$.

Proof. The Jacobian matrix of the map (ρ_1, ρ_2) is calculated as follows:

$$J(\rho_1, \rho_2) = \begin{pmatrix} 2a_2 & -2b_2 & 0 & 0 \\ a_3 & -b_3 & a_2 & -b_2 \end{pmatrix}$$

Then $\operatorname{rank} J(\rho_1, \rho_2) = 2$ by $\gamma''(t) \not\equiv 0$. This means that Q_1 is a submanifold in $\mathbb{R}^4$ with codimension two.

The Jacobian matrix of the map (ϕ_1, ϕ_2) is also calculated as follows:
$J(\phi_1, \phi_2) =$

$$\begin{pmatrix} \frac{\partial \phi_1}{\partial a_2} & a_2 \pm b_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} & 0 & \frac{\partial \phi_1}{\partial b_2} & \pm a_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} - b_2 & 0 \\ \frac{\partial \phi_2}{\partial a_2} & \frac{\partial \phi_2}{\partial a_3} & 4(a_2 \pm b_2)|{a_2}^2 - {b_2}^2|^{\frac{3}{2}} & \frac{\partial \phi_2}{\partial b_2} & \frac{\partial \phi_2}{\partial b_3} & \frac{\partial \phi_2}{\partial b_4} \end{pmatrix}$$

Then $\operatorname{rank} J(\phi_1, \phi_2) = 2$ by $\frac{1}{2}k(s_0) \neq 0$. In fact, $\frac{1}{2}k(s_0) \neq 0$ if and only if $a_2 \neq \pm b_2$. Then $4(a_2 \pm b_2)|{a_2}^2 - {b_2}^2|^{\frac{3}{2}} \neq 0$, and either $a_2 \pm b_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}}$ or $\pm a_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} - b_2 \neq 0$. If $a_2 \pm b_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} = \pm a_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} - b_2 = 0$, we have $|{a_2}^2 - {b_2}^2| = 1$, this is equivalent to the condition that $a_2 \pm b_2 = 0$ by $a_2 \pm b_2|{a_2}^2 - {b_2}^2|^{\frac{1}{2}} = 0$. This is a contradiction. This completes the proof. □

We can use Theorem 5.1, Lemmas 5.2 and 5.3 for the proof of Theorem A exactly the same way as the proof of Corollary 9.7 in [4]. So we omit the detail here.

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