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Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane

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Abstract

We prove the existence of absolutely continuous invariant measures for piecewise real-analytic expanding maps on bounded regions in the plane.

1 Introduction

Expanding properties of dynamical systems give rise to chaotic behavior of the orbits. On the other hand, they often lead to good ergodic properties such as the existence of absolutely continuous invariant measures. One typical example is the fork-lore theorem[7] that shows the existence of a smooth ergodic measure for every expanding C^2 self-map on a closed manifold. Hence, one interest in the study of chaotic dynamical systems is the relations between expanding properties and the ergodic properties they produce.

Lasota and Yorke showed, in their famous work[1], the existence of absolutely continuous invariant measures for piecewise C^2 expanding maps on intervals. They made use of the Perron-Frobenius operator and functions of bounded variation, and their idea has been used extensively in the study of one dimensional dynamical systems. This paper concerns the generalization of their result towards higher dimension. Though it is natural to expect similar results, it has been turned out that things are not simple in higher dimension. In fact, at present, we do not know whether piecewise C^2 expanding maps on bounded regions in higher dimensional Euclidean space always have absolutely continuous invariant measures. The main difficulty in higher dimension is the fact that the partition of the domain into the regions where an iteration of the map is smooth can be very complicated.

Keller treated piecewise C^2 expanding maps on bounded regions in the plane in his thesis[2, 3] and gave some criterion for the existence of absolutely continuous invariant measure. The most effective result we have so far is that of Góra and Boyarski[4], which gives a lower bound for the expansion rate that assures the existence of absolutely continuous invariant measures. Their result is valid for arbitrary dimension. But their lower bound depends on the minimal angle on the boundaries of the regions in the partition associated to the map. See [5] for a modification of their result.

In this article, we consider the problem for piecewise real-analytic maps on bounded regions in the plane. (We will give the definition of piecewise real-analytic maps in the next section.) The real-analytic property somewhat relax the difficulty we mentioned above. In fact, we can prove the following theorem as the main result of this paper.

Theorem 1 *An absolutely continuous invariant finite measure exists for piecewise real-analytic expanding maps on bounded regions in the plane.*

This result improves a theorem of Keller in his thesis[2, 3], which give the same conclusion under one additional assumption that the map is piecewise conformal.

The author learned from Gerhard Keller that Jerome Buzzi at Marseille obtained a similar result when he was preparing the manuscript of this paper.

2 Piecewise real-analytic map

We call a map $c : [a, b] \to \mathbf{R}^2$ a real-analytic curve if it is a restriction of a real-analytic map defined on a neighborhood of $[a, b]$ and satisfies $c'(t) \neq 0$ for $t \in [a, b]$. In what follows, we will assume

$$\|c'(t)\| \equiv 1 \quad \text{for } t \in [a, b] \tag{1}$$

for real-analytic curves, by real-analytic change of the variable t. Also we will denote the image of a real-analytic curve $c : [a, b] \to \mathbf{R}^2$ by the same symbol c, as an abuse of the symbol.

A continuous map $c : [a, b] \to \mathbf{R}^2$ is called a piecewise real-analytic curve if there is a sequence $a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = b$ such that the restrictions $c|_{[\xi_i, \xi_{i+1}]}$, $0 \leq i < n$, are real-analytic curves.

Let D be a region on the plane $\mathbf{R}^2$ whose boundary consists of finite simple closed piecewise real-analytic curves. We consider a finite (quasi-)partition $\xi = \{D_i\}_{i=1}^k$ of the domain D such that

- $D_i \subset D$ is a region whose boundary is a finite union of simple closed piecewise real-analytic curves,
- $D_i \cap D_j = \phi$ if $i \neq j$, and
- $\cup_{i=1}^k \overline{D}_i = \overline{D}$ where $\overline{D}$ and $\overline{D}_i$ denote the closures of D and D_i respectively.

We call such partition a real-analytic partition of D. We denote $E = \cup_{i=1}^k \partial D_i = \overline{D} - \cup_{i=1}^k D_i$.

For a real-analytic partition $\xi = \{D_i\}_{i=1}^k$, we can choose a finite set of real-analytic curves $\{\gamma_i : [a_i, b_i] \to E \subset \mathbf{R}^2\}_{i=1}^m$ in E with the following properties

- each γ_i is a simple curve, that is, has no multiple point,
- the boundary of each region D_j, $1 \le j \le k$, is a union of the image of some γ_i's, and
- the images of curves γ_i, $1 \le j \le k$, except their end points are mutually disjoint, that is,

$$\gamma_i(t) \neq \gamma_j(s) \quad \text{if } i \neq j,\ t \in (a_i, b_i) \text{ and } s \in (a_j, b_j).$$

We call these curves the dividing curves of the partition ξ. Remark that there are only finitely many points that belong to more than two dividing curves.

A map $f : D \to D$ is called a piecewise real-analytic map on D if there is a real-analytic partition $\xi = \{D_i\}_{i=1}^k$ of D as above such that each restriction $f|_{D_i}$ of f to D_i, $1 \le i \le k$, can be extended to a neighborhood of $\overline{D}_i$ as a real-analytic map. We will denote, by f_{D_i}, the real-analytic extension of $f|_{D_i}$ to a neighborhood of $\overline{D}_i$.

For a tangent vector v at $x \in D - E$, we define its expansion rate $\rho(v, f)$ by

$$\rho(v, f) = \frac{\|Df(v)\|}{\|v\|}.$$

The expansion rate $\rho(f)$ of the map f is the infimum of the expansion rate over all non-zero vectors at all points in $D - E$. If $\rho(f) > 1$ for a piecewise real-analytic map f, the map f is called a piecewise real-analytic expanding map. We will fix a piecewise real-analytic expanding map f, the partitions $\xi = \{D_i\}_{i=1}^k$ and the dividing curves $\{\gamma_i : [a_i, b_i] \to \mathbf{R}^2\}_{i=1}^m$ throughout this paper.

The most important consequence from the above definitions is the fact that the iterations f^n of a piecewise real-analytic expanding map f are also piecewise real-analytic expanding maps with expansion rate $\rho(f^n) = \rho(f)^n$, which is not true in C^r category. This is the main reason why we restrict our object to piecewise real-analytic maps.

3 Germs of real-analytic curves

Let p be a point on the plane $\mathbf{R}^2$, and let $c_i : [0, \epsilon_i] \to \mathbf{R}^2$, $i = 1, 2$, be two real-analytic curves satisfying $c_i(0) = p$ and (1). We say that these two curves give the same germ at p if $c_1(t) = c_2(t)$ for $0 \le t < \min\{\epsilon_1, \epsilon_2\}$. This is an equivalence relation between real-analytic curves c satisfying $c(0) = p$ and (1). The equivalence classes are called germs of real-analytic curves at p.

We say that two open subsets U_1 and U_2 on the plane $\mathbf{R}^2$ give the same germ at p if there exists $\delta > 0$ such that $U_1 \cap B(p, \delta) = U_2 \cap B(p, \delta)$, where $B(p, \delta) = \{x \in \mathbf{R}^2; \|x - p\| < \delta\}$. This is also an equivalence relation. We call the equivalence classes germs of open subsets at p.

Let β_1 and β_2 be distinct germs of real-analytic curves at p, and let $b_i : [0, \epsilon_i] \to \mathbf{R}^2$, $i = 1, 2$ be simple real-analytic curves that represent

β_i respectively. If $\delta > 0$ is sufficiently small, the open set $B(p,\delta)\backslash(b_1 \cup b_2)$ consists of two connected components. The germs of open subset represented by the connected component of $B(p,\delta)\backslash(b_1 \cup b_2)$ that is located in counterclockwise direction of the curve b_1 is called the region between β_1 and β_2. From this definition, the region U between β_1 and β_2 and that between β_2 and β_1 are complementary. The germs of real-analytic curves β_1 and β_2 are called the boundary curves of the region U.

Let $\mathrm{Angle}_1(\beta_1,\beta_2) \in [0, 2\pi]$ be the angle that is formed by the region between β_1 and β_2 at p. So we have $\mathrm{Angle}_1(\beta_1,\beta_2) = 2\pi - \mathrm{Angle}_1(\beta_2,\beta_1)$. If $\mathrm{Angle}_1(\beta_1,\beta_2) \neq 0$ we define $\mathrm{Ord}(\beta_1,\beta_2) = 1$. On the other hand, if $\mathrm{Angle}_1(\beta_1,\beta_2) = 0$, we define $\mathrm{Ord}(\beta_1,\beta_2)$ as the contact order of the two germs of real-analytic curves β_1 and β_2 at p, that is,

$$\begin{aligned}
\mathrm{Ord}(\beta_1,\beta_2) &= \lim_{t\to+0} \frac{\log \min\{\|b_1(t) - b_2(s)\| \mid s \in [0,\epsilon_2)\}}{\log t} \\
&= \lim_{t\to+0} \frac{\log \min\{\|b_2(t) - b_1(s)\| \mid s \in [0,\epsilon_1)\}}{\log t}.
\end{aligned}$$

When $\mathrm{Ord}(\beta_1,\beta_2) = d > 1$, we define $\mathrm{Angle}_d(\beta_1,\beta_2)$ by

$$\begin{aligned}
\mathrm{Angle}_d(\beta_1,\beta_2) &= \lim_{t\to+0} \frac{\min\{\|b_1(t) - b_2(s)\| \mid s \in [0,\epsilon_2)\}}{t^d} \\
&= \lim_{t\to+0} \frac{\min\{\|b_2(t) - b_1(s)\| \mid s \in [0,\epsilon_1)\}}{t^d}.
\end{aligned}$$

For the region U between β_1 and β_2, we define $\mathrm{Ord}(U) = \mathrm{Ord}(\beta_1,\beta_2)$ and $\mathrm{Angle}_d(U) = \mathrm{Angle}_d(\beta_1,\beta_2)$. We will need the following elementary lemmas.

Lemma 2 *Let $H : W \to \mathbf{R}^2$ be a real-analytic map defined on a neighborhood W of a point $p \in \mathbf{R}^2$. Assume that $\|DH(p)w\|/\|w\| \geq 1$ for all tangent vectors $w \neq 0$ at p. Let β_1 and β_2 be germs of real-analytic curves at p, and let v_i, $i = 1, 2$, be the unit tangent vectors of them at the point p respectively. Let U be the region between β_1 and β_2.*

(a) *If $0 < \mathrm{Angle}_1(\beta_1,\beta_2) < 2\pi$, we have*

$$|\det DH(p)| = \left|\frac{\sin(\mathrm{Angle}_1(H(U)))}{\sin(\mathrm{Angle}_1(U))}\right| \cdot \rho(v_1,H)\rho(v_2,H). \tag{2}$$

(b) *If $\mathrm{Ord}(U) = d > 1$, we have*

$$|\det DH(p)| = \frac{\mathrm{Angle}_d(H(U))}{\mathrm{Angle}_d(U)} \cdot \rho(v_1,H)^{d+1}. \tag{3}$$

Remark 3 In the claim (b), $v_1 = v_2$ and $\rho(v_1,H) = \rho(v_2,H)$.

proof. In the case $\mathrm{Ord}(U) = 1$, the formula (2) says nothing but the fact that the absolute value of the determinant of H at p is the ratio between the area of the infinitesimal parallelogram at p spanned by the vectors v_1 and v_2 and that of its image under $DH(p)$. Let us consider the case

$\mathrm{Ord}(U) = d > 1$. Since two curves b_i, $i = 1, 2$, representing the germs β_i are almost parallel in small neighborhoods of the point p, the minimum $\min\{\|b_1(t) - b_2(s)\| | s \in [0, \epsilon_2)\}$ is attained when $b_1(t) - b_2(s)$ is almost orthogonal to the vector $v_1 = v_2$ when t is small. Hence we can see, by elementary geometric argument,

$$\begin{aligned}\lim_{t \to +0} \frac{\min\{\|H \circ b_1(t) - H \circ b_2(s)\| \mid s \in [0, \epsilon_2)\}}{\min\{\|b_1(t) - b_2(s)\| \mid s \in [0, \epsilon_2)\}} &= \langle DH(v_1^\perp), DH(v_1)^\perp \rangle \\ &= \frac{|\det DH(p)|}{\rho(v_1, H)}\end{aligned}$$

where $v_1^\perp$ and $DH(v_1)^\perp$ are the unit vectors that are orthogonal to the vectors v_1 and $DH(v_1)$ respectively and $\langle \cdot, \cdot \rangle$ is the inner product in ordinary sense. Taking into account the change of the variable t so that the map $H \circ b_1(t)$ satisfies the condition (1), we obtain the claim (b). $\square$
Under the assumption of lemma 2, we define $\rho(U, H)$ as the maximum of the expansion rate $\rho(v, H) = \|DH(v)\|/\|v\|$ over all tangent vectors v at p that is in between v_1 and v_2 or, in other words, that is contained in the closure of the angle formed by U at p.

Lemma 4 *Under the same assumption as in the last lemma, we have*

$$\frac{\mathrm{Angle}_{\mathrm{Ord}(U)}(H(U))}{\mathrm{Angle}_{\mathrm{Ord}(U)}(U)} \leq 2\pi \cdot \frac{|\det DH(p)|}{\rho(U, H)\rho_0} \tag{4}$$

where $\rho_0 \geq 1$ is the minimum of the expansion rate $\rho(w, H)$ over all tangent vectors $w \neq 0$ at p.

proof. If $\mathrm{Ord}(U) > 1$, (4) is obvious from the last lemma. Let us consider the case $\mathrm{Ord}(U) = 1$. We first prove, for $K = 2\pi$,

$$\frac{\mathrm{Angle}_1(H(U))}{\mathrm{Angle}_1(U)} \leq K \cdot \frac{|\det DH(p)|}{\rho(v_1, H)\rho_0}. \tag{5}$$

If $\mathrm{Angle}_1(U) > \pi/2$, we get (5) for $K = 4$ because the left hand side is smaller than $2\pi/(\pi/2) = 4$ while the right hand side except K is not smaller than 1. If $\mathrm{Angle}_1(U) \leq \pi/2$ and $\mathrm{Angle}_1(H(U)) \leq \pi/2$, we get (5) for $K = \pi/2$ from the last proposition because

$$2/\pi \leq \sin(x)/x \leq 1 \quad \text{for } 0 < x \leq \pi/2.$$

Finally, we consider the case when $\mathrm{Angle}_1(U) \leq \pi/2$ and $\mathrm{Angle}_1(H(U)) > \pi/2$. In this case, we can take a unit tangent vector v between v_1 and v_2 such that $DH(p)v$ is orthogonal to $DH(p)v_1$. Let γ be a germ of real-analytic curve passing through U that is tangent to the vector v at p, and let V be the region between β_1 and γ. Then we can apply the above argument to V and get (5) with U replaced by V for $K = \pi/2$. Since

$$\frac{\mathrm{Angle}_1(H(U))}{\mathrm{Angle}_1(U)} \leq \frac{2\pi}{\mathrm{Angle}_1(V)} = 4\frac{\mathrm{Angle}_1(H(V))}{\mathrm{Angle}_1(V)}$$

we get (5) for $K = 2\pi$. Therefore we have (5) for $K = 2\pi$ in any case.

Remark that we can replace v_1 by v_2 in (5) by symmetry. Now let us take a vector v at p that is in between v_1 and v_2 and satisfy $\rho(v, H) = \rho(U, f)$. If $v = v_1$ or $v = v_2$, the conclusion of the lemma is nothing but (5) or that with v_1 replaced by v_2. Otherwise, we consider a germ γ of a real-analytic curve that is tangent to v at p. The germ of curve γ divide U into two regions U_1 and U_2. Applying (5) to these regions, we get

$$\begin{aligned}\frac{\mathrm{Angle}_1(H(U))}{\mathrm{Angle}_1(U)} &\leq \max\left\{\frac{\mathrm{Angle}_1(H(U_1))}{\mathrm{Angle}_1(U_1)}, \frac{\mathrm{Angle}_1(H(U_2))}{\mathrm{Angle}_1(U_2)}\right\} \\ &\leq 2\pi \cdot \frac{|\det DH(p)|}{\rho(v, H)\rho_0}.\end{aligned}$$

The lemma is proved. □

4 Weighted multiplicity

In this section we introduce what we call weighted multiplicity that count the multiplicity of the intersection of dividing curves $\{\gamma_i\}_{i=1}^m$ with appropriate weight. Let p be a point in $E = \cup_{i=1}^k \partial D_i$. Let $\gamma_i : [a_i, b_i] \to \mathbf{R}^2$ be a dividing curve. If the curve γ_i pass through the point p, it gives germs of real-analytic curves at p in the following manner: If $\gamma_i(t) = p$ for $t \in (a_i, b_i)$, the curve γ_i gives two germs of real-analytic curve at p represented by the curves $s \mapsto \gamma_i(t+s)$ and $s \mapsto \gamma_i(t-s)$. If $\gamma_i(a_i) = p$ (resp. $\gamma_i(b_i) = p$), the curve γ_i gives one germ represented by a curve $s \mapsto \gamma_i(a_i + s)$ (resp. $s \mapsto \gamma_i(b_i - s)$).

Let $\mathcal{B}(p) = \{\beta_i(p)\}_{i=1}^{m(p)}$ be the collection of the distinct germs of real-analytic curves given in such way by all dividing curves. Remark that $m(p) = 2$ for all points $p \in E$ except for finite points. These germs of real-analytic curves are called the germs of curves at p given by the dividing curves. We always assume that the germs of curves $\beta_i(p)$, $i = 1, 2, \cdots, m(p)$, are arranged in counterclockwise order around the point p.

Let U_i, $1 \leq i < m(p)$, be the region between $\beta_i(p)$ and $\beta_{i+1}(p)$, and let $U_{m(p)}$ be that between $\beta_{m(p)}(p)$ and $\beta_1(p)$. We denote the set of these regions by $\mathcal{U}(p) = \{U_i(p)\}_{i=1}^{m(p)}$. For $U \in \mathcal{U}(p)$, let f_U be the germ of a real-analytic map at p that is obtained as the real-analytic extension of the restriction of f to a representative of U.

We define the weight $W(U_i(p))$ of the region $U_i(p) \in \mathcal{U}(p)$ by

$$W(U_i(p)) = \frac{\|Df_{U_i(p)}(p)v_1\|/\|v_1\| + \|Df_{U_i(p)}(p)v_2\|/\|v_2\|}{|\det Df_{U_i(p)}(p)|}$$

for $1 \leq i \leq m(p)$, where v_1 and v_2 are the tangent vectors of the boundary curves of $U_i(p)$ at p. The weighted multiplicity $M(p, f)$ at a point $p \in E$ is defined by

$$M(p, f) = \sum_{i=1}^{m(p)} W(U_i(p)).$$

The weighted multiplicity $M(f)$ of a piecewise real-analytic expanding map f is the supremum of $M(f,p)$ over all points $p \in E$. Remark again that $M(f,p) \leq 4\rho(f)^{-1}$ for all $p \in E$ except for finite points. Weighted multiplicity $M(f)$ is the quantity that we most concern in the argument below.

5 Functions of bounded variation

We use the theory of bounded variation functions in higher dimensional space, which is developed in the book [6]. We recall some definitions and properties of functions of bounded variation from [6].

Let U be an open subset of the plane $\mathbf{R}^2$. Let $C^r(U,\mathbf{R}^2)$ be a set of bounded vector-valued C^r functions $g = (g_1, g_2) : U \to \mathbf{R}^2$ and let $C_0^r(U,\mathbf{R}^2)$ be the subset of $C^r(U,\mathbf{R}^2)$ that consists of functions with compact support. Similarly, let $\Omega^r(U)$ be the set of 1-forms $\Psi = \Psi_1 dx + \Psi_2 dy$ of class C^r on U and let Ω_0^r be the subset of $\Omega^r(U)$ that consists of 1-forms with compact support. We denote the d-dimensional Hausdorff measure by μ_d.

We define the variation $\mathrm{Var}(\varphi, U)$ of the function $\varphi \in L^1(U)$ as the supremum of

$$\int_U \varphi(z)\mathrm{Div}g(z)d\mu_2(z) \tag{6}$$

over all $g = (g_1, g_2) \in C_0^1(U,\mathbf{R}^2)$ satisfying $\|g(z)\| \leq 1$ for $z \in U$, where

$$\mathrm{Div}g(x,y) = \frac{\partial}{\partial x}g_1(x,y) + \frac{\partial}{\partial y}g_2(x,y).$$

A function $\varphi \in L^1(U)$ is said to be of bounded variation if $\mathrm{Var}(f,\mathbf{R}^2) < \infty$. We denote, by $\mathcal{BV}(U)$, the set of functions $\varphi \in L^1(U)$ of bounded variation. Sometimes it is convenient to write the variation $\mathrm{Var}(\varphi, U)$ as

$$\mathrm{Var}(f,U) = \sup\left\{\int_U \varphi d\Psi \ ; \Psi \in \Omega_0^1(U) \text{ and } \|\Psi(z)\| \leq 1 \text{ for } z \in U.\right\} \tag{7}$$

where $\|\Psi(z)\| = \sqrt{\Psi_1^2(z) + \Psi_2^2(z)}$ for $\Psi = \Psi_1 dx + \Psi_2 dy$. We can obtain this formula from the correspondence between $g = (g_1, g_2) \in C_0^1(U,\mathbf{R}^2)$ and $\Psi = -g_2 dx + g_1 dy \in \Omega_0^1(U)$.

Remark that, for $\varphi \in \mathcal{BV}(U)$, the functional

$$\Phi_\varphi : g \in C_0^1(U,\mathbf{R}^2) \mapsto \int_U \varphi \mathrm{Div} g d\mu_2 \in \mathbf{R}$$

satisfies

$$|\Phi(g_1) - \Phi(g_2)| \leq \left(\sup_{z\in U} \|g_1(z) - g_2(z)\|\right) \cdot \mathrm{Var}(\varphi, U).$$

Hence Φ can be extended uniquely as a continuous linear functional on $C_0^0(U,\mathbf{R}^2)$. We can consider this extension as a vector-valued Radon measure on U with total variation $\mathrm{Var}(\varphi, U)$. (See [8, Ch.6] for example.) We denote this vector-valued Radon measure by $D\varphi$. Let $\int_U \langle g, D\varphi\rangle$ be the integration of a vector-valued function $g \in C^0(U, \mathbf{R}^2)$ with respect to the vector-valued measure $D\varphi$. Let $|D\varphi|$ be the measure that is obtained as the total variation of $D\varphi$. Obviously we have $\int_U \langle g, D\varphi\rangle \leq \int \|g\| \cdot |D\varphi|$ for $g \in C^0(U, \mathbf{R}^2)$, where $\|g\|$ denote the function $\|g\|(x) = \|g(x)\|$ on U.

The bounded variation norm of the function $g \in \mathcal{BV}(U)$ is defined as

$$\|\varphi\|_{BV(U)} = \mathrm{Var}(\varphi, U) + \int_U |\varphi| d\mu_2.$$

This norm makes $\mathcal{BV}(U)$ a Banach space. See [6, Remark 1.12]. We make use of the following fact when we prove the existence of an absolutely continuous invariant measure in section 6.

Proposition 5 *Let $U \subset \mathbf{R}^2$ be a bounded open set with C^1 boundary. Then sets of functions in $\mathcal{BV}(U)$ that are uniformly bounded in the bounded variation norm $\|\cdot\|_{BV(U)}$ are relatively compact in $L^1(U)$.*

Another important property of functions of bounded variation is that they give traces on the boundary. Let $U \subset \mathbf{R}^2$ be a bounded region whose boundary is a finite union of real-analytic simple closed curves. We denote by $L^1(\partial U)$ the set of functions that is integrable with respect to one dimensional Hausdorff measure μ_1. We put $B(x,r) = \{y \in \mathbf{R}^2 \mid \|x - y\| < r\}$. Then we have

Proposition 6 *For $\varphi \in \mathcal{BV}(U)$, there is a unique function $\varphi^- \in L^1(\partial U)$ such that*

$$\lim_{r\to 0} \mu_2(B(x,r))^{-1} \int_{B(x,r)\cap U} |\varphi(z) - \varphi^-(x)| d\mu_2(z) = 0$$

for μ_1-almost all $x \in \partial U$. Moreover,

(a) *for $\zeta \in C_0^1(\mathbf{R}^2, \mathbf{R}^2)$, we have*

$$\int_{\partial U} \varphi^-(z)\langle\zeta(z), \nu(z)\rangle d\mu_1(z) = \int_U \varphi(z)\mathrm{Div}\zeta(z) d\mu_2(z) + \int_U \langle\zeta, D\varphi\rangle$$

where $\nu(z)$ is the unit outer normal vector for the boundary ∂U at z,

(b) *if we define $\varphi(z) = 0$ for $z \notin U$, we have*

$$\mathrm{Var}(\varphi, \mathbf{R}^2) = \mathrm{Var}(\varphi, U) + \int_{\partial U} |\varphi^-(x)| d\mu_1(x).$$

The function $\varphi^- \in L^1(\partial U)$ in the above proposition is called the trace of φ on the boundary ∂U. We refer theorem 1.19 of [6] for proposition 5, and theorem 2.10 and Remark 2.14 of [6] for proposition 6.

Remark 7 In theorem 2.10 and Remark 2.14 of [6], the boundary of the region U is assumed to be Lipschitz. Hence proposition 6 is not a direct consequence of that theorem when there are cusps on the boundary of U. But, with slight modification in the proof, we can derive proposition 6, because the proposition is essentially a local one.

6 An existence theorem for absolutely continuous invariant measures

Let $f : D \to D$ be a piecewise real-analytic expanding map and let $\xi = \{D_i\}_{i=1}^k$ be the partition of the domain D associated to it as in section 2. In this section we prove

Theorem 8 *If f satisfies*

(a) $M(f) + \rho(f)^{-1} < 1$ *and,*

(b) *the continuous extension of each restriction $f|_{D_i}$, $1 \leq i \leq k$, to the closure $\overline{D}_i$ is injective,*

then there exists an absolutely continuous invariant finite measure for f.

Theorem 8 above is a modification of the result of Góra and Boyarski in [4], and the essential part of the proof below is a repetition of the argument in [4]. We define Perron-Frobenius operator $P_f : L^1(D) \to L^1(D)$ by

$$P_f(\varphi)(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|\det Df(y)|}$$

where the sum is taken over all $y \in \cup_{i=1}^k D_i$ such that $f(y) = x$. Remark that if there exists a non-negative valued function $h \neq 0$ in $L^1(D)$ such that $P_f(\varphi) = \varphi$, the measure $h \cdot \mu_2$ is an absolutely continuous invariant finite measure for f. From the definition of Perron-Frobenius operator, we have

$$\int_D P_f g d\mu_2 = \int_D g d\mu_2 \tag{8}$$

for non-negative valued function $g \in L^1(D)$. In what follows, we consider each element $\varphi \in L^1(D)$ as an element of $L^1(\mathbf{R}^2)$ by defining $\varphi(x) = 0$ for $x \notin D$. The following is the key in the proof of theorem 8.

Proposition 9 *For any $\epsilon > 0$, there exists a constant $C > 0$ such that*

$$\mathrm{Var}(P_f\varphi, \mathbf{R}^2) \leq (M(f) + \rho^{-1}(f) + \epsilon)\mathrm{Var}(\varphi, \mathbf{R}^2) + C\|\varphi\|_{L^1} \tag{9}$$

for $\varphi \in \mathcal{BV}(D)$.

This kind of inequality was appeared in the original work of Lasota and Yorke and can be seen in the papers of Keller [2, 3] and Góra&Boyarski[4].

First, we prove that proposition 9 implies theorem 8. Take a small number $\epsilon > 0$ such that $M(f) + \rho(f)^{-1} + \epsilon < 1$. Then take the constant C in proposition 9 for that ϵ. Let $\varphi \in \mathcal{BV}(D)$ be a non-negative valued function such that $\int_D \varphi d\mu_2 = 1$. From (9) and (8), we obtain

$$\mathrm{Var}(P_f^n\varphi, \mathbf{R}^2) \leq (1 - M(f) - \rho(f)^{-1} - \epsilon)^{-1}C + \mathrm{Var}(\varphi, \mathbf{R}^2)$$

by induction. Hence the set of functions $\{\zeta_n = n^{-1}\sum_{i=0}^{n-1} P_f^i\varphi\}_{n=1}^\infty$ are contained in $\mathcal{BV}(D)$ and uniformly bounded in the bounded variation norm $\|\cdot\|_{BV(\mathbf{R}^2)}$ on $\mathbf{R}^2$. Applying theorem 5 to a bounded open subset

$U \supset D$ with C^1 boundary, we can find a subsequence ζ_{n_k} that converges to a function $\varphi_\infty \in L^1(D)$ in L^1 norm. Obviously φ_∞ is non-negative valued. We have $\int_D \varphi_\infty d\mu_2 = 1$ from (8). Moreover, φ_∞ is a fixed point of Perron-Frobenius operator P_f because

$$\begin{aligned}\|P_f\varphi_\infty - \varphi_\infty\|_{L^1} &= \lim_{k\to\infty}\left\|\frac{1}{n_k}\sum_{i=0}^{n_k-1} P_f^{i+1}\varphi - \frac{1}{n_k}\sum_{i=0}^{n_k-1} P_f^i\varphi\right\|_{L^1} \\ &\leq \lim_{k\to\infty}\frac{1}{n_k}\|P_f^{n_k}\varphi - \varphi\|_{L^1} \leq \lim_{k\to\infty}\frac{2}{n_k} = 0.\end{aligned}$$

Therefore $\varphi_\infty \cdot \mu_2$ is an absolutely continuous invariant finite measure for f. Theorem 8 is proved.

Now let us go into the proof of proposition 9. We first study the situation that the image of a dividing curve $\gamma_i : [a_i, b_i] \to \mathbf{R}^2$ is contained in the boundary of some region $D_j \in \xi$. From proposition 6, a function $\varphi \in \mathcal{BV}(D)$ viewed as a function on D_j gives the trace φ_j^- on the curve γ_i. We consider one side of the tubular neighborhood of the curve γ_i,

$$\Gamma_{ij} : [a_i, b_i] \times [0, \delta] \to \mathbf{R}^2, \quad (t, s) \mapsto \gamma_i(t) + s \cdot \nu(t)$$

where $\nu(t)$ is the unit inner normal vector for the boundary ∂D_j at $\gamma_i(t)$ and $\delta > 0$ is a small constant that we will specify in the argument below. We first take $\delta > 0$ so small that Γ_{ij} is a diffeomorphism. We will denote the image of Γ_{ij} by the same symbol Γ_{ij}.

Let $V_i(x)$ be the unit tangent vector of the curve γ_i at $x \in \gamma_i$. We define a real-analytic function $h_{ij} : \gamma_i \to \mathbf{R}$ by

$$h_{ij}(x) = \frac{\rho(V_i(x), f_{D_j})}{|\det Df_{D_j}(x)|}.$$

Let $\pi : [a_i, b_i] \times [0, \delta] \to [a_i, b_i]$ be the projection. We define a function

$$\tilde{h}_{ij} = (0, h_{ij} \circ \Gamma_{ij} \circ \pi) : [a_i, b_i] \times [0, \delta] \to \mathbf{R}^2.$$

Then we have

Lemma 10 *If $\varphi \in \mathcal{BV}(D)$, the composition $\varphi \circ \Gamma_{ij}$ viewed as a function on the open rectangle $(a_i, b_i) \times (0, \delta)$ is of bounded variation. We have*

$$\begin{aligned}\int_{\gamma_i} h_{ij} \cdot \varphi_j^- d\mu_1 &\leq \frac{1}{\delta}\int_{[a_i,b_i]\times[0,\delta]} \|\tilde{h}_{ij}\| \cdot |\varphi \circ \Gamma_{ij}| d\mu_2 \\ &\quad + \int_{(a_i,b_i)\times(0,\delta)} \|\tilde{h}_{ij}\| \cdot |D(\varphi \circ \Gamma_{ij})|.\end{aligned}$$

proof. We can get the first claim easily from the formula (7). For $y \in (0, \delta)$, let us consider the rectangle $R_y = [a_i, b_i] \times [0, y]$. The function $\varphi \circ \Gamma_{ij}$, viewed as a function on the interior of R_y, gives the trace G_y^- on the boundary ∂R_y of the rectangle. Remark that the restriction of G_y^- on the edge $[a_i, b_i] \times \{0\}$ does not depend on y and equals to the function

$\varphi_i^- \circ \Gamma_{ij}$ on $[a_i, b_i] \times \{0\}$ from the definition of the trace. Obviously, we have

$$\int_{[a_i,b_i]\times\{0\}} G_y^- \cdot h_{ij} \circ \Gamma_{i,j} \circ \pi d\mu_1 = \int_{\gamma_i} h_{ij} \cdot \varphi_j^- d\mu_1.$$

Let us define a function $B : (a_i, b_i) \times (0, \delta) \to \mathbf{R}$ by $B(x, y) = G_y^-(x, y)$. Then, from Lebesgue's theorem [9, Theorem 1.3.8], $B(x, y) = \varphi \circ \Gamma_{ij}(x, y)$ for almost every $(x, y) \in (a_i, b_i) \times (0, \delta)$. Applying proposition 6 to $\varphi \circ \Gamma_{ij}(x, y)$ and $\tilde{h}$ on R_y, we obtain

$$\left| \int_{[a_i,b_i]\times\{y\}} B \cdot h_{ij} \circ \Gamma_{ij} \circ \pi d\mu_1 - \int_{\gamma_i} h_{ij} \cdot \varphi_j^- d\mu_1 \right| \le \int_{intR_y} \|\tilde{h}_{ij}\| \cdot |D(\varphi \circ \Gamma_{ij})| \tag{10}$$

because $\mathrm{Div}\tilde{h}_{ij} \equiv 0$. Since $\|\tilde{h}_{ij}\| = h_{ij} \circ \Gamma_{ij} \circ \pi$, we get

$$\delta \left(\int_{\gamma_i} h_{ij} \cdot \varphi_j^- d\mu_1 - \int_{(a_i,b_i)\times(0,\delta)} \|\tilde{h}_{ij}\| \cdot |D(\varphi \circ \Gamma_{ij})| \right) \le \int_{[a_i,b_i]\times[0,\delta]} \|\tilde{h}_{ij}\| \cdot |\varphi \circ \Gamma_{ij}| d\mu_2$$

from Fubini's theorem. This implies the lemma. □

Let us take a small number $\eta > 0$ such that $(1+\eta)^2 M(f) < M(f) + \epsilon$. Remark that, if $\delta > 0$ is sufficiently small, the map Γ_{ij} is almost isometry on $[a_i, b_i] \times [0, \delta]$. Hence we can take $\delta > 0$ so small that

$$(1+\eta)^{-1}\|v\| < \|D\Gamma_{ij}(v)\| \le (1+\eta)\|v\| \tag{11}$$

for all tangent vectors v at all points in $[a_i, b_i] \times [0, \delta]$. Let us define a function $\hat{h}_{ij} : \mathbf{R}^2 \to \mathbf{R}$ by

$$\hat{h}_{ij}(z) = \begin{cases} h_{ij} \circ \Gamma_{ij} \circ \pi \circ \Gamma_{ij}^{-1} & \text{for } x \in \Gamma_{ij}; \\ 0 & \text{for } x \notin \Gamma_{ij}. \end{cases}$$

Since $\hat{h}_{ij} \circ \Gamma_{ij} = \|\tilde{h}_{ij}\|$ for $x \in \Gamma_{ij}$, we obtain, from (11),

$$\int_{(a_i,b_i)\times(0,\delta)} \|\tilde{h}_{ij}\| \cdot |D(\varphi \circ \Gamma_{ij})| \le (1+\eta) \int_{int\Gamma_{ij}} \hat{h}_{ij} |D\varphi|.$$

Therefore we get, from the last proposition,

Proposition 11 *We have*

$$\int_{\gamma_i} h_{ij} \cdot \varphi_j^- d\mu_1 \le (1+\eta)^2 \left(\frac{1}{\delta} \int_{\Gamma_{ij}} \hat{h}_{ij} |\varphi| d\mu_2 + \int_{\mathbf{R}^2} \hat{h}_{ij} |D\varphi| \right).$$

Next we prove the following proposition.

Proposition 12 *Let U be one of the regions in the partition $\xi = \{D_j\}_{j=1}^k$. Let $V(x)$ be the unit tangent vector for the boundary ∂U at $x \in \partial U$. Let $\varphi \in \mathcal{BV}(U)$ be non-negative valued. Let φ^- be the trace of φ on the boundary ∂U. Then we have*

$$\begin{aligned}\mathrm{Var}(P_f\varphi, \mathbf{R}^2) \;\le\; & \rho(f)^{-1}\mathrm{Var}(\varphi, U) + C(f,U)\|\varphi\|_{L^1} \\ & + \int_{\partial U} \frac{\rho(V(x), f_U)}{|\det Df_U(x)|}\varphi^-(x)d\mu_1(x).\end{aligned}$$

where $C(f,U)$ is a constant depending only on the restriction $f|_U$ of f (defined in (12) below).

proof. From Stokes' theorem, we have

$$\begin{aligned}\int_{\mathbf{R}^2} P_f\varphi d\Psi &= \int_{\mathbf{R}^2} \frac{\varphi}{|\det Df|} df^*\Psi \\ &= \int_{\mathbf{R}^2} \varphi\, d\!\left(\frac{f^*\Psi}{|\det Df|}\right) - \int_{\mathbf{R}^2} \varphi\, d\!\left(\frac{1}{|\det Df|}\right) \wedge f^*\Psi\end{aligned}$$

for $\Psi \in \Omega_0^1(f(U))$. Hence we get, from the formula (7),

$$\mathrm{Var}(P_f\varphi, f(U)) \le \rho(f)^{-1}\mathrm{Var}(\varphi, U) + C(f,U)\|\varphi\|_{L^1}$$

if we put $\sigma(f,U) = \sup\{\|Df(v)\|/\|v\| \mid 0 \ne v \in T_x\mathbf{R}^2, x \in U\}$ and

$$C(f,U) = \sigma(f,U)\sup\{\|D((\det Df)^{-1})(x)\| \mid x \in U\}. \tag{12}$$

On the other hand, we have

$$\int_{\partial f(U)} (P_f\varphi)^- d\mu_1 = \int_{\partial U} \frac{\rho(V(x), f_U)}{|\det Df_U(x)|}\varphi^-(x)d\mu_1(x).$$

From these and proposition 6(b), We obtain the conclusion. □

Now we complete the proof of proposition 9. From proposition 12,

$$\begin{aligned}\mathrm{Var}(P_f\varphi, \mathbf{R}^2) &\le \sum_{j=1}^k \mathrm{Var}(P_f(\varphi\cdot\chi_{D_j}), \mathbf{R}^2) \\ &\le \sum_{j=1}^k \left(\frac{\mathrm{Var}(\varphi, D_j)}{\rho(f)} + C(f, D_j)\|\varphi\|_{L^1} + \sum_{i\sim j}\int_{\gamma_i} h_{ij}\varphi_j^- d\mu_1\right)\end{aligned}$$

where $\sum_{i\sim j}$ is the sum over i satisfying $\gamma_i \subset D_j$. We have

$$\sum_{j=1}^k \mathrm{Var}(\varphi, D_j) \le \mathrm{Var}(\varphi, \mathbf{R}^2).$$

Hence, in order to prove proposition 9, we show

$$\sum_{j=1}^k \sum_{i\sim j} \int_{\gamma_i} h_{ij}\varphi_j^- d\mu_1 \le (M(f)+\epsilon)\mathrm{Var}(\varphi, \mathbf{R}^2) + K\|\varphi\|_{L^1}$$

for some constant $K > 0$. But, from proposition 11, it is sufficient to show

$$(1+\eta)^2 \sum_{j=1}^{k} \sum_{i\sim j} \int_{\mathbf{R}^2} \hat{h}_{ij} |D\varphi| \leq (M(f)+\epsilon)\mathrm{Var}(\varphi, \mathbf{R}^2)$$

or, more simply,

$$(1+\eta)^2 \sum_{j=1}^{k} \sum_{i\sim j} \hat{h}_{ij}(x) \leq M(f)+\epsilon \quad \text{for } x \in D. \tag{13}$$

Notice that $\hat{h}_{ij}(x) = h_{ij}(x)$ on the dividing curves γ_i. From the definition of weighted multiplicity $M(p, f)$, we have

$$\sum_{j=1}^{k} \sum_{i\sim j} \hat{h}_{ij}(x) \leq M(x, f)$$

for $x \in E$, if δ is sufficiently small. Let F be the set of points that is contained in more than two dividing curves. From the choice of η, we can take a small open neighborhood W of the finite set F in $\overline{D}$ such that the left hand side of (13) is not larger than $(M(f)+\epsilon)$ on W. If δ is sufficiently small, the intersections of two distinct subsets in $\{\Gamma_{ij} \mid \gamma_i \subset D_j\}$ are contained in W. By continuity, we easily see that the left hand side of (13) is smaller than $M(f)+\epsilon$ for $x \in \overline{D} - W$ if δ is sufficiently small. Therefore (13) holds for sufficiently small δ. Proposition 9 is proved.

7 Estimates for the weighted multiplicity for the iterations

In this last section, we complete the proof of theorem 1 by considering the iterations of a piecewise real-analytic expanding map f. First, remark that a map has an absolutely continuous finite invariant measure if and only if an iteration of it does. Since f^n is a piecewise real-analytic map with expansion rate $\rho(f)^n$, we can assume

$$\rho(f) > 64\pi$$

without loss of generality. We prove the following theorem.

Theorem 13 $M(f^n) \to 0$.

Subdividing the partition ξ by real-analytic curves artificially, we can assume that f satisfies the condition (b) in theorem 8. It means that all iterations of f also satisfy that condition. Thus, from theorem 13 above, iterations f^n of f satisfies the assumptions (a) and (b) of theorem 8 if n is sufficiently large. Therefore we can get theorem 1 from theorem 8.

Remark 14 The above argument and the proof of theorem 8 in section 6 show that the iteration f^n satisfies the inequality (9) with f replaced by f^n and $(M(f^n)+\rho^{-1}(f^n)+\epsilon)<1$ if n is sufficiently large. As is pointed out in the papers of Keller[3] and Góra&Boyarski[4], once we get such inequality, we can derive many properties of Perron-Frobenius operator P_f and those of the absolutely continuous invariant measures for f. We refer section 3 of [4] and the references given there.

Let us prepare some notations in order to prove theorem 13. Let $\xi_n = \{D_i^{(n)}\}_{i=1}^{k(n)}$ be the real-analytic partition associated to the piecewise real-analytic map f^n. Let $E(n) = \overline{D} - \cup_{i=1}^{k(n)} D_i^{(n)} = \cup_{i=1}^{k(n)} \partial D_i^{(n)}$. For $p \in E(n)$, let $\{\beta_i(p,n)\}_{i=1}^{m(p,n)}$ be the germs of real-analytic curves at $p \in E(n)$ given by the dividing curves of the partition ξ_n. We assume that these germs of curves arranged in counterclockwise order around p as before. Let $U_i(p,n)$ be the region between $\beta_i(p,n)$ and $\beta_{i+1}(p,n)$ for $1 \le i \le m(p,n)$. Let us denote $\mathcal{U}(p,n) = \{U_i(p,n)\}_{i=1}^{m(p,n)}$. For $U \in \mathcal{U}(p,n)$, let f_U^n be the germ of real-analytic map at p obtained as a real-analytic continuation of the restriction of f^n to a representative of U.

Let Δ be the maximum of $\mathrm{Ord}(U_i(p,1))$ over all $1 \le i \le m(p,1)$ and all $p \in E(1)$. Let θ and Θ be the minimum and maximum of

$$\mathrm{Angle}_{\mathrm{Ord}(U_i(p,1))}(f(U_i(p,1)))$$

over all $1 \le i \le m(p,1)$ and all $p \in E(1)$ respectively. Let μ be the maximum of $m(p,1)$ over all $p \in E(1)$. Thus Δ, θ, Θ and μ depend only on the single map f.

Let us consider a point $p \in E(n)$. Each $V \in \mathcal{U}(p,n+1)$ is contained in some $U \in \mathcal{U}(p,n)$ as a germ. Remark that, if $V \subset U$ and $V \neq U$, the image $f_U^n(p)$ is contained in $E(1)$ and a dividing curves of the partition $\xi = \xi_1$ passing through $f_U^n(p)$ divides $f^n(U)$ into more than two regions.

We say that $V \in \mathcal{U}(p,n+1)$ is a kid of $U \in \mathcal{U}(p,n)$ if $V \subset U$. If V is a kid of U and if V and U have at least one germ of real-analytic curve as a boundary curve in common additionally, we say V is a daughter of U. Especially, if $V = U$, V is a daughter of U. Obviously, each $U \in \mathcal{U}(p,n)$ has at most two daughters. If $\mathrm{Ord}(V) > \mathrm{Ord}(U)$, we say that V is a small kid of U.

The reason why we distinguish daughters is the following. Let $V \in \mathcal{U}(n+1,p)$ be a kid of $U \in \mathcal{U}(n,p)$ and assume that $U \neq V$. If V is not a daughter, $f^n(V)$ should coincide with an element of $\mathcal{U}(f_U^n(p),1)$. So $\mathrm{Angle}_{\mathrm{Ord}(V)}(f^{n+1}(V)) \le \Theta$. On the other hand, if V is a daughter and it is small, we can not expect such estimate on $\mathrm{Angle}_{\mathrm{Ord}(V)}(f^{n+1}(V))$. For the same reason, we put the following definition. An element U of $\mathcal{U}(p,n)$ is called special if $\mathrm{Ord}(U) > \Delta$ or if there is a chain $U_i \in \mathcal{U}(p,n-\ell+i)$, $i = 0,1,2,\cdots,\ell$, of regions with length $\ell+1 \ge 2$ such that

- $U_\ell = U$,
- U_1 is a small kid and daughter of U_0, and
- U_{i+1} is a daughter of U_i for $1 \le i < \ell$.

In order to estimate $M(f^n)$, we introduce what we call modified weight $\mathcal{W}(U)$ of $U \in \mathcal{U}(p,n)$ in the following manner. We fix a small number $0 < \eta < 1$ that will be specified later in the condition (16). We define the level $\ell(U)$ of $U \in \mathcal{U}(p,n)$, $p \in E(n)$, by

$$\ell(U) = \begin{cases} 2\min\{\mathrm{Ord}(U), \Delta+1\} & \text{if } U \text{ is not special;} \\ 2\min\{\mathrm{Ord}(U), \Delta+1\} - 1 & \text{if } U \text{ is special.} \end{cases}$$

Remark that we always have $\ell(V) \geq \ell(U)$ if V is a kid of U. If $U \in \mathcal{U}(p,n)$ is special, we define

$$\mathcal{W}(U) = \frac{\eta^{\ell(U)}\rho(U, f_U^n)}{|\det Df_U^n(p)|}.$$

(For the definition of $\rho(U, f_U^n)$, see section 3.) On the other hand, if U is not special, we define

$$\mathcal{W}(U) = \frac{\eta^{\ell(U)}\rho(U, f_U^n)}{|\det Df_U^n(p)|}\left[\frac{\mathrm{Angle}_{\mathrm{Ord}(U)}(f^n(U))}{\theta} + 1\right]$$

where $[\cdot]$ is Gauss' symbol. We put

$$\mathcal{M}(p, f^n) = \sum_{U \in \mathcal{U}(p,n)} \mathcal{W}(U, f^n) \quad \text{and} \quad \mathcal{M}(f^n) = \sup_{p \in E(n)} \mathcal{M}(p, f^n).$$

Clearly, we have $M(f^n) \leq 2\eta^{-2\Delta-2}\mathcal{M}(f^n)$. In order to prove theorem 13, it is enough to show the following proposition.

Proposition 15 *If we take $\eta > 0$ sufficiently small, we have*

$$\sum_{V:\text{a kid of } U} \mathcal{W}(V) \leq \mathcal{W}(U)/2 \tag{14}$$

for all $U \in \mathcal{U}(p,n)$, $p \in E(n)$, $n \geq 1$.

In fact, if this is true, we have

$$\mathcal{M}(p, f^{n+1}) \leq (1/2)\mathcal{M}(p, f^n) \leq (1/2)\mathcal{M}(f^n)$$

for $p \in E(n)$. On the other hand, we have

$$\mathcal{M}(p, f^{n+1}) \leq \rho(f)^{-n}\mathcal{M}(f^n(p), f) \leq (1/2)^n\mathcal{M}(f)$$

for $p \in E(n+1) - E(n)$. These show $\mathcal{M}(f^n) \leq (1/2)^n\mathcal{M}(f)$ inductively. Therefore $M(f^n) \leq (1/2)^{n-1}\eta^{-2\Delta-2}\mathcal{M}(f) \to 0$ as $n \to \infty$.

We prove proposition 15. Let us consider a region $U \in \mathcal{U}(p,n)$ and its kids. We assume that $\mathrm{Ord}U \leq \Delta$ until the end of this proof where we treat the case $\mathrm{Ord}U > \Delta$. We classify the kids V of U into the following four classes:

1. V is a daughter of U, and V is a small kid of U,
2. V is not a daughter of U, and V is a small kid of U,
3. V is a daughter of U, and V is not a small kid of U, and

4. V is not a daughter of U, and V is not a small kid of U.

We estimate the sums of modified weights over the kids V in each class.

First we consider the kids in the class 1. Kids V in this class is special. Hence we have

$$\mathcal{W}(V) \leq \frac{\rho(f^n(V), f_{f^n(U)})}{|\det Df_{f^n(U)}(f_U^n(p))|}\mathcal{W}(U) \leq \rho(f)^{-1}\mathcal{W}(U).$$

Since the number of kids in this class is at most 2, the sum of $\mathcal{W}(V)$ over the kids V in this class is not larger than $2\rho(f)^{-1}\mathcal{W}(U) < (1/8)\mathcal{W}(U)$.

We consider the class 2. Note that $\ell(V) \geq \ell(U) + 1$ in this case. The number of the kids in this class is not larger than μ. For each kid V in this class, we have

$$\mathrm{Angle}_{\mathrm{Ord}(V)}(f^{n+1}(V)) \leq \Theta.$$

Thus it holds

$$\begin{aligned}\mathcal{W}(V) &\leq \eta(\Theta/\theta + 1)\frac{\rho(f^n(V), f_{f^n(V)})}{|\det Df_{f^n(V)}(f_V^n(p))|}\mathcal{W}(U)\\ &\leq \eta(\Theta/\theta+1)\mathcal{W}(U).\end{aligned}$$

Hence the sum of $\mathcal{W}(V)$ over the kids V in this class is not larger than $\mu\eta(\Theta/\theta + 1)\mathcal{W}(U)$.

We consider the case 3. Remark that U is special if and only if so is V. In the case U is special, we easily see that

$$\mathcal{W}(V) \leq \frac{\rho(f^n(V), f)}{|\det Df_U(p)|}\mathcal{W}(U) \leq \rho(f)^{-1}\mathcal{W}(U).$$

And the sum is not larger than $2\rho(f)^{-1}\mathcal{W}(U) < (1/8)\mathcal{W}(U)$. In the case U is not special, we have

$$\begin{aligned}\mathcal{W}(V) &\leq \frac{[\mathrm{Angle}_{\mathrm{Ord}(V)}(f^{n+1}(V))/\theta] + 1}{[\mathrm{Angle}_{\mathrm{Ord}(U)}(f^n(U))/\theta] + 1}\frac{\rho(f^n(V), f)}{|\det Df_{f^n(V)}(f_V^n(p))|}\mathcal{W}(U)\\ &\leq \left\{1 + \frac{\mathrm{Angle}_{\mathrm{Ord}(V)}(f^{n+1}(V))}{\mathrm{Angle}_{\mathrm{Ord}(V)}(f^n(V))}\right\}\frac{\rho(f^n(V), f)}{|\det Df_{f^n(V)}(f_V^n(p))|}\mathcal{W}(U).\end{aligned}$$

Here we used the fact $\mathrm{Angle}_{\mathrm{Ord}(U)}(f^n(U)) \geq \mathrm{Angle}_{\mathrm{Ord}(V)}(f^n(V))$ and an inequality

$$\frac{[y+1]}{[x+1]} \leq \frac{y+1}{\max\{x, 1\}} \leq \frac{y}{x} + 1 \quad \text{for } x > 0 \text{ and } y > 0.$$

Using lemma 4 in case $\frac{\mathrm{Angle}(f^{n+1}(V))}{\mathrm{Angle}(f^n(V))} > 1$, we get

$$\mathcal{W}(V) \leq 4\pi\rho(f)^{-1}\mathcal{W}(U).$$

Since the number of kids in this class is at most 2, the sum is not larger than $8\pi\rho(f)^{-1}\mathcal{W}(U) < (1/8)\mathcal{W}(U)$.

We see the class 4. In this case, let V_i, $i = 1, 2, \cdots, \ell$, be the kids of this class and let $d = \mathrm{Ord}(U)$. Remark that V_i, $i = 1, 2, \cdots, \ell$, are not special. In the case U is special, we have $\ell(V_i) \geq \ell(U) + 1$ for all i. Hence, in this case, we can see that the sum $\sum_{i=1}^{\ell} \mathcal{W}(V_i)$ is not larger than $\mu\eta(\Theta/\theta + 1)\mathcal{W}(U)$ by just the same argument as above for the class 2. Let us consider the case that U is not special. Obviously we have

$$\sum_{i=1}^{\ell} \mathrm{Angle}_d(f^n(V_i)) \leq \mathrm{Angle}_d(f^n(U)). \tag{15}$$

Since $\mathrm{Angle}_d(f^{n+1}(V_i)) \geq \theta$ for $i = 1, 2, \cdots, \ell$, we have

$$[\mathrm{Angle}_d(f^{n+1}(V_i))/\theta + 1] \leq 2\mathrm{Angle}_d(f^{n+1}(V_i))/\theta.$$

By using lemma 4, we obtain

$$\begin{aligned} \frac{\mathcal{W}(V_i)}{\mathcal{W}(U)} &\leq \frac{[\mathrm{Angle}_d(f^{n+1}(V_i))/\theta + 1]}{[\mathrm{Angle}_d(f^n(U)/\theta + 1]} \cdot \frac{\rho(f^n(V_i), f)}{|\det Df_{f^n(V_i)}(f^n_{V_i}(p))|} \\ &\leq \frac{[\mathrm{Angle}_d(f^{n+1}(V_i))/\theta + 1]}{[\mathrm{Angle}_d(f^n(U))/\theta + 1]} \cdot \frac{2\pi \mathrm{Angle}_d(f^n(V_i))}{\rho(f)\mathrm{Angle}_d(f^{n+1}(V_i))} \\ &\leq \frac{4\pi \mathrm{Angle}_d(f^n(V_i))}{\rho(f)\mathrm{Angle}_d(f^n(U))}. \end{aligned}$$

Thus we have, from (15),

$$\sum_{i=1}^{\ell} \mathcal{W}(V_i) \leq 4\pi\rho(f)^{-1}\mathcal{W}(U)$$

in this case.

Summing up all the above argument for the four classes, we obtain (14) for the case $\mathrm{Ord}(U) \leq \Delta$, if we take $\eta > 0$ so small that

$$\mu\eta(\Theta/\theta + 1) < 1/8 \tag{16}$$

because the sums of modified weights over each of four classes are smaller than 1/8.

Finally, let us consider the case $\mathrm{Ord}(U) > \Delta$. In this case every kid of U should be daughter. So the number of the kids is at most 2. Since U and its kid V are special, we have

$$\mathcal{W}(V) \leq \frac{\rho(f^n(V), f)}{|\det Df_{f^n(V)}(f^n_V(p))|}\mathcal{W}(U) \leq \rho(f)^{-1}\mathcal{W}(U).$$

Therefore the left hand side of (14) is smaller than $2\rho(f)^{-1}\mathcal{W}(U) < (1/2)\mathcal{W}(U)$. We completed the proof of proposition 15.

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