HOKKAIDO UNIVERSITY

Title	Dual class of a subvariety
Author(s)	Suwa, T.
Citation	Hokkaido University Preprint Series in Mathematics, 414, 1-19
Issue Date	1998-5-1
DOI	10.14943/83560
Doc URL	http://hdl.handle.net/2115/69164
Type	bulletin (article)
File Information	pre414.pdf

Instructions for use

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

DUAL CLASS OF A SUBVARIETY

Tatsuo Suwa

Series #414. May 1998

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#389 T. Tsukada, Reticular Lagrangian singularities, 41 pages. 1997.

#390 M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space, 12 pages. 1997.

#391 Y. Giga, M. Ohnuma and M.-H. Sato, On strong maximum principle and large time behaviour of generalized mean curvature flow with the Neumann boundary condition, 24 pages. 1997.

#392 T. Tsujishita and H. Watanabe, Monoidal closedness of the category of simulations, 24 pages. 1997.

#393 T. Arase, A remark on the quantale structure of multisets, 10 pages. 1997.

#394 N. H. Bingham and A. Inoue, Extension of the Drasin-Shea-Jordan theorem, 16 pages. 1997.

#395 N. H. Bingham and A. Inoue, Ratio Mercerian theorems with applications to Hankel and Fourier transforms, 30 pages. 1997.

#396 Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, 27 pages. 1997.

#397 K. Hirata and K. Sugano, On semisimple extensions of serial rings, 6 pages. 1997.

#398 D. Pei, Singularities of $\mathbb{R}P^2$-valued Gauss maps of surfaces in Minkowski 3-space, 15 pages. 1997.

#399 T. Mikami, Markov marginal problems and their applications to Markov optimal control, 28 pages. 1997.

#400 M. Tsujii, A simple proof for monotonicity of entropy in the quadratic family, 8 pages. 1998.

#401 M. Nakamura and T. Ozawa, Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth, 9 pages. 1998.

#402 D. Lehmann and T. Suwa, Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties, 19 pages. 1998.

#403 T. Nakazi and K. Okubo, ρ-contraction and 2×2 matrix, 6 pages. 1998.

#404 Y. Kohsaka, Free boundary problem for quasilinear parabolic equation with fixed angle of contact to a boundary, 28 pages. 1998.

#405 K. Yokoyama, Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions, 25 pages. 1998.

#406 F. Hiroshima, Ground states and spectrum of quantum electrodynamics of non-relativistic particles, 58pages. 1998

#407 N. Kawazumi and T. Uemura, Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries, 9pages. 1998.

#408 T. nakazi and K. Okubo, Generalized Numerical Radius And Unitary ρ-Dilation, 12pages. 1998.

#409 Y. Giga and K. Ito, Loss of convexity of simple closed curves moved by surface diffusion, 16pages. 1998.

#410 Y. Giga, K. Inui and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, 34pages. 1998.

#411 S. Izumiya, D. Pei and T. Sano, The lightcone gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, 16pages. 1998.

#412 M. Tsujii, Absolutely continuous invarient measures for piecewise real-analytic expanding maps on the plane, 18pages. 1998.

#413 J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, 40pages. 1998.

DUAL CLASS OF A SUBVARIETY

Tatsuo Suwa

Hokkaido University

Dedicated to the memory of N. Sasakura

Let M be a complex manifold of dimension n and E a holomorphic vector bundle of rank k over M. If s is a holomorphic section of E generically transverse to the zero section, it defines a subvariety V of pure codimension k in M possibly with singularities. It is "well-known" that, if M is compact, then the top Chern class $c_k(E)$ of E corresponds to the homology class $[V]$ under the Poincaré duality $P : H^{2k}(M;\mathbb{C}) \xrightarrow{\sim} H_{2n-2k}(M;\mathbb{C})$ (in fact this holds with $\mathbb{Z}$ coefficients). The nature of the proof of this fact depends on, of course, how one defines the class $c_k(E)$ (cf. [G] §5 for the projective non-singular case, [F] §14.1 for the general case in the algebraic category and [GH] Ch.1, §1 for the case $k = 1$ in the complex analytic category). In this article, we take up the definition of Chern classes via the Chern-Weil theory and give a relatively elementary proof of a more precise statement. Namely, we prove that there is a canonical localization $c_k(E, s)$, in the relative cohomology $H^{2k}(M, M \setminus V;\mathbb{C})$, of $c_k(E)$ with respect to s and, if V is compact (M may not be), it corresponds to $[V]$ under the Alexander duality $A : H^{2k}(M, M \setminus V;\mathbb{C}) \xrightarrow{\sim} H_{2n-2k}(V;\mathbb{C})$ (Theorem 3.2). If M is compact, we have the commutative diagram

$$\begin{array}{ccc}
H^{2k}(M, M \setminus V;\mathbb{C}) & \xrightarrow{j^*} & H^{2k}(M;\mathbb{C}) \\
\wr\downarrow A & & \wr\downarrow P \\
H_{2n-2k}(V,\mathbb{C}) & \xrightarrow{i_*} & H_{2n-2k}(M,\mathbb{C}),
\end{array}$$

where i and j denote, respectively, the inclusions $V \hookrightarrow M$ and $(M,\emptyset) \hookrightarrow (M, M\setminus V)$. Since $j^*(c_k(E,s)) = c_k(E)$, we recover the result we first mentioned.

As related topics, we discuss intersections of subvarieties. We also prove a duality theorem when V as above may not be compact, considering it as a relative cycle in M modulo $M \setminus S$ for a compact connected component S of its singular set (Theorem 6.1). This is effectively used in [BLSS].

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX

The proofs are done in the framework of Čech-de Rham cohomology. Note that the varieties defined as above include the hypersurfaces and the set-theoretic complete inetrsections in the projective space.

In section 1, we recall the Čech-de Rham cohomology and integration theory on it, describe the Poincaré and Alexander dualities and define the characteristic classes in the Čech-de Rham cohomology. We give a short discussion on the localization of the top Chern class in section 2. In section 3, we prove the duality theorem mentioned above . In section 4, we give an explicit formula for the residue of the top Chern class at an isolated zero in terms of Grothendieck residues. This gives an explicit formula for the local intersection number of divisors. In section 5, we discuss (refined) intersections of subvarieties. Combined with results in the previous sections, we reprove that the global intersection number of divisors intersectiong at isolated points is the sum of local intersection numbers. Finally, in section 6 we prove another type of duality theorem mentioned above.

1. Preliminaries

For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration theory on the Čech-de Rham cohomology is developed in [L1-3]. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [B], [GH] and [MS]. See also [S] for the material in this section.

(A) Čech-de Rham cohomology

Let M be a (connected) oriented C^∞ manifold of dimension m. For an open set U in M, we denote by $A^q(U)$ the space of complex valued C^∞ q-forms on U. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of M and set $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. We assume that I is an ordered set such that, if $U_{\alpha_0 \dots \alpha_p} \neq \emptyset$, the induced order on the subset $\{\alpha_0, \dots, \alpha_p\}$ is total. We let $I^{(p)}$ be the set of $(p+1)$-tuples $(\alpha_0, \dots, \alpha_p)$ with $\alpha_0 < \cdots < \alpha_p$ and denote by $C^p(\mathcal{U}, A^q)$ the direct product

$$C^p(\mathcal{U}, A^q) = \prod_{(\alpha_0, \dots, \alpha_p) \in I^{(p)}} A^q(U_{\alpha_0 \dots \alpha_p}).$$

Thus an element σ in $C^p(\mathcal{U}, A^q)$ assigns to each $(\alpha_0, \dots, \alpha_p)$ in $I^{(p)}$ an element $\sigma_{\alpha_0 \dots \alpha_p}$ in $A^q(U_{\alpha_0 \dots \alpha_p})$. The coboundary operator $\delta : C^p(\mathcal{U}, A^q) \longrightarrow C^{p+1}(\mathcal{U}, A^q)$ is defined as in the usual Čech cohomology theory. This together with the exterior derivative d make the collection $C^\bullet(\mathcal{U}, A^\bullet)$ a double complex. The simple complex associated to this is denoted by $(A^\bullet(\mathcal{U}), D)$ or simply by $A^\bullet(\mathcal{U})$. Thus $A^r(\mathcal{U}) = \bigoplus_{p+q=r} C^p(\mathcal{U}, A^q)$ and the differential $D : A^r(\mathcal{U}) \longrightarrow A^{r+1}(\mathcal{U})$ is given by

$$(D\sigma)_{\alpha_0 \dots \alpha_p} = \sum_{\nu=0}^{p} (-1)^\nu \sigma_{\alpha_0 \dots \widehat{\alpha_\nu} \dots \alpha_p} + (-1)^p \, d\sigma_{\alpha_0 \dots \alpha_p}.$$

We denote by $H^r(A^\bullet(\mathcal{U}))$ the cohomology of $(A^\bullet(\mathcal{U}), D)$ and call it the Čech-de Rham cohomology associated to the covering $\mathcal{U}$. It is known that the restriction map $A^r(M) \to C^0(\mathcal{U}, A^r) \subset A^r(\mathcal{U})$ induces an isomorphism

$$H^r(M; \mathbb{C}) \xrightarrow{\sim} H^r(A^\bullet(\mathcal{U})), \tag{1.1}$$

where $H^r(M;\mathbb{C})$ denotes the de Rham cohomology of M ([BT]).

We define the "cup product"

$$A^r(\mathcal{U}) \times A^s(\mathcal{U}) \longrightarrow A^{r+s}(\mathcal{U})$$

by assigning to σ in $A^r(\mathcal{U})$ and τ in $A^s(\mathcal{U})$ the element $\sigma \smile \tau$ in $A^{r+s}(\mathcal{U})$ given by

$$(\sigma \smile \tau)_{\alpha_0 \dots \alpha_p} = \sum_{\nu=0}^{p} (-1)^{(r-\nu)(p-\nu)} \sigma_{\alpha_0 \dots \alpha_\nu} \wedge \tau_{\alpha_\nu \dots \alpha_p}.$$

Then $\sigma \smile \tau$ is linear in σ and τ and we have

$$D(\sigma \smile \tau) = D\sigma \smile \tau + (-1)^r \sigma \smile D\tau.$$

Thus it induces the cup product

$$H^r(A^\bullet(\mathcal{U})) \times H^s(A^\bullet(\mathcal{U})) \longrightarrow H^{r+s}(A^\bullet(\mathcal{U}))$$

compatible, via (1.1), with the usual cup product in the de Rham cohomology.

In what follows, we use the following convention. Let $(\alpha_0, \dots, \alpha_p)$ be an element in I^{p+1}. If $U_{\alpha_1 \dots \alpha_p} \neq \emptyset$, there is a permutation ρ such that $(\alpha_{\rho(0)}, \dots, \alpha_{\rho(p)})$ is in increasing order. Then we set $\sigma_{\alpha_0 \dots \alpha_p} = \operatorname{sign} \rho \cdot \sigma_{\alpha_{\rho(0)} \dots \alpha_{\rho(p)}}$. If $U_{\alpha_1 \dots \alpha_p} = \emptyset$, we set $\sigma_{\alpha_0 \dots \alpha_p} = 0$. Note that this is consistent with the definitions of the coboundary operator and the cup product.

A system of honey-comb cells adapted to $\mathcal{U}$ ([L1-3]) is a collection $\{R_\alpha\}_{\alpha \in I}$ of m dimensional manifolds R_α with piecewise C^∞ boundary in M satisfying the following conditions :

(1) $R_\alpha \subset U_\alpha$ and $M = \bigcup_\alpha R_\alpha$.

(2) $\operatorname{Int} R_\alpha \cap \operatorname{Int} R_\beta = \emptyset$, if $\alpha \neq \beta$,

(3) If $U_{\alpha_0 \dots \alpha_p} \neq \emptyset$, $R_{\alpha_0 \dots \alpha_p} = \bigcap_{\nu=0}^p R_{\alpha_\nu} (= \bigcap_{\nu=0}^p \partial R_{\alpha_\nu})$ is an $(m-p)$ dimensional manifold with piecewise C^∞ boundary.

(4) If the set $\{\alpha_0, \dots, \alpha_p\}$ is maximal, $R_{\alpha_0 \dots \alpha_p}$ has no boundary.

In the above, $\operatorname{Int} R$ denotes the interior of a subset R in M and $\{\alpha_0, \dots, \alpha_p\}$ being maximal means that, if $U_{\alpha, \alpha_0, \dots, \alpha_p} \neq \emptyset$, then $\alpha \in \{\alpha_0, \dots, \alpha_p\}$. We orient $R_{\alpha_0 \dots \alpha_p}$ by the following rules :

(1) Each R_α has the same orientation as M and the boundary is oriented so that, if $(x_1, \dots, x_m)$ is a positive coordinate system on an open set U in M with $R_\alpha \cap U =$

$\{ x_m \geq 0 \}$, then the coordinate system $(x_1, \ldots, x_{m-1})$ on ∂R_α is positive or negative according as m is even or odd.
(2) If ρ is a permutation, $R_{\alpha_{\rho(0)}\ldots\alpha_{\rho(p)}} = \operatorname{sign}\rho \cdot R_{\alpha_0\ldots\alpha_p}$.
(3) $\partial R_{\alpha_0\ldots\alpha_p} = \sum_\alpha R_{\alpha_0\ldots\alpha_p\alpha}$.

Let $\{R_\alpha\}$ be a system of honey-comb cells adapted to $\mathcal{U}$. Suppose M is compact, then each R_α is compact and we define the integration

$$\int_M : A^m(\mathcal{U}) \to \mathbb{C}$$

by the sum

$$\int_M \sigma = \sum_{p=0}^{m} \left(\sum_{(\alpha_0,\ldots,\alpha_p)\in I^{(p)}} \int_{R_{\alpha_0\ldots\alpha_p}} \sigma_{\alpha_0\ldots\alpha_p} \right)$$

for σ in $A^m(\mathcal{U})$. Then we see that it induces the integration on the cohomology

$$\int_M : H^m(A^\bullet(\mathcal{U})) \to \mathbb{C},$$

which is compatible, via (1.1), with the usual integration on the de Rham cohomology.

(B) Duality theorems

If M is a compact oriented C^∞ manifold, the bilinear pairing

$$A^\ell(\mathcal{U}) \times A^{m-\ell}(\mathcal{U}) \to A^m(\mathcal{U}) \to \mathbb{C}$$

defined as the composition of the cup product and the integration induces the Poincaré duality

$$P_M : H^\ell(M;\mathbb{C}) \simeq H^\ell(A^\bullet(\mathcal{U})) \xrightarrow{\sim} H^{m-\ell}(A^\bullet(\mathcal{U}))^* \simeq H_{m-\ell}(M;\mathbb{C}).$$

In the above isomorphism, a class $[\sigma]$ in $H^\ell(A^\bullet(\mathcal{U}))$ corresponds to the class $[C]$ in $H_{m-\ell}(M;\mathbb{C})$ such that

$$\int_M \sigma \smile \tau = \int_C \tau$$

for all τ in $A^{m-\ell}(\mathcal{U})$ with $D\tau = 0$, where we choose the cycle C in its homology class so that it is transverse to each $R_{\alpha_0\ldots\alpha_p}$ and the integral in the right hand side is defined by

$$\sum_{p=0}^{m} \left(\sum_{(\alpha_0,\ldots,\alpha_p)\in I^{(p)}} \int_{R_{\alpha_0\ldots\alpha_p}\cap C} \tau_{\alpha_0\ldots\alpha_p} \right).$$

We may define, for an r-chain C transverse to each $R_{\alpha_0 \dots \alpha_p}$ and an s-cochain σ in $A^s(\mathcal{U})$, an $(r-s)$-chain $C \frown \sigma$, which assigns to an $(r-s)$-cochain τ in $A^{(r-s)}(\mathcal{U})$ the value $\int_C \sigma \smile \tau$. This induces the cap product

$$H_r(M;\mathbb{C}) \times H^s(A^\bullet(\mathcal{U})) \longrightarrow H_{r-s}(M;\mathbb{C}).$$

Then we may write

$$P_M([\sigma]) = [M] \frown [\sigma],$$

where $[M]$ denotes the fundamental class of M.

Now let M be an oriented manifold of dimension m again and S a closed set in M. Let $U_0 = M \setminus S$ and U_1 a neighborhood of S in M and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M with $0 < 1$. We denote by $A^r(\mathcal{U}, U_0)$ the kernel of the canonical projection $A^r(\mathcal{U}) \longrightarrow A^r(U_0)$. It is not dificult to see that

$$H^r(A^\bullet(\mathcal{U}, U_0)) \simeq H^r(M, M \setminus S; \mathbb{C}).$$

Let $\{R_0, R_1\}$ be a system of honey-comb cells adapted to $\mathcal{U}$. Recall that, if M is compact,

$$\int_M \sigma = \int_{R_0} \sigma_0 + \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}$$

for $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^m(\mathcal{U})$. Now suppose that only S is compact (M may not be). Then we may assume that R_1 is compact and we still have the integration

$$\int_M : A^m(\mathcal{U}, U_0) \longrightarrow \mathbb{C}$$

given by

$$\int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}$$

for $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^m(\mathcal{U}, U_0)$. This again induces the integration on the cohomology

$$\int_M : H^m(A^\bullet(\mathcal{U}, U_0)) \longrightarrow \mathbb{C}.$$

In the cup product $A^\ell(\mathcal{U}) \times A^{m-\ell}(\mathcal{U}) \longrightarrow A^m(\mathcal{U})$, we have

$$(\sigma_0, \sigma_1, \sigma_{01}) \smile (\tau_0, \tau_1, \tau_{01}) = (\sigma_0 \wedge \tau_0, \sigma_1 \wedge \tau_1, (-1)^r \sigma_0 \wedge \tau_{01} + \sigma_{01} \wedge \tau_1).$$

Hence, if $\sigma_0 = 0$, the right hand side depends only on σ_1, σ_{01} and τ_1. Thus we have a pairing $A^\ell(\mathcal{U}, U_0) \times A^{m-\ell}(U_1) \longrightarrow A^m(\mathcal{U}, U_0)$, which, followed by the integration, gives a bilinear pairing

$$A^\ell(\mathcal{U}, U_0) \times A^{m-\ell}(U_1) \longrightarrow \mathbb{C}.$$

If we further assume that U_1 is a regular neighborhood of S, this induces the Alexander duality

$$(1.2)\quad A_{M_S} : H^\ell(M, M \setminus S; \mathbb{C}) \simeq H^\ell(A^\bullet(\mathcal{U}, U_0)) \xrightarrow{\sim} H^{m-\ell}(U_1, \mathbb{C})^* \simeq H_{m-\ell}(S, \mathbb{C}).$$

Similarly we have

$$(1.3)\qquad H^{m-\ell}(S, \mathbb{C}) \simeq H^{m-\ell}(U_1, \mathbb{C}) \xrightarrow{\sim} H^\ell(A^\bullet(\mathcal{U}, U_0))^* \simeq H_\ell(M, M \setminus S; \mathbb{C}).$$

In the isomorphism (1.2), a class $[\sigma] = [(0, \sigma_1, \sigma_{01})]$ in $H^\ell(A^\bullet(\mathcal{U}, U_0))$ corresponds to the class $[C]$ in $H_{m-\ell}(S; \mathbb{C})$ such that

$$\int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 = \int_C \tau_1,$$

for all τ_1 in $A^{m-\ell}(U_1)$ with $d\tau_1 = 0$. Also, in the isomorphism (1.3), a class $[\tau_1]$ in $H^{m-\ell}(U_1, \mathbb{C})$ corresponds to the class $[C]$ in $H_\ell(M, M \setminus S; \mathbb{C})$ such that

$$(1.4)\qquad \int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 = \int_{R_1 \cap C} \sigma_1 + \int_{R_{01} \cap C} \sigma_{01},$$

for all $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^\ell(\mathcal{U}, U_0)$ with $D\sigma = 0$. If S is connected, then we have $H_m(M, M \setminus S; \mathbb{C}) \simeq H^0(S; \mathbb{C}) = \mathbb{C}$. We denote by $[M_S]$ the class in $H_m(M, M \setminus S; \mathbb{C})$ corresponding to $[1]$ in $H^0(S; \mathbb{C})$. We may also define the cap product

$$H_r(M, M \setminus S; \mathbb{C}) \times H^s(M, M \setminus S; \mathbb{C}) \longrightarrow H_{r-s}(S; \mathbb{C})$$

as before. Then we may write

$$A_{M_S}([\sigma]) = [M_S] \frown [\sigma].$$

When M is compact, we have the commutative diagram

$$(1.5)\qquad \begin{array}{ccc} H^\ell(M, M \setminus S; \mathbb{C}) & \xrightarrow{j^*} & H^\ell(M; \mathbb{C}) \\ \wr \downarrow A_{M_S} & & \wr \downarrow P_M \\ H_{m-\ell}(S, \mathbb{C}) & \xrightarrow{i_*} & H_{m-\ell}(M, \mathbb{C}), \end{array}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow M$ and $(M, \emptyset) \hookrightarrow (M, M \setminus S)$.

We also describe the Alexander duality in another situation we consider later. Let M be a complex manifold of (complex) dimension n and V a compact subvariety in M. Let $S = \mathrm{Sing}(V)$ be the singular set of V. Also, let $U_0 = M \setminus V$, U_1 a sufficiently small tubular neighborhood of $V' = V \setminus S$ and U_2 a sufficiently small regular neighborhood of S in M. We consider the coverings $\mathcal{U} = \{U_0, U_1, U_2\}$ of M and $\mathcal{U}' = \{U_1, U_2\}$ of $U = U_1 \cup U_2$, which may be assumed to be a regular neighborhood of V. An element σ in $A^\ell(\mathcal{U})$ is expressed as $(\sigma_0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{02}, \sigma_{12}, \sigma_{012})$. We denote by $A^\ell(\mathcal{U}, U_0)$ the subspace $\{\, \sigma \in A^\ell(\mathcal{U}) \mid \sigma_0 = 0 \,\}$ of $A^\ell(\mathcal{U})$. The Alexander duality

$$(1.6) \qquad H^\ell(M, M \setminus V; \mathbb{C}) \simeq H^\ell(A^\bullet(\mathcal{U}, U_0)) \xrightarrow{\sim} H_{2n-\ell}(U; \mathbb{C}) \simeq H_{2n-\ell}(V; \mathbb{C})$$

is induced from the pairing

$$B : A^\ell(\mathcal{U}, U_0) \times A^{2n-\ell}(\mathcal{U}') \to \mathbb{C}$$

given by, for $\sigma = (0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{02}, \sigma_{12}, \sigma_{012})$, in $A^\ell(\mathcal{U}, U_0)$ and $\tau = (\tau_1, \tau_2, \tau_{12})$ in $A^{2n-\ell}(\mathcal{U}')$,

$$\begin{aligned} B(\sigma, \tau) = &\int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_2} \sigma_2 \wedge \tau_2 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 + \int_{R_{02}} \sigma_{02} \wedge \tau_2 \\ &+ \int_{R_{12}} (\sigma_1 \wedge \tau_{12} + \sigma_{12} \wedge \tau_2) + \int_{R_{012}} (-\sigma_{01} \wedge \tau_{12} + \sigma_{012} \wedge \tau_2), \end{aligned}$$

where $\{R_0, R_1, R_2\}$ is a system of honeycomb cells adapted to $\mathcal{U}$. Thus in the Alexander duality (1.6), the class $[\sigma]$ in $H^\ell(A^\bullet(\mathcal{U}, U_0))$ corresponds to the class $[C]$ in $H_{2n-\ell}(V; \mathbb{C})$ such that

$$(1.7) \qquad B(\sigma, \tau) = \int_{R_1 \cap C} \tau_1 + \int_{R_2 \cap C} \tau_2 + \int_{R_{12} \cap C} \tau_{12}.$$

for all τ in $A^{2n-\ell}(\mathcal{U}')$ with $D\tau = 0$.

(C) Characteristic classes in the Čech-de Rham cohomology

Let M be a C^∞ manifold of dimension m and E a C^∞ complex vector bundle of (complex) rank r on M. For a connection ∇ for E and for $i = 1, \dots, r$, we denote by $c_i(\nabla)$ the i-th Chern form defined by ∇. Thus it is a closed $2i$-form on M and its class $[c_i(\nabla)]$ in $H^{2i}(M; \mathbb{C})$ is the i-th Chern class $c_i(E)$ of E.

If we have $p+1$ connections $\nabla_0, \dots, \nabla_p$ for E there is a $(2i-p)$-form $c_i(\nabla_0, \dots, \nabla_p)$ alternating in the $p+1$ entries and satisfying

$$(1.8) \qquad \sum_{\nu=0}^{p} (-1)^\nu c_i(\nabla_0, \dots, \widehat{\nabla_\nu}, \dots, \nabla_p) + (-1)^p d\, c_i(\nabla_0, \dots, \nabla_p) = 0$$

([B], here we use a different sign convention).

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of M as in (A). For each α, we choose a connection ∇_α for E on U_α, and for the collection $\nabla_* = (\nabla_\alpha)_\alpha$, we define the element $c_i(\nabla_*)$ in $A^{2i}(\mathcal{U})$ by

$$c_i(\nabla_*)_{\alpha_0 \dots \alpha_p} = c_i(\nabla_{\alpha_0}, \dots, \nabla_{\alpha_p}).$$

Then we have $D\, c_i(\nabla_*) = 0$ by (1.8). Moreover, it is shown that the class of $c_i(\nabla_*)$ in $H^{2i}(A^\bullet(\mathcal{U}))$ does not depend on the choice of the collection of connections ∇_*. Comparing with the class defined by a global connection, we see that the class $[c_i(\nabla_*)]$ corresponds to the class $c_i(E)$ in $H^{2i}(M; \mathbb{C})$ under the isomorphism (1.1).

2. Localization of the top Chern class

Let $\pi : E \longrightarrow M$ be a C^∞ complex vector bundle of rank r over an oriented C^∞ manifold M of dimension m as in the previous section. We say that a connection ∇ for E is trivial with respect to a non-vanishing section s (simply, s-trivial), if $\nabla(s) = 0$. Note that if $\nabla_0, \dots, \nabla_p$ are s-trivial connections, we have the vanishing (cf. [2] Ch.II, Proposition 9.1)

$$c_r(\nabla_0, \dots, \nabla_p) = 0. \tag{2.1}$$

Let S be a closed set in M and suppose we have a C^∞ non-vanishing section s of E on $M \setminus S$. Then, from the above fact, we will see that there is a localization $c_r(E, s)$ in $H^{2r}(M, M \setminus S; \mathbb{C})$ of the top Chern class $c_r(E)$ in $H^{2r}(M, \mathbb{C})$.

Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Recall the Chern class $c_r(E)$ is represented by the cocycle $c_r(\nabla_*)$ in $A^{2r}(\mathcal{U})$ given by

$$c_r(\nabla_*) = (c_r(\nabla_0), c_r(\nabla_1), c_r(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an s-trivial connection, then $c_r(\nabla_0) = 0$ and thus the cocycle is in $A^{2r}(\mathcal{U}, U_0)$ and it defines a class in the relative cohomology $H^{2r}(M, M \setminus S; \mathbb{C})$, which we denote by $c_r(E, s)$. It is sent to the class $c_r(E)$ by the canonical homomorphism $j^* : H^{2r}(M, M \setminus S; \mathbb{C}) \longrightarrow H^{2r}(M; \mathbb{C})$. It does not depend on the choice of the connection ∇_1 or on the choice of the s-trivial connection ∇_0 ([S]). We call $c_r(E, s)$ the localization of $c_r(E)$ with respect to the section s at S. Suppose S is a compact set admitting a regular neighborhood. Then we have the Alexander duality (1.2)

$$A_{M_S} : H^{2r}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} H_{m-2r}(S, \mathbb{C}).$$

Thus the class $c_r(E, s)$ defines a class in $H_{m-2r}(S; \mathbb{C})$, which we call the residue of $c_r(E)$ at S with respect to s and denote by $\mathrm{Res}_{c_r}(s, E; S)$. This is called the "localized top Chern class" of E with respect to s in [F] §14.1.

Let R_1 be an m dimensional manifold with C^∞ boundary in U_1 containing S in its interior and set $R_0 = M \setminus \mathrm{Int}\, R_1$ so that $\{R_0, R_1\}$ is a system of honeycomb cells adapted to $\mathcal{U}$. Then the residue $\mathrm{Res}_{c_r}(s, E; S)$ is represented by an $(m - 2r)$-cycle C in S such that

$$\int_C \tau_1 = \int_{R_1} c_r(\nabla_1) \wedge \tau_1 + \int_{R_{01}} c_r(\nabla_0, \nabla_1) \wedge \tau_1$$

for any closed $(m - 2r)$-form τ_1 on U_1. In particular, if $2r = m$, the residue is a complex number given by

$$\mathrm{Res}_{c_r}(s, E; S) = \int_{R_1} c_r(\nabla_1) + \int_{R_{01}} c_r(\nabla_0, \nabla_1). \tag{2.2}$$

If we let $(S_\lambda)_\lambda$ be the connected components of S, we have

$$H_{m-2r}(S,\mathbb{C}) = \bigoplus_\lambda H_{m-2r}(S_\lambda,\mathbb{C})$$

Hence, for each λ, $c_r(E,s)$ defines a class in $H_{m-2r}(S_\lambda;\mathbb{C})$, which we call the residue of $c_r(E)$ at S_λ with respect to s and denote by $\mathrm{Res}_{c_r}(s,E;S_\lambda)$.

From the commutativity of (1.5), we have the following "residue formula".

Proposition 2.3. *In the above situation, if M is compact,*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_{c_r}(s,E;S_\lambda) = [M] \frown c_r(E) \qquad \text{in } H_{m-2r}(M;\mathbb{C}),$$

where i_λ denotes the inclusion $S_\lambda \hookrightarrow M$.

3. The duality

Let M be a complex manifold of complex dimension n and V a subvariety (reduced analytic subspace) of pure codimension k in M. In what follows we assume that there is a holomorphic vector bundle E of rank k over M and V is defined by a holomorphic section s of E generically transverse to the zero section. This means that, if $(s_1,\dots,s_k)$ is a holomorphic frame of E on an open set U in M and if we write $s=\sum_{i=1}^k f_i s_i$ with f_i holomorphic functions on U, V is defined by $f_1=\cdots=f_k=0$ in U and $df_1\wedge\cdots\wedge df_k$ is not identically equal to 0 on $V\cap U$. If we denote the singular set of V by $\mathrm{Sing}(V)$, $\mathrm{Sing}(V)\cap U$ is given by $df_1\wedge\cdots\wedge df_k=0$ in $V\cap U$. Thus V is a set-theoretic local complete intersection and the restriction $E|_{V'}$ of E to the regular part $V'=V\setminus\mathrm{Sing}(V)$ coincides with the normal bundle of V' in M. In this situation, we prove the following

Theorem 3.1. *If M is compact, the class $c_k(E)$ corresponds to $[V]$ under the Poincaré duality $H^{2k}(M;\mathbb{C}) \xrightarrow{\sim} H_{2n-2k}(M;\mathbb{C})$. Thus we have*

$$[M]\frown c_k(E) = [V] \qquad \text{in } H_{2n-2k}(M;\mathbb{C}).$$

In fact, this follows from the following more "precise" theorem, where the things are localized at V and we need only the compactness of V but not of M itself (cf. (1.5) and the introduction).

Recall that we have the localization $c_k(E,s)$ in $H^{2k}(M,M\setminus V;\mathbb{C})$ of $c_k(E)$ with respect to the section s, as discussed in section 2.

Theorem 3.2. *Let V be a subvariety of dimension $n-k$ in M as above. If V is compact, the class $c_k(E,s)$ corresponds to $[V]$ under the Alexande duality $H^{2k}(M,M\setminus V;\mathbb{C}) \xrightarrow{\sim} H_{2n-2k}(V;\mathbb{C})$. Thus we have*

$$[M_V]\frown c_k(E,s) = [V] \qquad \text{in } H_{2n-2k}(V;\mathbb{C}).$$

Proof. Let S denote the singular set $\mathrm{Sing}(V)$ of V. Also, let U_0, U_1 and U_2 be as in the last paragraph of section 1 (B). It suffices to prove that there is a representative σ , $D\sigma = 0$, of $c_k(E, s)$ in $A^{2k}(\mathcal{U}, U_0)$ such that for any τ in $A^{2n-2k}(\mathcal{U}')$ with $D\tau = 0$, we have (1.7) with $C = V$. Now let σ be an element in $A^{2k}(\mathcal{U}, U_0)$ with $D\sigma = 0$ so that we have

$$(3.3)\quad d\sigma_1 = 0, \quad d\sigma_2 = 0, \quad d\sigma_{01} = \sigma_1, \quad d\sigma_{02} = \sigma_2, \quad d\sigma_{12} = \sigma_2 - \sigma_1, \quad \text{and} \quad d\sigma_{012} = -\sigma_{12} + \sigma_{02} - \sigma_{01}.$$

Also let τ be an element in $A^{2n-2k}(\mathcal{U}')$ with $D\tau = 0$ so that we have

$$(3.4)\qquad d\tau_1 = 0, \quad d\tau_2 = 0 \quad \text{and} \quad d\tau_{12} = \tau_2 - \tau_1.$$

Thus τ_1 is a closed $2(n-k)$-form on U_1, which is a tubular neighborhood of $V' = V \setminus S$. Denoting by π the projection $U_1 \to V'$, we have an isomorphism $\pi^* : H^{2n-2k}(V'; \mathbb{C}) \xrightarrow{\sim} H^{2n-2k}(U_1; \mathbb{C})$. Hence there is a closed $2(n-k)$-form θ on V' and a $(2n-2k-1)$-form ρ_1 on U_1 such that

$$(3.5)\qquad \tau_1 = \pi^*\theta + d\rho_1.$$

Also, τ_2 is a closed $2(n-k)$-form on U_2. Since U_2 is homotopically equivalent to S, which is less than $2(n-k)$-dimensional, we have $H^{2n-2k}(U_2; \mathbb{C}) = 0$. Hence there is a $(2n-2k-1)$-form ρ_2 on U_2 such that

$$(3.6)\qquad \tau_2 = d\rho_2.$$

Using (3.3), (3.4) and the Stokes theorem and noting that $\partial R_1 = -R_{01} + R_{12}$ and $\partial R_{01} = R_{012}$, we compute

$$\int_{R_1} \sigma_1 \wedge d\rho_1 = \int_{R_1} d(\sigma_1 \wedge \rho_1) = -\int_{R_{01}} \sigma_1 \wedge \rho_1 + \int_{R_{12}} \sigma_1 \wedge \rho_1 \qquad \text{and}$$

$$\int_{R_{01}} \sigma_{01} \wedge d\rho_1 = \int_{R_{01}} d\sigma_{01} \wedge \rho_1 - \int_{R_{01}} d(\sigma_{01} \wedge \rho_1) = \int_{R_{01}} \sigma_1 \wedge \rho_1 - \int_{R_{012}} \sigma_{01} \wedge \rho_1.$$

Similarly we have, noting that $\partial R_2 = -R_{02} - R_{12}$, $\partial R_{02} = -R_{012}$, $\partial R_{12} = R_{012}$, and $\partial R_{012} = 0$,

$$\begin{aligned}
\int_{R_2} \sigma_2 \wedge d\rho_2 &= -\int_{R_{02}} \sigma_2 \wedge \rho_2 - \int_{R_{12}} \sigma_2 \wedge \rho_2 \\
\int_{R_{02}} \sigma_{02} \wedge d\rho_2 &= \int_{R_{02}} \sigma_2 \wedge \rho_2 + \int_{R_{012}} \sigma_{02} \wedge \rho_2 \\
\int_{R_{12}} \sigma_{12} \wedge d\rho_2 &= \int_{R_{12}} (\sigma_2 - \sigma_1) \wedge \rho_2 - \int_{R_{012}} \sigma_{12} \wedge \rho_2 \\
\int_{R_{012}} \sigma_{012} \wedge d\rho_2 &= \int_{R_{012}} (\sigma_{12} - \sigma_{02} + \sigma_{01}) \wedge \rho_2.
\end{aligned}$$

Hence, if we denote by I_1 the left hand side of (1.7), we have

$$I_1 = \int_{R_1} \sigma_1 \wedge \pi^*\theta + \int_{R_{01}} \sigma_{01} \wedge \pi^*\theta - \int_{R_{12}} \sigma_1 \wedge \rho_{12} + \int_{R_{012}} \sigma_{01} \wedge \rho_{12},$$

where $\rho_{12} = \rho_2 - \rho_1 - \tau_{12}$, which is a $(2n-2k-1)$-form on $U_1 \cap U_2$. Note that from (3.4), (3.5) and (3.6), we have

$$d\rho_{12} = \pi^*\theta \qquad \text{on} \quad U_{12}.$$

The chain R_{12} is in the interior of the $(2n-1)$-dimensional manifold $U_1 \cap \partial R_2$, which may be assumed to retract to $V \cap \partial R_2 = R_{12} \cap V$ by the projection π so that we have the commutative diagram

$$\begin{array}{ccc} U_1 \cap \partial R_2 & \xrightarrow{\tilde{i}} & U_1 \cap U_2 \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ V \cap \partial R_2 & \xrightarrow{i} & V' \cap U_2, \end{array}$$

where i and $\tilde{i}$ denote the inclusions. We have $d\tilde{i}^*\rho_{12} = \tilde{i}^*d\rho_{12} = \tilde{i}^*\pi^*\theta = \pi^*i^*\theta = 0$, since $i^*\theta$ is a $2(n-k)$-form on $V \cap \partial R_2$, which is a $(2n-2k-1)$-dimensional manifold. Hence we see that there exist a $(2n-2k-1)$-form ρ on $V \cap \partial R_2$ and a $(2n-2k-2)$-form ω_{12} on $U_1 \cap \partial R_2$ such that

$$\rho_{12} = \pi^*\rho + d\omega_{12} \qquad \text{on} \quad U_1 \cap \partial R_2. \tag{3.7}$$

We have, as before

$$\int_{R_{12}} \sigma_1 \wedge d\omega_{12} = \int_{R_{012}} \sigma_1 \wedge \omega_{12} \quad \text{and} \quad \int_{R_{012}} \sigma_{01} \wedge d\omega_{12} = \int_{R_{012}} \sigma_1 \wedge \omega_{12}.$$

Hence we obtain

$$I_1 = \int_{R_1} \sigma_1 \wedge \pi^*\theta + \int_{R_{01}} \sigma_{01} \wedge \pi^*\theta - \int_{R_{12}} \sigma_1 \wedge \pi^*\rho + \int_{R_{012}} \sigma_{01} \wedge \pi^*\rho.$$

Next, we compute the right hand side I_2 of (1.7). From

$$\int_{R_1 \cap V} \tau_1 = \int_{R_1 \cap V} (\pi^*\theta + d\rho_1) = \int_{R_1 \cap V} \theta + \int_{R_{12} \cap V} \rho_1 \qquad \text{and}$$

$$\int_{R_2 \cap V} \tau_2 = -\int_{R_{12} \cap V} \rho_2,$$

and using (3.7), we have

$$I_2 = \int_{R_1 \cap V} \theta - \int_{R_{12} \cap V} \rho.$$

Now we let ∇_0 be a connection for E on U_0, trivial with respect to the section s. On U_1, the bundle E is C^∞ equivalent to the pull-back $\pi^* N_{V'}$ of the normal bundle $N_{V'}$ of V' in M. We take a connection ∇ for $N_{V'}$ on V' and let ∇_1 be the connection for E on U_1 obtained as the pull-back; $\nabla_1 = \pi^*\nabla$. On U_2, let ∇_2 be an arbitrary connection for E. The class $c_k(E, s)$ is then represented by the cocycle σ with $\sigma_0 = c_k(\nabla_0) = 0$, $\sigma_1 = c_k(\nabla_1)$ and $\sigma_{01} = c_k(\nabla_0, \nabla_1)$. In fact we have $\sigma_2 = c_k(\nabla_2)$ and so forth, but as we have seen above all the terms involving ∇_2 cancel out. First we have $\sigma_1 = c_k(\nabla_1) = \pi^* c_k(\nabla)$. Hence $\sigma_1 \wedge \pi^*\theta = \pi^*(c_k(\nabla) \wedge \theta)$, but this is 0, since $c_k(\nabla) \wedge \theta$ is a $2n$-form on V', which is real $2(n-k)$-dimensional. By a similar reason, we also have $\sigma_1 \wedge \pi^*\rho = 0$. We denote by π_1 and π_{01} the restrictions of π to R_1 and R_{01}, respectively. We may assume that $\pi_1 : R_1 \longrightarrow R_1 \cap V$ is a closed $2k$-disk bundle and that $\pi_{01} : R_{01} \longrightarrow R_1 \cap V$ is an S^{2k-1}-bundle. Recall that $R_{01} = -\partial R_1$.

We prove that

$$(3.8) \qquad (\pi_{01})_* \sigma_{01} = 1$$

so that we get, by the projection formula,

$$\int_{R_{01}} \sigma_{01} \wedge \pi^*\theta = \int_{R_1 \cap V} (\pi_{01})_* \sigma_{01} \cdot \theta = \int_{R_1 \cap V} \theta.$$

This is done as in the proof of [S] Ch.III Theorem 4.4. First, we identify U_1 with a neighborhood of the zero section in $E|_{V'}$. We also identify $E|_{U_1}$ with $\pi^*(E|_{V'})$. Let p be a point in V'. We may choose a neighborhood W of p in V' and a frame $(s_1, \dots, s_k)$ of E over W such that, if we identify $E|_W$ with $W \times \mathbb{C}^k$ by this frame, we have $s(x, z) = \sum_{i=1}^k z_i \cdot \pi^* s_i(x, z)$ for a point (x, z) in $U = \pi^{-1} W \subset E|_W = W \times \mathbb{C}^k$, where $z = (z_1, \dots, z_k)$ denotes a coordinate system on $\mathbb{C}^k$ and $(\pi^* s_1, \dots, \pi^* s_k)$ is the frame of E on U obtained by pulling back $(s_1, \dots, s_k)$ by π. We identify $E|_U$ with $U \times \mathbb{C}^k$ by this frame. Note that $\pi^* s_i(x, z) = ((x, z), s_i(x))$ in this identification and $s(x, z) = (x, z, z)$ in the identification $E|_U = U \times \mathbb{C}^k = W \times \mathbb{C}^k \times \mathbb{C}^k$.

We may assume that ∇ is obtained by extending the connection on W trivial with respect to $(s_1, \dots, s_k)$. Thus ∇_1 is trivial with respect to the frame $\mathbf{s} = (\pi^* s_1, \dots, \pi^* s_k)$. Let $U^{(i)}$ be the open set in U defined by $z_i \neq 0$. Then $\{U^{(i)}\}_{i=1}^k$ is a covering of $U \setminus V$. On $U^{(i)}$, $\mathbf{s}_i = (\pi^* s_1, \dots, \pi^* s_{i-1}, s, \pi^* s_{i+1}, \dots, \pi^* s_k)$ is a frame of E. Let $\nabla^{(i)}$ be the connection for E on $U^{(i)}$ trivial with respect to $\mathbf{s}_i$. Set $\rho_i = |z_i|^2/\|z\|^2$, $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_k|^2}$. On $U^{(i)}$, $\nabla^{(i)} \pi^* s_\ell = 0$ for $\ell \neq i$ and

$\nabla^{(i)}\,\pi^* s_i = -1/z_i \cdot \sum_{j=1}^k dz_j \otimes \pi^* s_j$. Thus we obtain an operator $\rho_i \nabla^{(i)}$ on $U \setminus V$ by setting

$$\rho_i \nabla^{(i)}\,\pi^* s_\ell = \begin{cases} -\frac{\bar{z}_i}{\|z\|^2} \sum_{j=1}^k dz_j \otimes \pi^* s_j, & \text{for } \ell = i \\ 0, & \text{for } \ell \neq i. \end{cases}$$

Moreover, from $\sum_{i=1}^k \rho_i \equiv 1$, we see that $\nabla_0 = \sum_{i=1}^k \rho_i \nabla^{(i)}$ is a connection for E on $U \setminus V$. Note that it is s-trivial, since each $\nabla^{(i)}$ is. We extend ∇_0 to an s-trivial connection for E on U_0. We denote by $\rho : E|_U = U \times \mathbb{C}^r \longrightarrow \mathbb{C}^k$ the projection. We may assume that $R_{01} \cap U = -(W \times S^{2k-1})$, where S^{2k-1} is the unit $(2k-1)$-sphere in $\mathbb{C}^k$, oriented as the boundary of the unit disk. By identifying the fiber of R_{01} over p with $-S^{2k-1}$, to prove (3.8), it suffices to prove

$$\int_{S^{2k-1}} c_k(\nabla_1, \nabla_0) = 1. \tag{3.9}$$

Exactly as in the proof of [S] Ch.III, Theorem 4.4, we can show that $c_k(\nabla_1, \nabla_0)$ is the pull-back by ρ of the Bochner-Martinelli kernel on $\mathbb{C}^k$, thus we get (3.9).

If we denote by π_{12} and π_{012} the restrictions of π to R_{12} and R_{012}, respectively, then $\pi_{12} : R_{12} \longrightarrow R_{12} \cap V$ is a closed $2k$-disk bundle and $\pi_{012} : R_{012} \longrightarrow R_{12} \cap V$ an S^{2k-1}-bundle. This time the orientation of R_{012} is same as that of ∂R_{12}, thus we have $(\pi_{012})_* \sigma_{01} = -1$ so that we get

$$\int_{R_{012}} \sigma_{01} \wedge \pi^* \rho = -\int_{R_{12} \cap V} \rho.$$

Therefore, we obtain $I_1 = I_2$. $\square$

Remark 3.10. As we noted, in the second half of the proof above, we use the techniques of the proof of [S] Ch.III, Theorem 4.4. In fact, Theorem 3.2 implies the latter for a holomorphic vector bundle over a compact complex manifold. Namely, let $\pi : E \longrightarrow M$ be a holomorphic vector bundle of rank k over a compact complex manifold M of dimension n. We denote by s_Δ the diagonal section of the pull-back bundle $\pi^* E$ over E. The section s_Δ is transverse to the zero section and the variety defined is the zero section of E, which is identified with M. Thus we have the localization $c_k(\pi^* E, s_\Delta)$ in $H^{2k}(E, E \setminus M; \mathbb{C})$ of $c_k(\pi^* E)$ with respect to s_Δ. Recall that we have the Thom class Ψ_E in $H^{2k}(E, E \setminus M; \mathbb{C})$ and the Euler class $e(E)$ in $H^{2k}(M, \mathbb{C})$ of E as a real bundle (cf. [S] Ch.II, 5). In this situation, we have

$$c_k(\pi^* E, s_\Delta) = \Psi_E \qquad \text{and} \qquad c_k(E) = e(E).$$

In fact, the second identity follows from the first. To show the first identity, consider the commutative diagram

$$\begin{array}{ccc} H^0(M; \mathbb{C}) & \xrightarrow[T_E]{\sim} & H^{2k}(E, E \setminus M; \mathbb{C}) \\ \wr\big\downarrow P_M & & \wr\big\downarrow A_{E_M} \\ H_{2n}(M, \mathbb{C}) & \xrightarrow{=} & H_{2n}(M, \mathbb{C}), \end{array}$$

where T_E denotes the Thom isomorphism. The identity follows from Theorem 3.2 and $\Psi_E = T_E([1])$.

4. Residue of the top Chern class at an isolated zero

Let $\pi : E \longrightarrow M$ be a holomorphic vector bundle of rank n over a complex manifold M of dimension n. Suppose we have a section s with an isolated zero at p in M. In this situation, we have $\mathrm{Res}_{c_n}(s, E; p)$ in $H_0(\{p\}; \mathbb{C}) = \mathbb{C}$. In the following, we compute this residue. Let U be an open neighborhood of p where the bundle E is trivial with holomorphic frame $(s_1, \dots, s_n)$. We may write $s = \sum_{i=1}^n f_i s_i$ with f_i holomorphic functions on U. In this case, we may express the residue in terms of the Grothendieck residue symbol.

Theorem 4.1. *In the above situation, we have*

$$\mathrm{Res}_{c_n}(s, E; p) = \mathrm{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}.$$

Remark 4.2. The right hand side above is defined as

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_\Gamma \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n},$$

where Γ denotes the n-cycle in U defined by

$$\Gamma = \{\, q \in U \mid |f_1(q)| = \cdots = |f_n(q)| = \varepsilon \,\}$$

for a small positive number ε. If we denote by D_i the divisor in U defined by f_i, $i = 1, \dots, n$, then this is the (local) intersection number $(D_1 \cdots D_n)_p$ of $D_1, \dots, D_n$ at p ([GH] Ch.5).

Proof of Theorem 4.1. This is done similarly as for [S] Ch.III, Theorem 5.5. Let $U_0 = U \setminus \{p\}$ and $U_1 = U$. On U_0, we let ∇_0 be an s-trivial connection for E and, on U_1, we let ∇_1 be the connection for E trivial with respect to the frame $\mathbf{s} = (s_1, \dots, s_n)$. We set

$$R_1 = \{\, q \in U \mid |f_1(q)|^2 + \cdots + |f_n(q)|^2 \le n\varepsilon^2 \,\}$$

for a small positive number ε. Since $c_n(\nabla_1) = 0$ and $R_{01} = -\partial R_1$, from (2.2) we have

$$\mathrm{Res}_{c_n}(s, E; p) = -\int_{\partial R_1} c_n(\nabla_0, \nabla_1). \tag{4.3}$$

Now we consider the covering $\mathcal{U} = \{U^{(1)}, \dots, U^{(n)}\}$ of $U_{01} = U_0$ defined by

$$U^{(i)} = \{\, q \in U_0 \mid f_i(q) \ne 0 \,\}$$

and work on the Čech-de Rham cohomology with respect to $\mathcal{U}$. On $U^{(i)}$, we may replace s_i in the frame $\mathbf{s}$ by s to obtain a frame $\mathbf{s}^{(i)}$ for E. We denote by $\nabla^{(i)}$ the connection for E on $U^{(i)}$ trivial with respect to the frame $\mathbf{s}^{(i)}$. Then we define an element τ in $A^{2n-2}(\mathcal{U})$ by

$$\tau_{i_0\cdots i_k} = c_n(\nabla_0, \nabla_1, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}),$$

which is a $(2n-k-2)$-form on $U^{(i_0)} \cap \cdots \cap U^{(i_k)}$. Since ∇_0 and $\nabla^{(i)}$ are all s-trivial, we have

$$c_n(\nabla_0, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}) = 0 \tag{4.4}$$

for $k \geq 0$. Also, if $0 \leq k \leq n-2$, ∇_1 and $\nabla^{(i_0)}, \dots, \nabla^{(i_k)}$ are all s_i-trivial for some i. Hence

$$c_n(\nabla_1, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}) = 0 \qquad \text{for} \qquad k = 0, \dots, n-2. \tag{4.5}$$

Now we compute $D\tau$. First for $k = 0$, we have, using (4.4) and (4.5),

$$\begin{aligned}(D\tau)_i &= dc_n(\nabla_0, \nabla_1, \nabla^{(i)}) = -c_n(\nabla_1, \nabla^{(i)}) + c_n(\nabla_0, \nabla^{(i)}) - c_n(\nabla_0, \nabla_1) \\ &= -c_n(\nabla_0, \nabla_1).\end{aligned}$$

For $k = 1, \dots, n-1$, we have, by (4.4),

$$\begin{aligned}(D\tau)_{i_0\dots i_k} &= \sum_{\nu=0}^{k} (-1)^\nu c_n(\nabla_0, \nabla_1, \nabla^{(i_0)}, \dots, \widehat{\nabla^{(i_\nu)}}, \dots, \nabla^{(i_k)}) \\ &\quad + (-1)^k dc_n(\nabla_0, \nabla_1, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}) \\ &= -c_n(\nabla_1, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}) + c_n(\nabla_0, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}) \\ &= -c_n(\nabla_1, \nabla^{(i_0)}, \dots, \nabla^{(i_k)}).\end{aligned}$$

Thus, using (4.5), we may summarize as

$$\begin{cases} (D\tau)_i & = -c_n(\nabla_0, \nabla_1) \\ (D\tau)_{i_0\cdots i_k} & = 0, \qquad\qquad \text{for} \quad k = 1, \dots, n-2 \\ (D\tau)_{1\cdots n} & = -c_n(\nabla_1, \nabla^{(1)}, \dots, \nabla^{(n)}). \end{cases}$$

Denoting by ι the inclusion map $\partial R_1 \hookrightarrow U_0$, we let $\iota^*\mathcal{U}$ be the covering of ∂R_1 by the open sets $\partial R_1 \cap U^{(i)}$. Then, as a system $(R^{(i)})_{i=1}^n$ of honey-comb cells adapted to $\iota^*\mathcal{U}$, we take

$$R^{(i)} = \{\, q \in \partial R_1 \mid \ |f_i(q)| \geq |f_j(q)| \quad \text{for } j \neq i \,\}$$

and, for $(i_0 \cdots i_k)$ with $1 \leq i_0 < \cdots < i_k \leq n$, we set $R^{(i_0 \cdots i_k)} = R^{(i_0)} \cap \cdots \cap R^{(i_k)}$, oriented as in section 1 (A). Considering the integration

$$\int_{\partial R_1} : A^{2n-1}(\iota^* \mathcal{U}) \to \mathbb{C},$$

we see that

$$0 = \int_{\partial R_1} D\tau = -\sum_{i=1}^{n} \int_{R^{(i)}} c_n(\nabla_0, \nabla_1) - \int_{R^{(1\ldots n)}} c_n(\nabla_1, \nabla^{(1)}, \ldots, \nabla^{(n)}).$$

Hence we get, by (4.3),

$$\mathrm{Res}_{c_n}(s, E; p) = \int_{R^{(1\ldots n)}} c_n(\nabla_1, \nabla^{(1)}, \ldots, \nabla^{(n)}).$$

If we compute the connection matrix $\theta^{(i)}$ of $\nabla^{(i)}$ with respect to the frame $\mathbf{s}$, we see that $\theta^{(i)}$ is an $n \times n$ matrix whose i-th row is given by $-\frac{1}{f_i}(df_1, \ldots, df_n)$ with all other rows equal to $(0, \ldots, 0)$. Let $\tilde{\nabla}$ denote the connection for the bundle $E \times \mathbb{R}^n$ over $\bigcap_{i=1}^{n} U^{(i)} \times \mathbb{R}^n$ given by $\tilde{\nabla} = (1 - \sum_{i=1}^{n} t_i)\nabla_1 + \sum_{i=1}^{n} t_i \nabla^{(i)}$. Then the connection matrix $\tilde{\theta}$ of $\tilde{\nabla}$ with respect to the frame $\mathbf{s}$ is given by

$$\tilde{\theta} = (1 - \sum_{i=1}^{n} t_i)\theta_1 + \sum_{i=1}^{n} t_i \theta^{(i)},$$

where θ_1 is the connection matrix of ∇_1 with respect to the frame $\mathbf{s}$ and is equal to zero. Denoting by $\tilde{\kappa}$ the curvature matrix of $\tilde{\nabla}$, we compute

$$\begin{aligned} c_n(\tilde{\kappa}) &= (-1)^n n! \left(dt_1 \wedge \frac{df_1}{f_1} \right) \wedge \cdots \wedge \left(dt_n \wedge \frac{df_n}{f_n} \right) \\ &= (-1)^{n + [\frac{n}{2}]} n!\, dt_1 \wedge \cdots \wedge dt_n \wedge \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}. \end{aligned}$$

We denote by Δ^n the standard n-simplex in $\mathbb{R}^n$ and by $\pi : M \times \Delta^n \to M$ the projection. Since $\int_{\Delta^n} dt_1 \wedge \cdots \wedge dt_n = \frac{1}{n!}$, we get,

$$c_n(\nabla_1, \nabla^{(1)}, \ldots, \nabla^{(n)}) = \left(\frac{\sqrt{-1}}{2\pi} \right)^n \pi_*(c_n(\tilde{\kappa})) = \frac{(-1)^{[\frac{n}{2}]}}{(2\pi\sqrt{-1})^n} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}.$$

Taking the orientations into account we have $\Gamma = (-1)^{[\frac{n}{2}]} R^{(1\ldots n)}$. Hence we have the theorem. □

5. Intersection of subvarieties

Let M be a complex manifold of dimension n. Also, for each $i = 1, \ldots, q$, let E_i be a holomorphic vector bundle of rank k_i over M and s_i a holomorphic section of E_i generically transverse to the zero section. We denote by V_i the subvariety defined by s_i, which is pure k_i codimensional. Then we have the localization $c_{k_i}(E_i, s_i)$ in $H^{2k_i}(M, M \setminus V_i; \mathbb{C})$ of $c_{k_i}(E_i)$ with respect to the section s_i as in section 2. Setting $S = \bigcap_{i=1}^q V_i$ and $k = \sum_{i=1}^q k_i$, we have the cup product

$$H^{2k_1}(M, M \setminus V_1; \mathbb{C}) \times \cdots \times H^{2k_q}(M, M \setminus V_q; \mathbb{C}) \to H^{2k}(M, M \setminus S; \mathbb{C}).$$

Let E be the direct sum $E = E_1 \oplus \cdots \oplus E_q$ and s the section of E given by $s = s_1 \oplus \cdots \oplus s_q$. Then the zero set of s is S and we have the localization $c_k(E, s)$ in $H^{2k}(M, M \setminus S; \mathbb{C})$ of $c_k(E)$ with respect to s. In the above cup product, we have

$$c_{k_1}(E_1, s_1) \cdots c_{k_q}(E_q, s_q) = c_k(E, s).$$

Suppose S is compact (V_i may not be). Then we have the Alexander duality

$$A_{M_S} : H^{2k}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} H_{2n-2k}(S; \mathbb{C}).$$

In view of Theorems 3.1 and 3.2, we define the (refined) intersection product $V_1 \cdots V_q$ of $V_1, \ldots, V_q$ to be the homology class $A_{M_S}(c_k(E, s))$ in $H_{2n-2k}(S; \mathbb{C})$ (cf. [F] §8.1). Thus, if $(S_\lambda)_\lambda$ denote the connected components of S, its λ component $(V_1 \cdots V_q)_\lambda$ is given by

$$(V_1 \cdots V_q)_\lambda = \mathrm{Res}_{c_k}(s, E; S_\lambda) \qquad \text{in} \quad H_{2n-2k}(S_\lambda; \mathbb{C})$$

and we have

$$(5.1) \qquad V_1 \cdots V_q = \sum_\lambda (V_1 \cdots V_q)_\lambda = \sum_\lambda \mathrm{Res}_{c_n}(s, E; S_\lambda) \qquad \text{in} \quad H_{2n-2k}(S; \mathbb{C}).$$

In particular, if $k = n$, $H_{2n-2k}(S_\lambda; \mathbb{C}) = \mathbb{C}$, hence $\mathrm{Res}_{c_n}(s, E; S_\lambda)$ is a complex number.

If s is also generically transverse to the zero section, then S is a subvariety of pure codimension k.

Recall that a divisor D on M is a formal finite sum $D = \sum_j n_j V_j$ with n_j integers and V_j irreducible subvarieties of codimension one (hypersurfaces). We denote by $|D|$ the support $\bigcup_j V_j$ of D. Note that every hypersurface is defined by a holomorphic section generically transverse to the zero section. Thus, for divisors $D_1, \ldots, D_q$, we may define the intersection product $D_1 \cdots D_q$ in $H_{2n-2q}(S; \mathbb{C})$, if $S = \bigcap_{i=1}^q |D_i|$ is compact. From (5.1) and Theorem 4.1, we have the following

Theorem 5.2. *Let M be a complex manifold of dimension n and let $D_1, \ldots, D_n$ be divisors on M. If $S = \bigcap_{i=1}^n |D_i|$ consists of finite isolated points, we have*

$$D_1 \cdots D_n = \sum_{p \in S} (D_1 \cdots D_n)_p,$$

where $(D_1 \cdots D_n)_p$ is the local intersection number at p (see Remark 4.2).

6. Duality for non-compact varieties

Let M be a complex manifold of dimension n, E a holomorphic vector bundle of rank k over M and V a subvariety of codimension k defined by a section s of E generically transverse to the zero section, as before. Also let S be a compact connected component of $\mathrm{Sing}(V)$ and U_1 a sufficiently small regular neighborhood of S in M. We may think of $[V]$ as a class in $H_{2n-2k}(M, M \setminus S; \mathbb{C})$. We denote by $c_k(E)$ the class $c_k(E|_{U_1})$ in $H^{2k}(U_1, \mathbb{C}) \simeq H^{2k}(S, \mathbb{C})$. Recall that we have the duality (1.3)

$$H^{2k}(S, \mathbb{C}) \xrightarrow{\sim} H_{2n-2k}(M, M \setminus S; \mathbb{C}).$$

Theorem 6.1. *The class $c_k(E)$ corresponds to $[V]$ under the above duality.*

Proof. Let $c_k(E)$ also denote the Chern form with respect to some connection for E. It suffices to show that

$$\int_{R_1} \sigma_1 \wedge c_k(E) + \int_{R_{01}} \sigma_{01} \wedge c_k(E) = \int_{R_1 \cap V} \sigma_1 + \int_{R_{01} \cap V} \sigma_{01}$$

for any $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{2n-2k}(\mathcal{U}, U_0)$ with $D\sigma = 0$ (cf. (1.4)). We have $d\sigma_1 = 0$ and may consider the class $[\sigma_1]$ in the de Rham cohomology $H^{2n-2k}(U_1; \mathbb{C}) \simeq H^{2n-2k}(S; \mathbb{C})$, which is zero, since S is less than $2n-2k$ dimensional. Hence there is a $(2n-2k-1)$-form η_1 on U_1 such that $\sigma_1 = \eta_1$. We compute

$$\int_{R_1} \sigma_1 \wedge c_k(E) = -\int_{R_{01}} \eta_1 \wedge c_k(E) \qquad \text{and} \qquad \int_{R_1 \cap V} \sigma_1 = -\int_{R_{01} \cap V} \eta_1.$$

Hence it suffices to show

$$\int_{R_{01}} (\sigma_{01} - \eta_1) \wedge c_k(E) = \int_{R_{01} \cap V} (\sigma_{01} - \eta_1). \tag{6.2}$$

From $\sigma_1 - d\sigma_{01} = 0$, we have $d(\sigma_{01} - \eta_1) = 0$. Note that $c_k(E)$ is the Euler class $e(E)$ of E (restricted to R_{01}) and the submanifold $R_{01} \cap V$ of the compact manifold R_{01} is defined by a section transverse to the zero section of E. Therefore, (6.2) follows from the fact that $R_{01} \cap V$ is the Poincaré dual to $e(E)$. □

Remark 6.3. Let C be a relative cycle representing a class in $H_\ell(M, M \setminus S; \mathbb{C})$. Suppose C is transverse to R_{01} and V. Then, by a similar argument as above, we have

$$\int_{R_1 \cap C} \sigma_1 \wedge c_k(E) + \int_{R_{01} \cap C} \sigma_{01} \wedge c_k(E) = \int_{R_1 \cap C \cap V} \sigma_1 + \int_{R_{01} \cap C \cap V} \sigma_{01}$$

for any $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{\ell-2k}(\mathcal{U}, U_0)$ with $D\sigma = 0$.

References

[BB] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geom. **7** (1972), 279–342.

[B] R. Bott, *Lectures on characteristic classes and foliations*, Lectures on Algebraic and Differential Topology, Lecture Notes in Mathematics 279, Springer-Verlag, New York, Heidelberg, Berlin, 1972, pp. 1–94.

[BT] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics 82, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[BLSS] J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor classes of local complete intersections*, preprint.

[F] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1978.

[G] A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154.

[L1] D. Lehmann, *Intégration sur les variétés stratifiées*, C.R. Acad. Sci. Paris **307, Sér. I** (1988), 603–606.

[L2] D. Lehmann, *Variétés stratifiées C^∞: Intégration de Čech-de Rham et théorie de Chern-Weil*, Geometry and Topology of Submanifolds II, World Scientific, Singapore, 1990, pp. 205–248.

[L3] D. Lehmann, *Systèmes d'alvéoles et intégration sur le complexe de Čech-de Rham*, Publications de l'IRMA, 23, N^o VI, Université de Lille I, 1991.

[MS] J. Milnor and J. Stasheff, *Characteristic Classes*, Annales of Mathematics Studies 76, Princeton University Press, Princeton, 1974.

[S] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Mathématiques, Hermann, Paris, to appear.

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: suwa@math.sci.hokudai.ac.jp