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Piecewise expanding maps on the plane with singular ergodic properties

Masato TSUJII*

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Abstract

For $1 \leq r < \infty$, we construct a piecewise C^r expanding map $F : D \to D$ on the domain $D = (0,1) \times (-1,1) \subset \mathbf{R}^2$ with the following property: there exists an open set B in D such that the diameter of $F^n(B)$ converges to 0 as $n \to \infty$ and the empirical measure $n^{-1} \sum_{k=0}^{n-1} \delta_{F^k(x)}$ converges to the point measure δ_p at a point p as $n \to \infty$ for any point $x \in B$.

1 Introduction

In [3], Lasota and Yorke studied ergodic properties of piecewise C^2 expanding maps on the interval. They showed that Perron-Frobenius operators associated to such maps preserve compact subsets in the space of integrable functions on the interval, which consists of functions uniformly bounded with respect to the bounded variation norm, and proved the existence of absolutely continuous invariant measures as a consequence. Since their work, several people have studied the generalzation of their result to higher dimensional case[1, 2, 4, 5, 6, 7, 8]. The method that has been used in these studies is a direct generalizations of that in the original work[3]. In this paper, we give an example of piecewise C^r expanding maps which clarify the difference between

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multi dimensional case and one dimensional case. Especially, our example shows that the method in [3] is not valid for general piecewise C^r expanding maps in higher dimensional case.

To begin with, we give some definitions. A curve $c : [a, b] \to \mathbf{R}^2$ in the plane $\mathbf{R}^2$ is said to be a C^r curve if it can be extended to a neighborhood of the interval $[a, b]$ as a C^r map. A curve $c : [a, b] \to \mathbf{R}^2$ is said to be a piecewise C^r curve if there exists a sequence $a = a_0 < a_1 < \cdots < a_n = b$ such that the restrictions $c|_{[a_i, a_{i+1}]}$ of c to the intervals $[a_i, a_{i+1}]$, $0 \le i \le m-1$, are C^r curve. We say that a region in the plane has piecewise C^r boundary if the boundary of the region consists of finitely many simple closed piecewise C^r curves.

Definition 1 *Let $D \subset \mathbf{R}^2$ be a bounded region which has piecewise C^r boundary. A map $F : D \to D$ is said to be a piecewise C^r expanding map if there exists a family of regions $D_i \subset D$, $1 \le i \le k$, with the following properties:*

(i) *each region D_i, $1 \le i \le k$, has piecewise C^r boundary,*

(ii) *the regions D_i, $1 \le i \le k$, are mutually disjoint, and the union of the closures of them equals to the closure of D,*

(iii) *the restriction $F|_{D_i}$ of the map F to D_i is C^r and can be extended to a neighborhood of the closure of D_i as a C^r map, and*

(iv) *there exists a number $\rho > 1$ such that $\|DF(x)v\| \ge \rho\|v\|$ for any $x \in \cup_{i=1}^{k} D_i$ and any $v \in T_x\mathbf{R}^2$.*

As usual, we ignore the points whose image by some iteration of F falls into the boundaries $\cup_{i=1}^{k} \partial D_i$.

In this paper we construct an example of piecewise C^r expanding map F on the open rectangle $D = (0, 1) \times (-1, 1)$ for which there exists an open subset $B \subset D$ with the following properties:

(A) the diameter of the set $F^n(B)$ converges to 0 as $n \to 0$, and

(B) the empirical measures $n^{-1} \sum_{i=0}^{n-1} \delta_{F^i(x)}$ for $x \in B$ converges to the point measure δ_p at $p = (0, 0)$ as $n \to 0$.

This example shows that analogues of the argument in [3] is not true for general piecewise C^r expanding map in higher dimensional case, if $r < \infty$. It is interesting to compare this example with very recent works of Jérôme Buzzi[2] and the author[7] in which they showed the existence of absolutely continuous invariant measures for arbitraly piecewise real-analytic expanding maps on the plane by an argument similar to that in [3].

2 Construction of the example

For $1 \leq r < \infty$, we construct a map defined on the rectangle

$$D = [0,1] \times [-1,1] = \{(x,y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1,\ -1 \leq y \leq 1\}.$$

Let us fix an integer $\gamma > r$ and a small number $\varepsilon > 0$ satisfying

$$\varepsilon < \gamma^3(1 - r^{-1})^2 \log 2. \tag{1}$$

We put $\lambda = \exp(\varepsilon)$. Let $f : [0, \exp(-\varepsilon)] \to [0, \exp(-\varepsilon))$ be a function of class C^r. We divide the rectangle D into the following eight regions with piecewise C^r boundaries:(See Figure 1.)

$$\begin{aligned}
D_{0,+} &= \{(x,y) \in \mathbf{R}^2 \mid 0 < x < \exp(-\varepsilon),\ f(x) < y < \exp(-\varepsilon)\ \},\\
D_{0,-} &= \{(x,y) \in \mathbf{R}^2 \mid 0 < x < \exp(-\varepsilon),\ -\exp(-\varepsilon) < y < f(x)\ \},\\
D_{1,+} &= \{(x,y) \in \mathbf{R}^2 \mid \exp(-\varepsilon) < x < 1,\ 0 < y < \exp(-\varepsilon)\ \},\\
D_{1,-} &= \{(x,y) \in \mathbf{R}^2 \mid \exp(-\varepsilon) < x < 1,\ -\exp(-\varepsilon) < y < 0\ \},\\
D_{2,+} &= \{(x,y) \in \mathbf{R}^2 \mid 0 < x < \exp(-\varepsilon),\ \exp(-\varepsilon) < y < 1\ \},\\
D_{2,-} &= \{(x,y) \in \mathbf{R}^2 \mid 0 < x < \exp(-\varepsilon),\ -1 < y < -\exp(-\varepsilon)\ \},\\
D_{3,+} &= \{(x,y) \in \mathbf{R}^2 \mid \exp(-\varepsilon) < x < 1,\ \exp(-\varepsilon) < y < 1\ \},\\
D_{3,-} &= \{(x,y) \in \mathbf{R}^2 \mid \exp(-\varepsilon) < x < 1,\ -1 < y < -\exp(-\varepsilon)\ \}.
\end{aligned}$$

Let $\eta : [0, \exp(-\varepsilon)] \to [0,1]$ be a function of class C^r satisfying

$$f(x) - \eta(x) \geq 0 \qquad \text{for } x \in [0, \exp(-\varepsilon)].$$

On the regions $D_{0,\pm}$ and $D_{1,+}$, we define

$$F(x,y) = \begin{cases} (\lambda x, \lambda(y - \eta(x))) & \text{for } (x,y) \in D_{0,+}; \\ (\lambda x, \lambda y) & \text{for } (x,y) \in D_{0,-}; \\ (\lambda y, \lambda(x - \exp(-\varepsilon))) & \text{for } (x,y) \in D_{1,+}. \end{cases}$$

Actually, the definition of the map F on the other regions is irrelevant to our argument below as far as F is a piecewise C^r expanding map. For completeness, we define

$$F(x,y) = \begin{cases} (-\lambda y, \lambda(x - \exp(-\varepsilon))) & \text{for } (x,y) \in D_{1,-}; \\ (\lambda x, \lambda(y - \exp(-\varepsilon)) & \text{for } (x,y) \in D_{2,+}; \\ (\lambda x, \lambda(y + \exp(-\varepsilon)) & \text{for } (x,y) \in D_{2,-}; \\ (\lambda(x - \exp(-\varepsilon)), \lambda(y - \exp(-\varepsilon)) & \text{for } (x,y) \in D_{3,+}; \\ (\lambda(x - \exp(-\varepsilon)), \lambda(y + \exp(-\varepsilon)) & \text{for } (x,y) \in D_{3,-}. \end{cases}$$

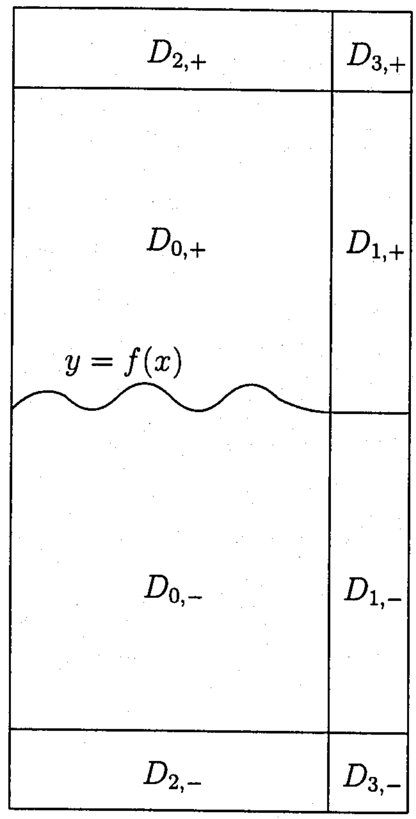

Figure 1: The partition of D

This construction of the map F depends on the functions f and η which we will specify below. Remark that the map F is piecewise C^r expanding map if η is sufficiently small in C^1 sense.

Put $I_k = [\exp(-\varepsilon k), \exp(-\varepsilon(k-1))]$ for $k \geq 1$ and let J_k be the left one-third of I_k, that is,

$$J_k = [\exp(-\varepsilon k), \exp(-\varepsilon k) + (\exp(-\varepsilon(k-1)) - \exp(-\varepsilon k))/3].$$

For $m \geq 1$ and $0 \leq k \leq \gamma^m - 1$, we define open rectangles

$$R(k,m) = \left\{ (x,y) \;\middle|\; \begin{array}{ll} B(k,m) & < y < B(k,m) + H(k,m), \\ L(k,m) & < x < L(k,m) + W(k,m) \end{array} \right\}$$

by setting $B(k,m)$, $L(k,m)$, $W(k,m)$ and $H(k,m)$. First we set

$$B(k,m) = \exp(\varepsilon k) \left(\sum_{i=0}^{\infty} \exp\left(-\varepsilon \sum_{j=0}^{2i+1} \gamma^{m+j} \right) \right),$$

$$L(k,m) = \exp(\varepsilon k) \left(\exp(-\varepsilon \gamma^m) + \sum_{i=0}^{\infty} \exp\left(-\varepsilon \sum_{j=0}^{2i+2} \gamma^{m+j} \right) \right)$$

for $0 \leq k \leq \gamma^m - 1$. Then we have

$$B(k+1,m) = \lambda B(k,m), \qquad L(k+1,m) = \lambda L(k,m)$$

for $0 \leq k \leq \gamma^m - 2$, and

$$\begin{aligned} B(0,m+1) &= \lambda(L(\gamma^m - 1, m) - \exp(-\varepsilon)), \\ L(0,m+1) &= \lambda B(\gamma^m - 1, m). \end{aligned}$$

Observe that, if m is sufficiently large, we have

$$B(k,m) \leq B(\gamma^m - 1, m) < \exp(-\varepsilon \gamma^{m+1}) \tag{2}$$

and

$$B(\gamma^{m+1} - k, m+1) < B(\gamma^m - k, m)/2, \tag{3}$$

$$0 < L(\gamma^m - k, m) - \exp(-\varepsilon k) < \frac{\exp(-\varepsilon(k-1)) - \exp(-\varepsilon k)}{6} \tag{4}$$

for $1 \leq k \leq \gamma^m$.

Next we define $H(k, m)$. When $0 \leq k < \gamma^m - \gamma^{m-1}$, we set

$$H(k,m) = \exp\left((\varepsilon - \log 2)k + \sum_{i=0}^{m-1} \varepsilon\gamma^i - (1-\gamma^{-1}) \log 2 \sum_{i=1}^{[m/2]} \gamma^{m-2i} \right).$$

On the other hand, when $\gamma^m - \gamma^{m-1} \leq k \leq \gamma^m - 1$, we set

$$H(k,m) = \exp\left(\varepsilon k + \sum_{i=0}^{m-1} \varepsilon\gamma^i - (1-\gamma^{-1}) \log 2 \sum_{i=0}^{[m/2]} \gamma^{m-2i} \right).$$

We define $W(k, m)$, $0 \leq k \leq \gamma^m - 1$, by

$$W(k,m) = \exp\left(\varepsilon k + \sum_{i=0}^{m-1} \varepsilon\gamma^i - (1-\gamma^{-1}) \log 2 \sum_{i=0}^{[(m-1)/2]} \gamma^{m-2i-1} \right).$$

Then we can check

$$\begin{aligned} W(k+1,m) &= \lambda W(k,m) && \text{for } 0 \leq k \leq \gamma^m - 2, \\ H(k+1,m) &= (\lambda/2)H(k,m) && \text{for } 0 \leq k < \gamma^m - \gamma^{m-1}, \\ H(k+1,m) &= \lambda H(k,m) && \text{for } \gamma^m - \gamma^{m-1} \leq k < \gamma^m - 2 \end{aligned}$$

and

$$\begin{aligned} W(0,m+1) &= \lambda H(\gamma^m - 1, m), \\ H(0,m+1) &= \lambda W(\gamma^m - 1, m). \end{aligned}$$

Observe that, from the condition (1) on ε, we have

$$W(\gamma^m - k, m) < \frac{\exp(-\varepsilon(k-1)) - \exp(-\varepsilon k)}{6}, \tag{5}$$

$$H(\gamma^m - k, m) < B(\gamma^m - k, m) < \exp(-\varepsilon\gamma^{m+1}) \tag{6}$$

for $1 \leq k \leq \gamma^m$, if m is sufficiently large.

For $k \geq 1$, let us put

$$K_k = \left[\frac{2\exp(-\varepsilon k) + \exp(-\varepsilon(k+1))}{3}, \frac{\exp(-\varepsilon k) + \exp(-\varepsilon(k-1))}{2} \right].$$

These intervals are mutually disjoint. Each K_k contains J_k, and the difference $K_k - J_k$ consists of two intervals with length larger than

$$\Delta(k) = (\exp(-\varepsilon k) - \exp(-\varepsilon(k+1)))/6.$$

Now we define the functions f and η. Let M be a large integer that we will specify below.

(I) Define $f(x) = 0$ and $\eta(x) = 0$ for $\exp(-\varepsilon\gamma^M) \le x \le \exp(-\varepsilon)$.

(II) Define $f(x) = 0$ and $\eta(x) = 0$ for $x \notin \cup_{k=1}^{\infty} K_k$.

(III) For $x \in J_k$ with $k > \gamma^M$, we set

$$f(x) = B(\gamma^m - k, m) + (1/2)H(\gamma^m - k, m),$$
$$\eta(x) = (1/2)H(\gamma^m - k, m).$$

where m is the integer satisfying $\gamma^{m-1} < k \le \gamma^m$.

It remains to define the functions on $K_k - J_k$ with $k \ge \gamma^M$. Let us consider the interval K_k with $k \ge \gamma^M$. Let $m(k)$ be the integer satisfying $\gamma^{m(k)-1} < k \le \gamma^{m(k)}$. From (2) and (6), each of two functions f and η takes constant value which is smaller than $2\exp(-\varepsilon\gamma^{m(k)+1})$ from (2) and (6). On the other hand, we have

$$\Delta(k) > \varepsilon \exp(-\varepsilon(k+1))/6 > \varepsilon \exp(-\varepsilon\gamma^{m(k)} - \varepsilon)/6.$$

Since we set $\gamma > r$, the ratio $2\exp(-\varepsilon\gamma^{m(k)+1})/\Delta(k)^r$ converges to 0 as $k \to \infty$. Hence we can define the functions f and η on the sets $K_k - J_k$, $k \ge \gamma^M$, so that

- the functions f and η are of class C^r on K_k,
- the value and the derivatives up to order r of the functions f and η vanish at the end points of the interval K_k, and
- the C^r norm of the functions f and η on the interval J_k converge to 0 as $k \to \infty$.

Then the functions f and η are C^r on the interval $[0, \exp(-\varepsilon)]$. Moreover, if we take M sufficiently large, the function η can be taken so small in C^1 sense that F is a piecewise C^r expanding map.

We have defined a piecewise C^r expanding map $F : D \to D$. Observe that the following hold if M is sufficiently large:

(i) For $k > \gamma^M$, the rectangles $R(\gamma^m - k, m)$, $m \ge m(k)$, are contained in $J_k \times [0,1]$ from (4) and (5), where $m(k)$ is defined as above.

(ii) For $k > \gamma^M$, the graph of f restricted to the interval J_k is a horizontal segment that passes through the center of the rectangle $R(\gamma^{m(k)} - k, m(k))$. Other rectangles $R(\gamma^m - k, m)$, $m > m(k)$, in $J_k \times [0,1]$ are below the graph of $y = f(x)$ and, hence, contained in the region $D_{0,-}$, from (3) and (6). (See Figure 2.)

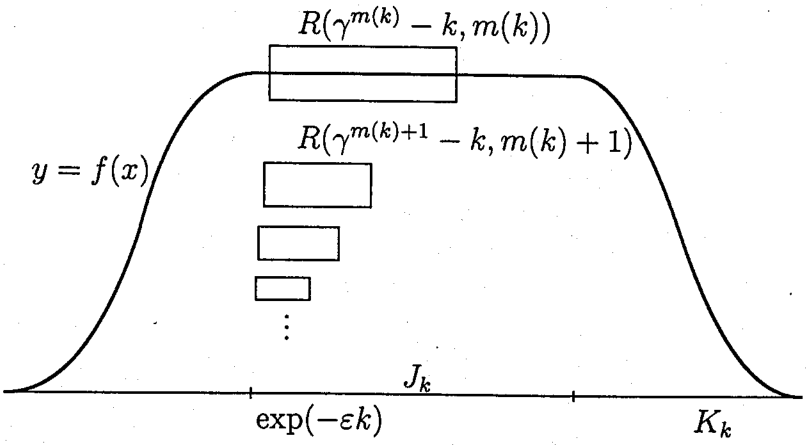

Figure 2: Rectangles in $J_k \times [0,1]$.

(iii) If $0 \le k < \gamma^m - \gamma^{m-1}$ and $m > M$, F maps each of two open rectangles $R(k,m) \cap D_{0,+}$ and $R(k,m) \cap D_{0,-}$ onto $R(k+1,m)$.

(iv) If $\gamma^m - \gamma^{m-1} \le k \le \gamma^m - 2$ and $m > M$, F maps $R(k,m)$ onto $R(k+1,m)$.

(v) F maps $R(\gamma^m - 1, m)$ onto $R(0, m+1)$ if $m > M$.

Therefore we have, for $m > M$,

- $F(R(k,m)) \subset R(k+1,m)$ for $0 \le k \le \gamma^m - 2$, and
- $F(R(\gamma^m - 1, m)) = R(0, m+1)$.

The properties (A) and (B) hold for $B = R(0, M+1)$.

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