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Existence of a forward self–similar stagnation flow of the Navier–Stokes equations

Dedicated to Professor Rentaro Agemi on his sixtieth birthday

Masayoshi ISHIKAWA and Shin'ya MATSUI*

Abstract

In this paper we construct an exact forward self–similar solution, representing a stagnation flow, for the Navier-Stokes equations by solving a third order ordinary differential equations.

1 Introduction

The solution of the form

$$u(x,t) = \frac{\sqrt{\nu}}{\sqrt{2(T-t)}} \cdot U\left(\frac{x}{\sqrt{2\nu(T-t)}}\right), p(x,t) = \frac{\nu}{\sqrt{2(T-t)}} \cdot P\left(\frac{x}{\sqrt{2\nu(T-t)}}\right)$$

is called (Leray's) backward self–similar solutions of the Navier–Stokes equations:

$$\text{(NS)} \quad \left\{ \begin{array}{c} u_t - \nu\Delta u + (u, \nabla)u + p = 0, \\ \mathrm{div} u = 0 \end{array} \right.$$

for $x \in \mathbb{R}^3$ and $0 < t < T$, where $U = U(X) \in \mathbb{R}^3$ and $P = P(X)$ with $X \in \mathbb{R}^3$. In 1934 Leray [5] proposed to study backward self–similar solutions to construct blowing up solutions. His original question whether backward self–similar solutions with finite energy could be blow up, was answered negatively by [6] and [9] recently. They showed that backward self–similar solutions with the global energy estimates (in [6]) or the local energy estimates (in [9]) must be zero.

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In the interesting paper [8], Okamoto obtained so many exact solutions of backward and forward self–similar solutions (see, also [7]). Here, forward self–similar solutions are solutions in the form:

$$u(x,t) = \frac{\sqrt{\nu}}{\sqrt{2(T+t)}} \cdot U\left(\frac{x}{\sqrt{2\nu(T+t)}}\right), p(x,t) = \frac{\nu}{\sqrt{2(T+t)}} \cdot P\left(\frac{x}{\sqrt{2\nu(T+t)}}\right),$$

where $U = (U_1(X), U_2(X), U_3(X))$ and $P = P(X)$ satisfy the following equations:

$$(\mathrm{L}) \quad \left\{ \begin{array}{c} \Delta U - \nabla P = -U - (X, \nabla)U + (U, \nabla)U, \\ \mathrm{div} U = 0 \end{array} \right.$$

for $X \in \mathbb{R}^3$. It is known that forward self–similar solutions with finite energy do not exist ([3]). However, there is a systematic way based on global existence theorem, to construct forward self-similar solutions (with finite energy) as studied in [1, 3, 4]. Their solutions are not explicitly written by elementary functions or solutions of ordinary differential equations. In the section 12 in [8] he defined a forward self–similar stagnation flow of (NS) in the following manner. In (L), putting

$$U_1 = xf'(y), \quad U_2 = -f(y), \quad U_3 = 0$$

with $X = (x, y, z)$, then $f(y)$ satisfies

$$(\mathrm{L}_1) \left\{ \begin{array}{c} -2f'(y) + (f'(y))^2 - yf''(y) - f(y)f''(y) - f'''(y) = -2k + k^2 \text{ for } y > 0, \\ f(0) = f'(0) = 0, \quad f'(\infty) = k, \end{array} \right.$$

where k is any number. The pressure P is determined by $P = -f'(y) - yf(y) - f^2(y)/2 + (-k + k^2/2)(x^2 + y^2)$. He, however, did not show existence of solutions to (L_1). The purpose of the present paper is to construct solutions of (L_1) for $k = 1$ and to show results of numerical computation to (L_1) for $k = 1$. For this purpose we reduce (L_1) to a second order ordinary differential equations. Setting

$$v(y) = f'(y) - k, \tag{1}$$

we obtain

$$(\mathrm{L}_2) \quad \left\{ \begin{array}{c} v''(y) + ((1+k)y + w(y))v'(y) + 2(1-k)v(y) = v^2(y) \quad \text{for } y > 0, \\ v(0) = -k, \ v(\infty) = 0, \ w(y) = \int_0^y v(s)ds. \end{array} \right.$$

If $1 - k > 0$, Cauchy problem of linearized equation of (L_2) might have oscillate solutions. So, it seems to be impossible to construct solutions of (L_2) because of the condition $v(\infty) = 0$. On the other hand, according to our numerical computations, linearized problems have so many blow up solution in the case of $1-k<0$. That is why we solve the equations (L_2) for $k=1$.

To construct solutions, we employ the shooting method which is closely related to [2] although the equations discussed there are different. We consider the Cauchy problem:

$$(\mathrm{E}_\alpha) \quad \begin{cases} v''(y)+(2y+w(y))v'(y) \;=\; v^2(y) \quad \text{for } y>0, \\ v(0)=-1, \;\; v'(0)=\alpha, \;\; w(y)=\displaystyle\int_0^y v(s)ds. \end{cases}$$

We will find a suitable constant $\alpha=\alpha^*$ such that a solution of (E_{α^*}) satisfies $v(\infty)=0$.

We are now in position to state our main theorem.

Theorem 1 *There exists the global solution $v(y)$ of (E_α) for some $\alpha>0$, which tends to zero as $y\to\infty$. Thus, if $k=1$, the equations (L_1) has a solution.*

In the section 2 we prove Theorem 1. After the proof we show a result of numerical experiment for (E_α).

Finally, we are grateful to Professor Yoshikazu Giga for useful discussions and encouragement.

2 Existence of the exact solution

It is well known that the unique solution of (E_α) exists on $[0,T_\alpha)$ for some $T_\alpha>0$ and any $\alpha\in\mathbb{R}$, where T_α may not be finite. Here and hereafter, we assume $\alpha\geq 0$ and denote the solution of (E_α) by $v_\alpha(y)$.

Lemma 1 *Let $v_\alpha(y)$ be the solution of (E_α) on $[0,T_\alpha)$ for $\alpha\geq 0$. Then we have*

$$v'_\alpha(y) \;\geq\; \alpha e^{-A_\alpha(y)} \quad \text{for } y\in[0,T_\alpha), \tag{2}$$

where $A_\alpha(y)=y^2+\int_0^y w_\alpha(s)ds$ and $w_\alpha(y)=\int_0^y v_\alpha(s)ds$.

Proof. Since $v''_\alpha+(2y+w_\alpha)\cdot v'_\alpha=v^2_\alpha\geq 0$, we have

$$\left(e^{A_\alpha(y)}\cdot v'_\alpha(y)\right)' \;\geq\; 0$$

for $y\in[0,T_\alpha)$. This implies (2).

Next to this, we define

$$J\equiv\{\alpha\in[0,\infty);\ \text{There exists } Y_\alpha\in(0,T_\alpha) \text{ such that } v_\alpha(Y_\alpha)=0.\}$$

Here, if $\alpha>0$, then (2) implies that Y_α is unique as far as it exists.

Lemma 2 *The solution $v_0(y)$ of (E_0) has no zeros. That is, $0 \notin J$.*

Proof. We first note that

$$v_0(y) \; > \; -1 \text{ for any } y \in (0, T_0). \tag{3}$$

Indeed, suppose that there exists some $\overline{y} \in (0, T_0)$ such that $v_0(\overline{y}) \leq -1$. Since $v_0(0) = -1$ and (2), we obtain $v_0(y) = -1$ for any $y \in [0, \overline{y}]$. However, the constant is not the solution of (E_0). Thus we get (3).

We now prove our lemma. Suppose that $0 \in J$, that is, there exists $Y_0 \in (0, T_0)$ such that $v_0(y) < 0$ for $y \in [0, Y_0)$ and $v_0(Y_0) = 0$. Then we have $-1 < v_0(y) < 0$ for any $y \in (0, Y_0)$ by (3). Thus, we also obtain $w_0 \equiv \int_0^y v_0(s)ds < -y$ for $y \in (0, Y_0)$. These inequalities and (2) imply

$$v_0''(y) + yv_0'(y) = (-v_0(y)) \cdot (-v_0(y)) + (-w_0(y) - y) \cdot v_0'(y) < -v_0(y)$$

for $y \in (0, Y_0)$ using (E_0). So we get $\{e^{y^2/2}v_0'(y)\}' + e^{y^2/2}v_0(y) < 0$ for $y \in (0, Y_0)$. Multiplying $e^{-y^2/2}$ to this inequality and integrating on $[0, Y_0]$, we have

$$\int_0^{Y_0} e^{-y^2/2} \cdot \{e^{y^2/2}v_0'(y)\}'dy + \int_0^{Y_0} v_0(y)dy \; < \; 0. \tag{4}$$

By the integration by parts, the first term in the left–hand side of (4) equals

$$\begin{aligned} &v_0'(Y_0) + \int_0^{Y_0} yv_0'(y)dy \\ &\qquad = v_0'(Y_0) + Y_0 \cdot v_0(Y_0) - \int_0^{Y_0} v_0(y)dy = v_0'(Y_0) - \int_0^{Y_0} v_0(y)dy, \end{aligned}$$

since $v_0'(0) = v_0(Y_0) = 0$. Hence, (4) yields $v_0'(Y_0) < 0$, which contradicts (2). We have proves $0 \notin J$.

Lemma 3 *The set J is open in $[0, \infty)$.*

Proof. Let $\alpha \in J$. We note that $\alpha > 0$ by Lemma 2. From the definition of J, there exists unique Y_α such that $v_\alpha(Y_\alpha) = 0$. By (2) there exists some positive δ such taht $0 < v_\alpha(Y_\alpha + \delta) < \infty$. Thus, the continuity of $v_\alpha(y)$ with repect to y and α implies that there exists positive ε such that

$$v_\beta(Y_\alpha + \delta) \; > \; 0 \text{ for any } \beta \in (\alpha - \varepsilon, \alpha + \varepsilon).$$

Therefore, we have $v_\beta(Y_\beta) = 0$ for some $Y_\beta > 0$, as $v_\beta(0) = -1$ and (2) hold. This means that $\beta \in J$. This proves our lemma.

Lemma 4 *If $\alpha > 2/\pi$, then $\alpha \in J$. That is, the set J is not empty.*

Proof. Suppose that there exists $\beta > 2/\sqrt{\pi}$ with $\beta \notin J$. By (2) we have $v_\beta(y) < 0$ for any $y \geq 0$ and $w_\beta(y) = \int_0^y v_\beta(s)ds < 0$. Thus, the equation ($\mathrm{E}_\beta$) and (2) imply

$$v''_\beta(y) + 2yv'_\beta(y) \;=\; -w_\beta(y) \cdot v'_\beta(y) + v_\beta(y)^2 \;>\; 0.$$

So we obtain

$$\left(e^{y^2} v'_\beta(y)\right)' \;=\; e^{y^2} \cdot \left(v''_\beta(y) + 2yv'_\beta(y)\right) \;>\; 0.$$

Integrating this inequality twice on $[0, y]$, we have

$$v_\beta(y) \;>\; v_\beta(0) + v'_\beta(0) \cdot \int_0^y e^{-s^2} ds \;=\; -1 + \beta \cdot \int_0^y e^{-s^2} ds.$$

Letting $y \to \infty$ at the last inequality, then

$$\lim_{y\to\infty} v_\beta(y) \;\geq\; -1 + \beta \cdot \frac{\sqrt{\pi}}{2}$$

holds. From our assumption $\beta \notin J$ it follows that $\lim_{y\to\infty} v_\beta(y) \leq 0$. Thus we have $\beta \leq 2/\sqrt{\pi}$. This is contradiction. Thus we get our lemma.

The next lemma proves Theorem 1.

Lemma 5 *Put $\alpha^* = \inf J$. Then the solution v_{α^*} exists globaly and satisfies $\lim_{y\to\infty} v_{\alpha^*}(y) = 0$.*

Proof. First, we note that $\alpha^* \notin J$ by Lemma 4. Thus, since the solution of (E_{α^*}) satisfies $v_{\alpha^*}(y) < 0$ for $y \in [0, T_{\alpha^*})$ by (2), then $T_{\alpha^*} = \infty$.

Let a sequence $\alpha_j \in J$ satisfy $\alpha_j \to \alpha^*$ as $j \to \infty$. From the definition of J, there exists $Y_{\alpha_j} > 0$ such that $v_{\alpha_j}(Y_{\alpha_j}) = 0$ for any j. Then we have

$$Y_{\alpha_j} \to \infty \text{ as } j \to \infty. \tag{5}$$

Indeed, suppose that the sequence $\{Y_{\alpha_j}\}$ is bounded. Then, for some subsequence $\{\alpha_{j(k)}\}$ and $Y_* \in [0, \infty)$ we have $Y_{\alpha_{j(k)}} \to Y_*$ as $k \to \infty$. Thus we get

$$0 = v_{\alpha_{j(k)}}(Y_{\alpha_{j(k)}}) \to v_{\alpha^*}(Y_{\alpha^*}) \text{ as } k \to \infty.$$

This means $v_{\alpha^*}(Y_{\alpha^*}) = 0$, that is $\alpha^* = \inf J \in J$. However, this contradicts Lemma 4. Hence we have (5).

Finally, we shall show $\lim_{y\to\infty} v_{\alpha^*}(y) = 0$, using (5). If there exists some $\varepsilon > 0$ such that $\lim_{y\to\infty} v_{\alpha^*}(y) = -\varepsilon < 0$. Then by (5) we obtain

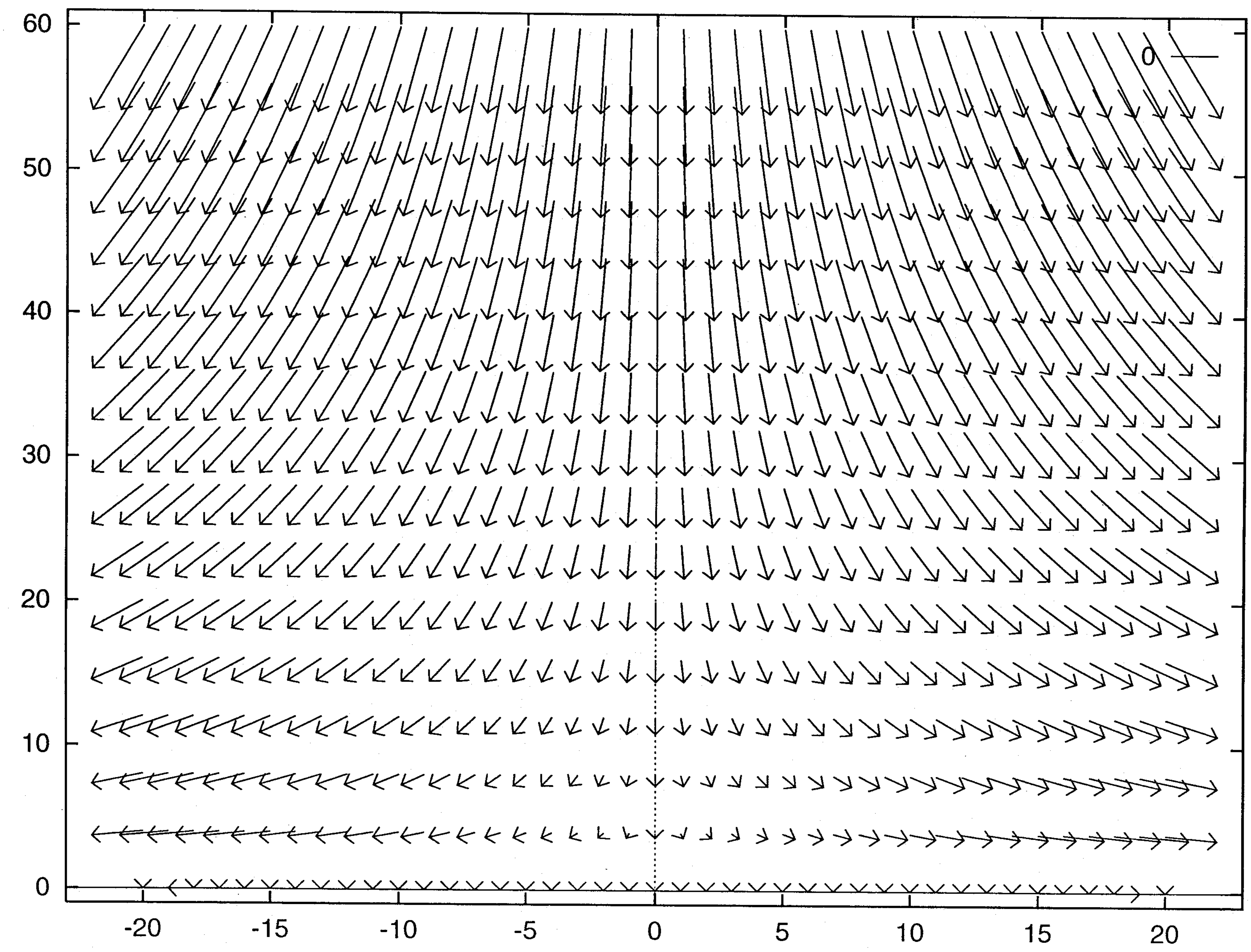
$$v_{\alpha_j}(Y_{\alpha_j}) < -\frac{\varepsilon}{2} \text{ for any } j \geq N$$

for some natural number N. On the other hand, by the definition of Y_{α_j} we have $v_{\alpha_j}(Y_{\alpha_j}) = 0$. This is a contradiction. The proof is now complete.

The following figure is the result of our numerical experiment for the following equation

$$\begin{cases} -2f'(y) + (f'(y))^2 - yf''(y) - f(y)f''(y) - f'''(y) = -1 \text{ for } y > 0, \\ f(0) = f'(0) = 0, \ f''(0) = 0.533, \end{cases}$$

which is equivalent to (E_α) with $\alpha = 0.533$, using the Runge–Kutta methods. In this figure each arrows indicate the vector fields $(xf'(y), -f(y))$ at each point. Here, the horizontal line is the x axis and the vertical line is the y axis.



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