Title	The rectifying developable and the spherical Darboux image of a space curve
Author(s)	Izumiya, S.; Katsumi, H.; Yamasaki, T.
Citation	Hokkaido University Preprint Series in Mathematics, 419, [1]-16
Issue Date	1998-8-1
DOI	10.14943/83565
Doc URL	http://hdl.handle.net/2115/69169
Type	bulletin (article)
File Information	pre419.pdf

THE RECTIFYING DEVELOPABLE AND THE SPHERICAL DARBOUX IMAGE OF A SPACE CURVE

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Series #419. August 1998

HOKKAIDO UNIVERSITY
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THE RECTIFYING DEVELOPABLE AND THE SPHERICAL DARBOUX IMAGE OF A SPACE CURVE

DEDICATED TO THE MEMORY OF PROFESSOR YOSUKE OGAWA

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Abstract. In this paper we study singularities of certain surfaces and curves associated with the family of rectifying planes along space curves. We establish the relationships between singularities of these subjects and geometric invariants of curves which are deeply related to the order of contact with helices.

1. Introduction. There are several articles concerning singularities of the tangent developable (i.e., the envelope of osculating planes) and the focal developable (i.e., the envelope of normal planes) of a space curve([2,4–12]). In these paper the relationships between singularities of these surfaces and classical geometric invariants of space curves have been studied. The notion of the distance-squared functions on space curves is useful for the study of singularities of focal developable [7,10,11]. For tangent developable, there are other techniques to study singularities[2,4–6,8,9]. The classical invariants of extrinsic differential geometry can be interpreted as "singularities" of these developable; however, the authors cannot find any articles concerning on singularities of *the rectifying developable* (i.e., the envelope of rectifying planes) of a space curve. The rectifying developable is an important surface in the following sense: the space curve γ is always a geodesic of the rectifying developable of itself (cf., [3] Page 308).

In this paper we introduce the notion of *volumelike distance functions* (*or binormal*

1991 *Mathematics Subject Classification*: Primary 58C27; Secondary 53A04.
The paper is in final form and no version of it will be published elsewhere.

directed distance functions) on space curves. This function is quite useful for the study of generic singularities of rectifying developable of space curves. We also introduce the notion of *volumelike height functions* (*or tangential height functions*) on space curves, which induce the notion of *rectifying Gaussian surfaces* and *spherical Darboux images* and these singularities are deeply related to the geometry of spherical tangential images of curves.

As a consequence, we apply ordinary techniques of singularity theory for these functions and describe the relationships between the singularities of the above three subjects and differential geometric invariants of space curves.

The main results in this paper are Theorems 2.1 and 2.2. We describe the geometric meaning of Theorem 2.2 in §4. The proof of Theorem 2.2 is given in §5. In §6 we consider generic properties of space curves. Since the calculations for the proof of Theorem 2.1 are terribly long and tedious, so that we only give a sketch of the proof in this paper. The basic techniques we used in this paper depend heavily on those in the book of Bruce and Giblin [1], so the authors are grateful to both of them. The authors also wishes to thank Professors K. Tsukada and T. Fukui for valuable suggestions.

All curves and maps considered here are of class C^∞ unless otherwise stated.

2. Basic notions and the main results. We now review some basic concepts on classical differential geometry of space curves in Euclidean space $\mathbb{R}^3$. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, we denote $< \boldsymbol{x}, \boldsymbol{y} >$ as the standard inner product. Let $\gamma : I \to \mathbb{R}^3$ be a curve with $\dot{\gamma}(t) \neq 0$, where $\dot{\gamma}(t) = d\gamma/dt(t)$. We also denote the norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\|$. The *arc-length* of a curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt.$$

Then a parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. So we say that a curve γ is *parameterized by the arc-length* if it satisfies $\|\gamma'(s)\| = 1$. Let us denote $\boldsymbol{t}(s) = \gamma'(s)$ and we call $\boldsymbol{t}(s)$ a *unit tangent vector* of γ at s. We define the *curvature* of γ by $\kappa(s) = \sqrt{\|\gamma''(s)\|}$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $\boldsymbol{n}(s)$ of the curve γ at s is given by $\gamma''(s) = \kappa(s)\boldsymbol{n}(s)$. The unit vector $\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)$ is called a *unit binormal vector* of the curve γ at s. Then the following Frenet-Serret formula holds:

$$\begin{cases} \boldsymbol{t}'(s) &= \kappa(s)\boldsymbol{n}(s) \\ \boldsymbol{n}'(s) &= -\kappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s) \\ \boldsymbol{b}'(s) &= -\tau(s)\boldsymbol{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve γ at s. It is well-known that the point at where the torsion $\tau(s)$ vanishes is the point at where the curve has at least third order contact with the osculating plane. It corresponds to the degenerated singularities of the tangent developable of γ(cf., [2]). So we do not consider such a point for the study of singularities of rectifying developable of γ. We assume that $\tau(s) \neq 0$ throughout in this paper. For any unit speed curve $\gamma : I \longrightarrow \mathbb{R}^3$, we call $D(s) = \tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)$ a *Darboux vector* of γ (cf., [7] Section 5.2). By using the Darboux vector, the Frenet-Serret formula is rewritten as follows:

$$\begin{cases} \boldsymbol{t}'(s) &= D(s) \times \boldsymbol{t}(s) \\ \boldsymbol{n}'(s) &= D(s) \times \boldsymbol{n}(s) \\ \boldsymbol{b}'(s) &= D(s) \times \boldsymbol{b}(s) \end{cases}$$

Thus the Darboux vector plays an important role for the study of space curves. We define a vector $\tilde{D}(s) = (\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)$ and we call it *a modified Darboux vector* along γ. We also define a spherical curve $\boldsymbol{d} : I \longrightarrow S^2$ by $\boldsymbol{d}(s) = \frac{D(s)}{\|D(s)\|}$ and surfaces

$$RG(\gamma) = \{u\boldsymbol{t}(s) + \boldsymbol{b}(s) \mid u \in \mathbb{R}, s \in I\},$$

$$RD(\gamma) = \{\gamma(s) + u\tilde{D}(s) \mid u \in \mathbb{R}\}.$$

We call the image of $\boldsymbol{d}$ *the spherical Darboux image,* the surface $RG(\gamma)$ *the rectifying Gaussian surface* and the surface $RD(\gamma)$ *the rectifying developable of* γ. Our main result is the following:

THEOREM 2.1. *Let $Imm_\tau(S^1, \mathbb{R}^3)$ be the space of regular curves $\gamma : S^1 \longrightarrow \mathbb{R}^3$ with $\tau \neq 0$ and $\kappa \neq 0$ equipped with C^∞-topology. Then there exists a residual set $\mathcal{O} \subset Imm_\tau(S^1, \mathbb{R}^3)$ such that for any $\gamma \in \mathcal{O}$ the following properties hold:*
(1) *The number of the points $s \in S^1$ where $(\tau/\kappa)'(s) = 0$ is finite.*
(2) *There is no point $s \in S^1$ where $(\tau/\kappa)'(s) = (\tau/\kappa)''(s) = 0$.*
(3) *The number of the points $s \in S^1$ where $(\tau/\kappa)''(s) = 0$ is finite.*
(4) *There is no point $s \in S^1$ where $(\tau/\kappa)''(s) = (\tau/\kappa)'''(s) = 0$.*

THEOREM 2.2. *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a unit speed curve. Then we have the following:*
(1) *The spherical Darboux image is locally diffeomorphic to the ordinary cusp C at $\boldsymbol{d}(s_0)$ if and only if $(\tau/\kappa)'(s_0) = 0$ and $(\tau/\kappa)''(s_0) \neq 0$.*
(2) (a) *The rectifying Gaussian surface is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at $u_0\boldsymbol{t}(s_0) + \boldsymbol{b}(s_0)$ if and only if $u_0 = (\tau/\kappa)(s_0)$ and $(\tau/\kappa)'(s_0) \neq 0$.*
(b) *The rectifying Gaussian surface is locally diffeomorphic to the swallow tail SW at $u_0\boldsymbol{t}(s_0) + \boldsymbol{b}(s_0)$ if and only if $u_0 = (\tau/\kappa)(s_0)$, $(\tau/\kappa)'(s_0) = 0$ and $(\tau/\kappa)''(s_0) \neq 0$.*
(3) (a) *The rectifying developable is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at $\gamma(s_0) + u_0\tilde{D}(s_0)$ if and only if $(\tau/\kappa)'(s_0) \neq 0$, $(\tau/\kappa)''(s_0) \neq 0$ and $u_0 = \frac{1}{(\tau/\kappa)'(s_0)}$.*
(b) *The rectifying developable is locally diffeomorphic to the swallow tail SW at $\gamma(s_0) + u_0\tilde{D}(s_0)$ if and only if $(\tau/\kappa)'(s_0) \neq 0$, $(\tau/\kappa)''(s_0) = 0$, $(\tau/\kappa)'''(s_0) \neq 0$ and $u_0 = \frac{1}{(\tau/\kappa)'(s_0)}$.*
Here, $C = \{(x_1, x_2) | {x_1}^2 = {x_2}^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallow tail.

The geometric meanings of the singularities of Image $\boldsymbol{d}$, $RG(\gamma)$ and $RD(\gamma)$ will be discussed in §4.

3. Families of smooth functions on a space curve. In this section we introduce two different families of functions on a space curve which are useful for the study of singularities of $\boldsymbol{d}$, $RG(\gamma)$ and $RD(\gamma)$. Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$.

Volumelike distance functions. We now define a three-parameter family of smooth functions on I:

$$F : I \times \mathbb{R}^3 \to \mathbb{R}$$

by $F(s, u) = |\boldsymbol{t}(s)\ \boldsymbol{n}(s)\ \gamma(s) - u| = < \gamma(s) - u, \boldsymbol{b}(s) >$. Here, $|\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}|$ denotes the determinant of the matrix $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})$ and $< \boldsymbol{a}, \boldsymbol{b} >$ denotes the standard inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$. We call F *a volumelike distance function* (*or a binormal directed distance function*) on γ. We denote that $f_u(s) = F(s, u)$ for any $u \in \mathbb{R}^3$.

PROPOSITION 3.1. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then*
(1) $f_y'(s) = 0$ *if and only if* $\gamma(s) - u \in < \boldsymbol{t}(s), \boldsymbol{b}(s) >$.
(2) $f_y'(s) = f_u''(s) = 0$ *if and only if* $\gamma(s) - u = \mu((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s))$.
(3) $f_u'(s) = f_u''(s) = f_u'''(s) = 0$ *if and only if* $\gamma(s) - u = \frac{1}{(\tau/\kappa)'(s)}((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s))$ *and* $(\tau/\kappa)'(s) \neq 0$.
(4) $f_u'(s) = f_u''(s) = f_u'''(s) = f_u^{(4)}(s) = 0$ *if and only if*

$$\gamma(s) - u = \frac{1}{(\tau/\kappa)'(s)}((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)),$$

$(\tau/\kappa)'(s) \neq 0$ *and* $(\tau/\kappa)''(s) = 0$.
(5) $f_u'(s) = f_u''(s) = f_u'''(s) = f_u^{(4)}(s) = f_u^{(5)}(s) = 0$ *if and only if*

$$\gamma(s) - u = \frac{1}{(\tau/\kappa)'(s)}((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)),$$

$(\tau/\kappa)'(s) \neq 0$ *and* $(\tau/\kappa)''(s) = 0$ *and* $(\tau/\kappa)'''(s) = 0$.

Proof. By the Frenet-Serret formula, we have the following calculation:

(a) $$f_u'(s) = \tau(s)|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u|.$$

(b) $$\begin{aligned} f_u''(s) = &-\tau^2(s)|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,\gamma(s) - u| + \kappa(s)\tau(s)\,|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| \\ &+\tau'(s)|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u|. \end{aligned}$$

(c) $$\begin{aligned} f_u'''(s) = &-3\tau(s)\tau'(s)|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,\gamma(s) - u| \\ &+(2\kappa(s)\tau'(s) + \kappa'(s)\tau(s))|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| \\ &+(\tau''(s) - \kappa^2(s)\tau(s) - \tau^3(s))|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| + \kappa(s)\tau(s). \end{aligned}$$

(d) $$\begin{aligned} f_u^{(4)}(s) = &(-4\tau(s)\tau''(s) - 3(\tau'(s))^2 + \kappa^2(s)\tau^2(s) + \tau^4(s))|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,\gamma(s) - u| \\ &+(3\kappa(s)\tau''(s) + 3\kappa'(s)\tau'(s) \\ &+\kappa''(s)\tau(s) - \kappa(s)\tau^3(s) - \kappa^3(s)\tau(s))|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| \\ &+(\tau'''(s) - 3\kappa(s)\kappa'(s)\tau(s) - 6\tau^2(s)\tau'(s) \\ &-3\kappa^2(s)\tau'(s))|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| \\ &+(2\kappa'(s)\tau(s) + 3\kappa(s)\tau'(s)). \end{aligned}$$

(e) $$\begin{aligned} f_u^{(5)}(s) = &(-5\tau(s)\tau'''(s) - 10\tau'(s)\tau''(s) + 5\kappa^2(s)\tau(s)\tau'(s). \\ &+10\tau^3(s)\tau'(s) + 5\kappa(s)\kappa'(s)\tau^2(s))|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,\gamma(s) - u| \\ &+(4\kappa(s)\tau'''(s) + 6\kappa'(s)\tau''(s) + 4\kappa''(s)\tau'(s) - 9\kappa(s)\tau^2(s)\tau'(s) \\ &-4\kappa^3(s)\tau'(s) + \kappa'''(s)\tau(s) - \kappa'(s)\tau^3(s) \\ &-6\kappa^2(s)\kappa'(s)\tau(s))|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| \\ &+(\tau^{(4)}(s) - 6\kappa^2(s)\tau''(s) - 10\tau^2(s)\tau''(s) \\ &-12\kappa(s)\kappa'(s)\tau'(s) - 15\tau(s)(\tau'(s))^2 \\ &+2\kappa^2(s)\tau^3(s) + \tau^5(s) - 4\kappa(s)\kappa''(s)\tau(s) \\ &+\kappa^4(s)\tau(s) - 3(\kappa'(s))^2\tau(s))|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,\gamma(s) - u| \\ &+(6\kappa(s)\tau''(s) + 8\kappa'(s)\tau'(s) + 3\kappa''(s)\tau(s) - \kappa(s)\tau^3(s) - \kappa^3(s)\tau(s)). \end{aligned}$$

We also calculate as follows:

$$(\tau/\kappa)'(s) = (1/\kappa^2(s))(\tau'(s)\kappa(s) - \tau(s)\kappa'(s)),$$

$$(\tau/\kappa)''(s) = (1/\kappa^3(s))\{(\tau''(s)\kappa(s) - \tau(s)\kappa''(s))\kappa(s) - 2\kappa'(s)(\tau'(s)\kappa(s) - \tau(s)\kappa'(s))\},$$

$$\begin{aligned}(\tau/\kappa)'''(s) &= (1/\kappa^4(s))\{\kappa^3(s)\tau'''(s) + 6\kappa(s)\kappa'(s)\kappa''(s)\tau(s) - \kappa^2(s)\kappa'''(s)\tau(s)\\ &\quad -3\kappa^2(s)\kappa''(s)\tau'(s) - 3\kappa^2(s)\kappa'(s)\tau''(s)\\ &\quad +6\kappa(s)\kappa'^2(s)\tau'(s) - 6\kappa'^3(s)\tau(s)\}.\end{aligned}$$

Since $\tau(s) \neq 0$, the assertion (1) follows from the above formula (a). By (1) and (b), we have $\gamma(s_0) - u = \nu \boldsymbol{t}(s_0) + \mu \boldsymbol{b}(s_0)$ and

$$-\tau^2(s)\,|\boldsymbol{t}(s)\;\boldsymbol{n}(s)\;\mu\,\boldsymbol{b}(s)| + \kappa(s)\tau(s)\,|\boldsymbol{n}(s)\;\boldsymbol{b}(s)\;\nu\,\boldsymbol{t}(s)| = -\tau^2(s)\,\mu + \kappa(s)\tau(s)\,\nu = 0.$$

Thus we have $\nu = (\tau/\kappa)(s)\mu$.

By (2) and (c), we have $\gamma(s) - u = \mu((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)$ and

$$\begin{aligned}&-\tau(s)\tau'(s)\,|\boldsymbol{t}(s)\;\boldsymbol{n}(s)\;\mu\,\boldsymbol{b}(s)| + \kappa'(s)\tau(s)\,|\boldsymbol{n}(s)\;\boldsymbol{b}(s)\;(\tau/\kappa)(s)\boldsymbol{t}(s)| + \kappa(s)\tau(s)\\ &= \left(-\tau(s)\tau'(s) + (\kappa'(s)\tau^2(s)/\kappa(s))\right)\mu + \kappa(s)\tau(s) = 0.\end{aligned}$$

It follows that $\frac{\kappa(s)\tau'(s) - \kappa'(s)\tau(s)}{\kappa(s)}\,\mu = \kappa(s)$. Hence, we have $(\tau/\kappa)'(s)\cdot\mu = 1$.

By (3) and (d), we have $(\tau/\kappa)'(s) \neq 0$ and $\gamma(s) - u = \frac{1}{(\tau/\kappa)'(s)}((\tau(s)/\kappa(s))\boldsymbol{t}(s) + \boldsymbol{b}(s))$. Moreover, by the direct calculations as above, we can show the assertion (4). By the similar but rather long calculation, we can also show the assertion (5). ∎

Volumelike height functions. We define a two-parameter family of smooth functions on I:

$$G : I \times S^2 \to I\!R$$

by $G(s,v) = |\boldsymbol{n}(s)\;\boldsymbol{b}(s)\;v| =< \boldsymbol{t}(s), v >$ We call G *a volumelike height function (or a tangential height function)* on γ. We denote that $g_v(s) = G(s,v)$.

PROPOSITION 3.2. *Let $\gamma : I \longrightarrow I\!R^3$ be a unit speed curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then*

(1) $g_v'(s) = 0$ *if and only if there exist real numbers $\lambda, \mu \in I\!R$, such that $v = \lambda\boldsymbol{t}(s) + \mu\boldsymbol{b}(s)$, $\lambda^2 + \mu^2 = 1$.*

(2) $g_v'(s) = g_v''(s) = 0$ *if and only if* $v = \pm\dfrac{\kappa(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}}\left((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)\right)$.

(3) $g_v'(s) = g_v''(s) = g_v'''(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}}\left((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)\right), (\tau/\kappa)'(s) = 0.$$

(4) $g_v'(s) = g_v''(s) = g_v'''(s) = g_v^{(4)}(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}}\left((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)\right), (\tau/\kappa)'(s) = (\tau/\kappa)''(s) = 0.$$

(5) $g_v'(s) = g_v''(s) = g_v'''(s) = g_v^{(4)}(s) = g_v^{(5)}(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}}\left((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)\right)$$

and $(\tau/\kappa)'(s) = (\tau/\kappa)''(s) = (\tau/\kappa)'''(s) = 0.$

Proof. By the Frenet-Serret formula, we have the following calculation:

(a) $g_v'(s) = -\kappa(s)|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,v|.$

(b) $g_v''(s) = -\kappa'(s)|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,v| - \kappa^2(s)|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,v| + \kappa(s)\tau(s)|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,v|.$

(c) $g_v'''(s) = (-\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s))|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,v| - 3\kappa(s)\kappa'(s)|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,v|$
$+(2\kappa'(s)\tau(s) + \kappa(s)\tau'(s))|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,v|.$

(d) $g_v^{(4)}(s) = (-\kappa'''(s) + 6\kappa^2(s)\kappa'(s) + 3\kappa'(s)\tau^2(s) + 3\kappa(s)\tau(s)\tau'(s))|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,v|$
$+(-4\kappa(s)\kappa''(s) + \kappa^4(s) + \kappa^2(s)\tau^2(s) - 3\kappa'^2(s))|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,v|$
$+(3\kappa''(s)\tau(s) - \kappa^3(s)\tau(s) - \kappa(s)\tau^3(s) + 3\kappa'(s)\tau'(s) + \kappa(s)\tau''(s))|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,v|.$

(e) $g_v^{(5)}(s) = (-\kappa^{(4)}(s) + 15\kappa(s)\kappa'^2(s) + 10\kappa^2(s)\kappa''(s) + 6\kappa''(s)\tau^2(s)$
$+12\kappa'(s)\tau(s)\tau'(s) + 3\kappa(s)\tau'^2(s) + 4\kappa(s)\tau(s)\tau''(s) - \kappa^5(s) - 2\kappa^3(s)\tau^2(s)$
$-\kappa(s)\tau^4(s))|\boldsymbol{t}(s)\,\boldsymbol{b}(s)\,v| + (-5\kappa(s)\kappa'''(s) + 10\kappa^3(s)\kappa'(s) + 5\kappa(s)\kappa'(s)\tau^2(s)$
$+5\kappa^2(s)\tau(s)\tau'(s) - 10\kappa'(s)\kappa''(s))|\boldsymbol{n}(s)\,\boldsymbol{b}(s)\,v|$
$+(4\kappa'''(s)\tau(s) - 9\kappa^2(s)\kappa'(s)\tau(s) - 4\kappa'(s)\tau^3(s) - 6\kappa(s)\tau^2(s)\tau'(s)$
$+6\kappa''(s)\tau'(s) - \kappa^3(s)\tau'(s) + 4\kappa'(s)\tau''(s) + \kappa(s)\tau'''(s))|\boldsymbol{t}(s)\,\boldsymbol{n}(s)\,v|.$

The assertion (1) is trivial by the formula (a) in the above calculation.
(2) By (1), we have $v = \lambda\boldsymbol{t}(s) + \mu\boldsymbol{b}(s)$. It follows from (b) that $g_v''(s) = \kappa(s)(\kappa(s)\lambda - \tau(s)\mu)$. Since $\kappa(s) \neq 0$, $g_v''(s) = 0$ if and only if $\lambda = \frac{\tau(s)}{\kappa(s)}\mu$ Therefore we have $v = (\tau/\kappa)(s)\mu\boldsymbol{t}(s) + \mu\boldsymbol{b}(s) = \mu\left((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)\right)$. By the assumption that $v \in S^2$ and $\kappa(s) \geq 0$, we have $\mu = \pm\frac{\kappa(s)}{\sqrt{\tau^2(s)+\kappa^2(s)}}$.
(3) We now substitute the formula $v = \pm\dfrac{\kappa(s)}{\sqrt{\tau^2(s)+\kappa^2(s)}}((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s))$ into (c), then we have

$$(-\kappa'(s)\tau(s) + \kappa(s)\tau'(s))\left(\pm\frac{\kappa(s)}{\sqrt{\tau^2(s)+\kappa^2(s)}}\right) = 0.$$

Since $\pm\dfrac{\kappa(s)}{\sqrt{\tau^2(s)+\kappa^2(s)}} \neq 0$ and $(\tau/\kappa)'(s) = (1/\kappa^2(s))(\tau'(s)\kappa(s) - \tau(s)\kappa'(s))$, the assertion (3) follows.
(4) We also substitute the formula (3) into (d), then we have

$$(-\kappa''(s)\tau(s) + \kappa(s)\tau''(s))\left(\pm\frac{\kappa(s)}{\sqrt{\tau^2(s)+\kappa^2(s)}}\right) = 0.$$

If $(\tau/\kappa)'(s) = 0$, then we can show that $(\tau/\kappa)''(s) = (1/\kappa^2(s))(\tau''(s)\kappa(s) - \tau(s)\kappa''(s))$. Since $\pm\dfrac{\kappa(s)}{\sqrt{\tau^2(s)+\kappa^2(s)}} \neq 0$, we have the assertion (4).
(5) By the similar calculation, we have the assertion (5). ∎

4. Helices and the spherical tangential image of a curve. In this section we study the geometric properties of the rectifying developable, the spherical Darboux image and the rectifying Gaussian surface of space curves. By the propositions in the last section, we can recognize that the function $(\tau/\kappa)'(s)$ and the modified Darboux vector $\tilde{D}(s) = (\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)$ are the important subjects. If $(\tau/\kappa)'(s) \equiv 0$ then the curve $\gamma(s)$ has been classically known as *a helix*. For a unit speed regular curve $\gamma : I \longrightarrow \mathbb{R}^3$, the unit tangent curve $\boldsymbol{t} : I \longrightarrow S^2$ is called *the spherical tangential image* of γ. We can easily calculate that the geodesic curvature of the spherical tangential image $\boldsymbol{t}$ is equal to the function τ/κ, which has been called *the conical curvature* of γ (cf., [7] Section 5.2). Moreover, we have the following proposition.

PROPOSITION 4.1. *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a regular curve. Then γ is a helix if and only if the modified Darboux vector $\tilde{D}(s)$ is a constant vector. In this case we have the following assertions:*

(1) *The spherical tangential image $\boldsymbol{t}(s)$ of γ is a circle on the unit sphere S^2 and the direction of the center of the circle is given by the constant vector $\boldsymbol{d}(s) \equiv \boldsymbol{e}$.*

(2) *The rectifying developable of γ is a cylindrical surface given by $\gamma(s) + u\boldsymbol{e}$.*

Proof. By the Frenet-Serret formula, we can show that $\tilde{D}'(s) = (\tau/\kappa)'(s)\boldsymbol{t}(s)$. Therefore, γ is a helix if and only if $\tilde{D}'(s) \equiv 0$. This condition is equivalent to the condition that $\tilde{D}(s)$ is a constant vector. In this case we have $<\boldsymbol{e}, \boldsymbol{t}(s)> = \cos\theta(s)$, where $\theta(s)$ is the angle between two vectors $\boldsymbol{t}(s), \boldsymbol{e}$. Since

$$\boldsymbol{e} = \pm\frac{\kappa(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}}\left((\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)\right),$$

we also have $<\boldsymbol{e}, \boldsymbol{t}(s)> = \frac{(\tau/\kappa)(s)}{\sqrt{(\tau/\kappa)^2(s)+1}}$, so $\cos\theta(s)$ is constant. This means that the spherical tangential image $\boldsymbol{t}(s)$ is a circle on the unit sphere S^2 and the center is directed by $\boldsymbol{e}$. The assertion (2) is clear by definition. ∎

Let $\gamma_i : I_i \longrightarrow \mathbb{R}^3$ $(i = 1, 2)$ be regular curves. We say that $\gamma_1(s_0)$ and $\gamma_2(t_0)$ *have at least $(k+1)$-points contact* if $\gamma_1^{(p)}(s_0) = \gamma_2^{(p)}(t_0)$ for $0 \le p \le k$. We also say that $\gamma_1(s_0)$ and $\gamma_2(t_0)$ *have $(k+1)$-points contact* if these have at least $(k+1)$-points contact and satisfy the relation $\gamma_1^{(k+1)}(s_0) \neq \gamma_2^{(k+1)}(t_0)$. By definition of the curvature and the torsion, we can show the following proposition.

PROPOSITION 4.2. *If $\gamma_1(s_0)$ and $\gamma_2(t_0)$ have $(k+1)$-points contact, then*

$(\tau/\kappa)_1^{(p)}(s_0) = (\tau/\kappa)_2^{(p)}(t_0)$ (for $0 \le p \le k-3$) and $(\tau/\kappa)_1^{(k-2)}(s_0) \neq (\tau/\kappa)_2^{(k-2)}(t_0)$.

We also have the following proposition.

PROPOSITION 4.3. *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a regular curve with $\kappa(s_0) \neq 0$ and $\tau(s_0) \neq 0$. Then there exists an open interval $s_0 \in J \subset I$ and a unique helix $\delta : J \longrightarrow \mathbb{R}^3$ such that $\delta(s_0) = \gamma(s_0)$, the curvature of $\delta(s)$ is $\kappa(s)$, the torsion of δ at s_0 is $\tau(s_0)$ and γ and δ have at least 4-point contact at s_0.*

The proof of Proposition 4.3 is given by solving the natural equation $\kappa_\delta(s) = \kappa(s)$, $\tau_\delta(s) = (\tau/\kappa)(s_0)\tau(s)$ under the initial condition $\delta(s_0) = \gamma(s_0), \delta'(s_0) = \gamma'(s_0)$ and $\delta'''(s_0) = \gamma'''(s_0)$. So we omit the detail. We call the helix δ in Proposition 4.3 *the osculating helix of* γ at s_0. We denote it by $\delta[s_0](s)$. By Proposition 4.1, the spherical tangential image $\boldsymbol{t}_{\delta[s_0]}(s)$ of the helix $\delta[s_0]$ is a circle whose center is directed by the spherical Darboux image $\boldsymbol{d}(s_0)$ of γ at s_0. We call $\boldsymbol{d}(s_0)$ *the center of conical curvature* of γ. We also call the locus of the center of conical curvature of γ *the spherical (or, conical) evolute of* γ. Under the above terminology, the spherical Darboux image is the spherical evolute of γ. Therefore the singularities of the spherical Darboux image (or, the rectifying Gaussian surface) describe how the shape of the curve γ is similar to a helix. On the other hand, the singularities of the rectifying developable describe how the shape of the curve γ is different from a helix.

5. Unfoldings of functions by one-variable. In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [1]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, x_0)$. We say that f has *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$ for $i = 1, \ldots, r$. Then F is called a *(p) versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ ($k-1 \leq r$). Under the same condition as the above, F is called a *versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k ($k \leq r$), where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *bifurcation set* $\mathcal{B}_F$ of F is the set

$$\mathcal{B}_F = \{x \in \mathbb{R}^r | \text{there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, x)\}$$

The *discriminant set* of F is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r | \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

Then we have the following well-known result (cf., [1]).

THEOREM 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0.*

(1) *Suppose that F is an (p) versal unfolding.*

(a) *If $k = 2$, then (s_0, x_0) is the fold point of $\pi | S_F$ and $\mathcal{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*

(b) *If $k = 3$, then $\mathcal{B}_F$ is diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(2) *Suppose that F is an versal unfolding.*

(a) *If $k = 1$, then $\mathcal{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*

(b) *If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(c) *If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

Here, $C = \{(x_1, x_2) | {x_1}^2 = {x_2}^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallow tail.

For the proof of Theorem 2.2, we have the following key propositions.

PROPOSITION 5.2. *Let $G : I \times S^2 \longrightarrow \mathbb{R}$ be the volumelike heightfunction on a unit speed curve $\gamma(s)$. If g_{v_0} has A_k-singularity ($k = 2, 3$) at s_0, then G is a (p)-versal unfolding of g_{v_0}.*

Proof. We denote by $\gamma(s)=(x_1(s),x_2(s),x_3(s))$ and $v=(y_1,y_2,\sqrt{1-y_1^2-y_2^2})$. By definition, we have

$$\begin{aligned} G(s,v) &= <\boldsymbol{t}(s),v> \\ &= y_1x_1'(s)+y_2x_2'(s)+\sqrt{1-y_1^2-y_2^2}x_3'(s). \end{aligned}$$

Let $j^{k-1}(\partial G/\partial y_i(s,y_0))(s_0)$ be the $(k-1)$-jet of $\partial G/\partial y_i$ at s_0; then we have

$$\begin{aligned} j^3\left(\frac{\partial G}{\partial y_i}(s,y_0)\right)(s_0) &= \left(x_i''(s_0)-\frac{y_{0,i}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0)\right)s \\ &\quad +\frac{1}{2}\left(x_i'''(s_0)-\frac{y_{0,i}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'''(s_0)\right)s^2 \\ &\quad +\frac{1}{6}\left(x_i^{(4)}(s_0)-\frac{y_{0,i}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3^{(4)}(s_0)\right)s^3. \end{aligned}$$

We distinguish two cases.

Case (1) When g_{v_0} has the A_2-singularity at s_0, we define the 1×2-matrix A as follows:

$$A=\begin{pmatrix} x_1''(s_0)-\frac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) & x_2''(s_0)-\frac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0)\end{pmatrix}$$

Since g_{v_0} has the A_2-singularity at s_0, we have

$$g_{v_0}'(s_0)=y_{0,1}x_1''(s_0)+y_{0,2}x_2''(s_0)+\sqrt{1-y_{0,1}^2-y_{0,2}^2}x_3''(s_0)=0.$$

By this relation, we have

$$\|A\|^2=x_1''^2(s_0)+x_2''^2(s_0)+\left(1+\frac{1}{1-y_{0,1}^2-y_{0,2}^2}\right)x_3''^2(s_0)\neq 0.$$

Therefore we have $\operatorname{rank} A=1$.

Case (2) When g_{v_0} has the A_3-singularity at s_0, we also require the 2×2-matrix

$$B=\begin{pmatrix} x_1''(s_0)-\frac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) & x_2''(s_0)-\frac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) \\ x_1'''(s_0)-\frac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'''(s_0) & x_2'''(s_0)-\frac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'''(s_0)\end{pmatrix}$$

to be nonsingular.

Since $\gamma''(s)=\kappa(s)\boldsymbol{n}(s)$, we have $\gamma'''(s)=-\kappa^2(s)\boldsymbol{t}(s)+\kappa'(s)\boldsymbol{n}(s)+\kappa(s)\tau(s)\boldsymbol{b}(s)$. By the assumption that g_{v_0}has the A_3-singularity at s_0, we have

$$v=\pm\frac{\kappa(s_0)}{\sqrt{\tau^2(s_0)+\kappa^2(s_0)}}\left((\tau/\kappa)(s_0)\boldsymbol{t}(s_0)+\boldsymbol{b}(s_0)\right).$$

By the direct calculation, we have

$$\begin{aligned}\det(B) &= \frac{1}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}} < \gamma''(s_0)\times\gamma'''(s_0), v > \\ &= \frac{1}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}\left(\pm\frac{\kappa(s_0)}{\sqrt{\tau^2(s_0)+\kappa^2(s_0)}}\right)\kappa(s_0)(\kappa^2(s_0)+\tau^2(s_0)) \neq 0.\end{aligned}$$

This means that $\operatorname{rank} B = 2$.

■

We define a function $\tilde{G} : I \times S^2 \times I\!R \longrightarrow I\!R$ by

$$\tilde{G}(s,v,w) = G(s,v) - w.$$

We also put $g_{v,w}(s) = \tilde{G}(s,v,w)$.

PROPOSITION 5.3. *If g_{v_0,w_0} has A_k-singularity ($k = 1,2,3$) at s_0, then G is a versal unfolding of g_{v_0,w_0}.*

Proof. Under the same notations as the above proposition, we have

$$\tilde{G}(s,v,y_3) = y_1x_1'(s) + y_2x_2'(s) + \left(\sqrt{1-y_1^2-y_2^2}\right)x_3'(s) - y_3$$

Let $j^{k-1}(\partial\tilde{G}/\partial y_i)(s,y_0)(s_0)$ be the $(k-1)$-jet of $\partial\tilde{G}/\partial y_i$ at s_0 ($i = 1,2,3$); then we have

$$\begin{aligned}\frac{\partial\tilde{G}}{\partial y_i}(s_0,y_0) + j^2\left(\frac{\partial\tilde{G}}{\partial y_i}(s,y_0)\right)(s_0) &= x_i'(s_0) - \frac{y_{0,i}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) \\ &+ \left(x_i''(s_0) - \frac{y_{0,i}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0)\right)s \\ &+ \frac{1}{2}\left(x_i'''(s_0) - \frac{y_{0,i}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'''(s_0)\right)s^2.\end{aligned}$$

We also distinguish three cases.

Case (1) When g_{v_0,w_0} has the A_1-singularity at s_0, the the rank of 1×3 matrix

$$\left(\begin{array}{ccc} x_1'(s_0) - \frac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) & x_2'(s_0) - \frac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) & -1 \end{array}\right)$$

is clearly 1.

Case (2) When g_{v_0,w_0} has the A_2-singularity at s_0, we require the 2×3-matrix

$$D = \left(\begin{array}{ccc} x_1'(s_0) - \frac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) & x_2'(s_0) - \frac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) & -1 \\ x_1''(s_0) - \frac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) & x_2''(s_0) - \frac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) & 0 \end{array}\right)$$

to have the maximal rank. By the case 1 in Proposition 5.2, the second line of D does not vanish, so that the rank of D is 2.

Case (3) When g_{v_0,w_0} has the A_3-singularity at s_0, we consider the 3×3 matrix

$$E=\begin{pmatrix} x_1'(s_0)-\dfrac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) & x_2'(s_0)-\dfrac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'(s_0) & -1 \\ x_1''(s_0)-\dfrac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) & x_2''(s_0)-\dfrac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3''(s_0) & 0 \\ x_1'''(s_0)-\dfrac{y_{0,1}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'''(s_0) & x_2'''(s_0)-\dfrac{y_{0,2}}{\sqrt{1-y_{0,1}^2-y_{0,2}^2}}x_3'''(s_0) & 0 \end{pmatrix}.$$

By the case 2 in Proposition 5.2, the rank of E is 3. ■

For the volumelike distance function F, we have the following proposition.

PROPOSITION 5.4. *If f_{u_0} has A_k-singularity $(k=2,3,4)$ at s_0, then F is a (p)-versal unfolding of f_{u_0}.*

Proof. Under the same notations as in the above proposition, we denote that $E_{ij}=x_i'(s)x_j''(s)-x_j'(s)x_i''(s)$. We also denote that $\boldsymbol{E}(s)=(E_{23},E_{31},E_{12})$, $\boldsymbol{F}(s)=\boldsymbol{E}'(s)$, $\tilde{\boldsymbol{G}}(s)=\boldsymbol{F}'(s)$ and $\boldsymbol{H}(s)=\tilde{\boldsymbol{G}}'(s)$. Then we have

$$\begin{aligned}
\boldsymbol{E}(s) &= \gamma'(s)\times\gamma''(s)\\
&= \kappa(s)\boldsymbol{b}(s).\\
\boldsymbol{F}(s) &= -\kappa(s)\tau(s)\boldsymbol{n}(s)+\kappa'(s)\boldsymbol{b}(s).\\
\tilde{\boldsymbol{G}}(s) &= \kappa^2(s)\tau(s)\boldsymbol{t}(s)+\left(-2\kappa'(s)\tau(s)-\kappa(s)\tau'(s)\right)\boldsymbol{n}(s)+\left(\kappa''(s)-\kappa(s)\tau^2(s)\right)\boldsymbol{b}(s).\\
\boldsymbol{H}(s) &= 2\kappa(s)\left(2\kappa'(s)\tau(s)+\kappa(s)\tau'(s)\right)\boldsymbol{t}(s)\\
&\quad+\left(-3\kappa''(s)\tau(s)-3\kappa'(s)\tau'(s)-\kappa(s)\tau''(s)+\kappa^3(s)\tau(s)+\kappa(s)\tau^3(s)\right)\boldsymbol{n}(s)\\
&\quad+\left(\kappa'''(s)-3\kappa'(s)\tau^2(s)-3\kappa(s)\tau(s)\tau'(s)\right)\boldsymbol{b}(s).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\kappa'(s)\boldsymbol{E}(s)-\kappa(s)\boldsymbol{F}(s)=\kappa^2(s)\tau(s)\boldsymbol{n}(s),\\
&\left(\kappa'(s)\boldsymbol{E}(s)-\kappa(s)\boldsymbol{F}(s)\right)'=\kappa'''(s)\left(\kappa(s)\boldsymbol{b}(s)\right)-\kappa(s)\boldsymbol{H}(s),\\
&\left(\kappa'(s)\boldsymbol{E}(s)-\kappa(s)\boldsymbol{F}(s)\right)''=\left(-5\kappa^2(s)\kappa'(s)\tau(s)-2\kappa^3(s)\tau'(s)\right)\boldsymbol{t}(s)\\
&\quad+\Big(2\kappa(s)\kappa''(s)\tau(s)+2\left\{\kappa'(s)\right\}^2\tau(s)+4\kappa(s)\kappa'(s)\tau'(s)\\
&\quad-\kappa^2(s)\tau''(s)+\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s)\Big)\boldsymbol{n}(s)\\
&\quad+\left(4\kappa(s)\kappa'(s)\tau^2(s)+3\kappa^2(s)\tau(s)\tau'(s)\right)\boldsymbol{b}(s).
\end{aligned}$$

By definition, we have

$$F(s,u)=(1/\kappa(s))\left\{(x_1(s)-u_1)E_{23}+(x_2(s)-u_2)E_{31}+(x_3(s)-u_3)E_{12}\right\}.$$

We denote that $\frac{\partial^r F}{\partial u^r}(s,u) = \left(\frac{\partial^r F}{\partial u_1^r}(s,u), \frac{\partial^r F}{\partial u_2^r}(s,u), \frac{\partial^r F}{\partial u_3^r}(s,u)\right)$, then we have

$$\frac{\partial F}{\partial u}(s,u) = -(1/\kappa(s))\boldsymbol{E}(s),$$

$$\begin{aligned}\frac{\partial^2 F}{\partial u^2}(s,u) &= \left(-(1/\kappa(s))\boldsymbol{E}(s)\right)' \\ &= (1/\kappa^2(s))\left(\kappa'(s)\boldsymbol{E}(s) - \kappa(s)\boldsymbol{F}(s)\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 F}{\partial u^3}(s,u) &= \left(1/\kappa^2(s)\right)'\left(\kappa'(s)\boldsymbol{E}(s) - \kappa(s)\boldsymbol{F}(s)\right) \\ &\quad +(1/\kappa^2(s))\left(\kappa''(s)\boldsymbol{E}(s) - \kappa(s)\tilde{\boldsymbol{G}}(s)\right).\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^4 F}{\partial u^4}(s,u) &= \left(1/\kappa^2(s)\right)''\left(\kappa'(s)\boldsymbol{E}(s) - \kappa(s)\boldsymbol{F}(s)\right) \\ &\quad +2\left(1/\kappa^2(s)\right)'\left(\kappa''(s)\boldsymbol{E}(s) - \kappa(s)\tilde{\boldsymbol{G}}(s)\right) \\ &\quad +(1/\kappa^2(s))\left(\kappa'''(s)\boldsymbol{E}(s) + \kappa''(s)\boldsymbol{F}(s) - \kappa'(s)\tilde{\boldsymbol{G}}(s) - \kappa(s)\boldsymbol{H}(s)\right).\end{aligned}$$

We now prove the (p)-versality of F. Let $j^k(\partial F/\partial u_i)(s_0)$ be the k-jet of $\partial F/\partial u_i$ at s_0 $(i = 1,2,3)$; then we have

$$j^3\left(\frac{\partial F}{\partial u_i}(s,\, u_0)\right)(s_0) = \left(\frac{\partial^2 F}{\partial u_i^2}(s_0,\, u_0)\right)s + \frac{1}{2}\left(\frac{\partial^3 F}{\partial u_i^3}(s_0,\, u_0)\right)s^2 + \frac{1}{6}\left(\frac{\partial^4 F}{\partial u_i^4}(s_0,\, u_0)\right)s^3.$$

We distinguish three cases.

Case (1). When f_{u_0} has the A_2-singularities at s_0, we have

$$\begin{aligned}\left(\frac{\partial^2 F}{\partial u_i^2}(s_0,\, u_0)\right) &= (1/\kappa^2(s_0))\left(\kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0)\right) \\ &= \tau(s_0)\boldsymbol{n}(s_0) \neq 0.\end{aligned}$$

This means that rank $\left(\frac{\partial^2 F}{\partial u_i^2}(s_0,\, u_0)\right) = 1$.

Case (2). When f_{u_0} has the A_3-singularities at s_0, we require the 2×3 matrix

$$Q = \begin{pmatrix} \kappa^2(s_0)\cdot\left(\frac{\partial^2 F}{\partial u_i^2}(s_0,\, u_0)\right) \\ \kappa^2(s_0)\cdot\left(\frac{\partial^3 F}{\partial u_i^3}(s_0,\, u_0)\right)\end{pmatrix}$$

to be nonsingular. By the previous notation, we have

$$\begin{aligned} Q &= \begin{pmatrix} (\kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0)), \\ \\ \kappa^2(s_0)(1/\kappa)'(s_0)(\kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0)) + (\kappa''(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\tilde{\boldsymbol{G}}(s_0))\end{pmatrix} \\ &= \begin{pmatrix} \kappa^2(s_0)\tau(s_0)\boldsymbol{n}(s_0), \\ \\ -\kappa^3(s_0)\tau(s_0)\boldsymbol{t}(s_0) + \kappa^2(s_0)\tau'(s_0)\boldsymbol{n}(s_0) + \kappa^2(s_0)\tau^2(s_0)\boldsymbol{b}(s_0)\end{pmatrix}.\end{aligned}$$

Let M be the Gramm-Schmidt matrix of Q, then we have

$$M = \kappa^8(s_0)\tau^2(s_0)\begin{pmatrix} 1 & \tau'(s_0) \\ \tau'(s_0) & \kappa^2(s_0)\tau^2(s_0) + \left\{\tau'(s)\right\}^2 + \tau^4(s_0) \end{pmatrix}.$$

So the determinant of M is given as follows:

$$det\, M = \kappa^8(s_0)\tau^4(s_0)\left(\kappa^2(s_0) + \tau^2(s_0)\right) \quad \neq 0.$$

Therefore we have rank $\begin{pmatrix} \left(\frac{\partial^2 F}{\partial u_i^2}(s_0, u_0)\right) \\ \frac{1}{2}\left(\frac{\partial^3 F}{\partial u_i^3}(s_0, u_0)\right) \end{pmatrix} = \operatorname{rank} Q = 2.$

Case (3). When f_{u_0} has the A_4-singularity at s_0, we require the 3×3 matrix

$$R = \begin{pmatrix} \kappa^2(s_0) \cdot \left(\frac{\partial^2 F}{\partial u_i^2}(s_0, u_0)\right), \\ \kappa^2(s_0) \cdot \left(\frac{\partial^3 F}{\partial u_i^3}(s_0, u_0)\right), \\ \kappa^2(s_0) \cdot \left(\frac{\partial^4 F}{\partial u_i^4}(s_0, u_0)\right) \end{pmatrix}$$

to be nonsingular. We also have

$$R = \begin{pmatrix} (\kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0)), \\ \kappa^2(s_0)(1/\kappa)'(s_0)(\kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0)) + (\kappa''(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\tilde{\boldsymbol{G}}(s_0)), \\ \Big\{\kappa^2(s_0)(1/\kappa)''(s_0)(\kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0)) \\ \quad +2\kappa^2(s_0)(1/\kappa)'(s_0)(\kappa''(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\tilde{\boldsymbol{G}}(s_0)) \\ \quad +(\kappa'''(s_0)\boldsymbol{E}(s_0) + \kappa''(s_0)\boldsymbol{F}(s_0) - \kappa'(s_0)\tilde{\boldsymbol{G}}(s_0) - \kappa(s_0)\boldsymbol{H}(s_0))\Big\} \end{pmatrix}.$$

It is enough to show that $\det(R) \neq 0$. By the direct calculation, we have

$$\begin{aligned} \det(R) &= \det\begin{pmatrix} \kappa'(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\boldsymbol{F}(s_0) \\ \kappa''(s_0)\boldsymbol{E}(s_0) - \kappa(s_0)\tilde{\boldsymbol{G}}(s_0) \\ \kappa'''(s_0)\boldsymbol{E}(s_0) + \kappa''(s_0)\boldsymbol{F}(s_0) - \kappa'(s_0)\tilde{\boldsymbol{G}}(s_0) - \kappa(s_0)\boldsymbol{H}(s_0) \end{pmatrix} \\ &= \kappa^6(s_0)\tau^3(s_0)\left(4\kappa'(s_0)\tau(s_0) + 3\kappa(s_0)\tau'(s_0)\right) \\ &\quad -\kappa^6(s_0)\tau^3(s_0)\left(5\kappa'(s_0)\tau(s_0) + 2\kappa(s_0)\tau'(s_0)\right) \\ &= \kappa^6(s_0)\tau^3(s_0)\left(\kappa(s_0)\tau'(s_0) - \kappa'(s_0)\tau(s_0)\right) \\ &\neq 0. \end{aligned}$$

This completes the proof of Proposition 5.4. ∎

Thoerem 2.2 follows from Propositions 3.1, 3.2, 5.2, 5.3, 5.4 and Theorem 5.1.

6. Generic properties of space curves. In this section we consider the notion of Monge-Taylor maps for space curves in Euclidean space (cf.,[1]). Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a regular curve, with I an open connected subset of the unit circle S^1, increasing t corresponding the anticlockwise orientation of S^1. We now choose a smooth family of unit vectors $\boldsymbol{n}(t)$, with $\boldsymbol{n}(t)$ normal to γ at t, so $\|\boldsymbol{n}(t)\| = 1$ and $< \boldsymbol{n}(t), \boldsymbol{t}(t) >= 0$

for all $t \in I$(See [1], Chapter 9). We can obtain a second smooth family of unit vectors $\boldsymbol{b}(t) = \boldsymbol{t}(t) \times \boldsymbol{n}(t)$ normal to γ at t. We now use the perpendicular lines spanned by $\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)$ as axes at $\gamma(t)$ with the unit points on the axes corresponding to the three given vectors. Note the curve $\gamma(t)$ not necessarily unit speed, with $\gamma(t_0) = 0$. We can write $\gamma(I)$ locally as a graph $\{(\xi, f_t(\xi), g_t(\xi))\}$, with $j^1 f_t(0) = j^1 g_t(0) = 0$. If V_k denotes the space of polynomials in ζ of degree ≥ 2 and $\leq k$ we have a map, the *Monge-Taylor map* for the space curve γ, $\mu_\gamma : I \longrightarrow V_k \times V_k$ given by $\mu_\gamma(t) = (j^k f_t(0), j^k g_t(0))$. ($V_k \times V_k$ can be identified with $\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} = \mathbb{R}^{2(k-1)}$ via the coordinates $(a_2, \dots, a_k, b_2, \dots, b_k)$.) Of course μ_γ depends rather heavily on our choice of unit normals $\boldsymbol{n}(t)$. Here, $a_i(t) = \frac{f_t(0)^{(i)}}{i!}$, $b_i(t) = \frac{g_t(0)^{(i)}}{i!}$ $(2 \leq i \leq k)$, that is

$$V_k \times V_k = \{(a_2\xi^2 + b_3\xi^3 + \dots + b_k\xi^k), (b_2\xi^2 + b_3\xi^3 + \dots + a_k\xi^k)\}.$$

Let P_k denote the set of maps $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $\psi(x, y, z) = (\psi_1(x, y, z), \psi_2(x, y, z), \psi_3(x, y, z))$ where $\psi_i(x, y, z)$ is a polynomial in x, y and z of degree $\leq k$. We may identify P_k and Euclidean space $\mathbb{R}^{(k+1)^2}$. This space provides the required deformations of the curve.

To simplify matters we now assume that the curve $\gamma(I)$ is compact, that is $I = S^1$. The identity map $1_{\mathbb{R}^3} : \mathbb{R}^3 \to \mathbb{R}^3$,, is of course an element of P_k (provided $k \geq 1$), and using the compactness of $\gamma(S^1)$ it easily follows that there is a open neighbourhood U of $1_{\mathbb{R}^3}$ in P_k with the property that if $\psi \in U$ then $\psi \circ \gamma : S^1 \longrightarrow \mathbb{R}^3$ is a regular curve. If we deform the original curve by the map ψ, then we can also obtain the required new smooth family of normal vectors $\boldsymbol{n}_\psi(t)$ as follows: since the map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism on some open set containing $\gamma(I)$, the vector $\boldsymbol{n}(t)$ will be sent to a new vector $D\psi(\gamma(t))\boldsymbol{n}(t)$ which will be neither zero nor tangent to $\psi \circ \gamma$ at t. Orthogonally projecting this vector onto the normal plane to $\psi \circ \gamma$ at t and normalizing, that is $\boldsymbol{n}_\psi(t) = \frac{D\psi(\gamma(t))\boldsymbol{n}(t) - <D\psi(\gamma(t))\boldsymbol{n}(t);\boldsymbol{t}_\psi>\boldsymbol{t}_\psi}{\|D\psi(\gamma(t))\boldsymbol{n}(t) - <D\psi(\gamma(t))\boldsymbol{n}(t),\boldsymbol{t}_\psi>\boldsymbol{t}_\psi\|}$, $< \boldsymbol{n}_\psi(t), \boldsymbol{n}_\psi(t) >= -1$. Where, $\boldsymbol{t}_\psi$ denotes the tangent vector of the curve $\psi \circ \gamma$ at t. We choose an open neighbourhood U of $1_{S^1} \in P_k$ consisting of polynomial maps which map an open set containing $\gamma(S^1)$ diffeomorphically to its image. We have now shown that there is a smooth map

$$\mu : S^1 \times U \longrightarrow V_k \times V_k$$

defined by $\mu(-, \psi) =$ Monge-Taylor map for the curve $\psi \circ \gamma$ using the family of normal vectors $\boldsymbol{n}_\psi(t)$. Then we have the following theorem (cf., Theorem 9.9 in [1]).

THEOREM 6.1. *Let Q be a submanifold in $V_k \times V_k = \mathbb{R}^{2k-2}$. For some open set $U_1 \subset U$ containing the identity map, the map $\mu : S^1 \times U_1 \to V_k \times V_k$ defined by $\mu(t, \psi) = \mu_{\psi \circ \gamma}(t)$ is transverse to Q.*

By the direct calculations, we have the following lemmas. The calculations are rather long and tedious so we omit details.

LEMMA 6.2. *Let $\gamma : S^1 \to \mathbb{R}_1{}^3$ be a curve defined by $\gamma(t) = (\xi, f_t(\xi), g_t(\xi))$ $= (\xi, a_2\xi^2 + a_3\xi^3 + \cdots, b_2\xi^2 + b_3\xi^3 + \cdots)$ with $\xi(t_0) = 0$. Then*
(1) $k = 0$ at t_0 if and only if $a_2{}^2 - b_2{}^2 = 0$.
(2)$(\kappa/\tau)'(t_0) = 0$ if and only if $f_1(a_2, a_3, a_4, b_2, b_3, b_4) = 0$, where

$$f_1 = 4(a_2^2 + b_2^2)(a_2b_4 - a_4b_2) - 9(a_2a_3 + b_2b_3)(a_2b_3 - a_3b_2).$$

(3) $(\kappa/\tau)''(t_0) = 0$ *if and only if* $f_2(a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5) = 0$, *where*

$$\begin{aligned} f_2 &= 12(a_2^2+b_2^2)^3(a_2b_3-a_3b_2) - 36(a_2^2+b_2^2)(a_2a_4+b_2b_4)(a_2b_3-a_3b_2) \\ &\quad -27(a_3^2+b_3^2)(a_2^2+b_2^2)(a_2b_3-a_3b_2) + 135(a_2a_3+b_2b_3)^2(a_2b_3-a_3b_2) \\ &\quad -72(a_2a_3+b_2b_3)(a_2b_4-a_4b_2)(a_2^2+b_2^2) + 12(a_3b_4-a_4b_3)(a_2^2+b_2^2)^2 \\ &\quad +20(a_2b_5-a_5b_2)(a_2^2+b_2^2)^2. \end{aligned}$$

(4) $(\kappa/\tau)'''(t_0) = 0$ *if and only if* $f_3(a_2, a_3, a_4, a_5, a_6, b_2, b_3, b_4, b_5, a_6) = 0$, *where*

$$\begin{aligned} f_3 &= -18\cdot 4(a_2a_3+b_2b_3)(a_2b_3-a_3b_2)(a_2^2+b_2^2)^3 \\ &\quad -60(a_2a_5+b_2b_5)(a_2b_3-a_3b_2)(a_2^2+b_2^2)^2 \\ &\quad -3^2\cdot 12(a_3a_4+b_3b_4)(a_2b_3-a_3b_2)(a_2^2+b_2^2)^2 \\ &\quad +3\cdot 15\cdot 12(a_2a_3+b_2b_3)(a_2a_4+b_2b_4)(a_2b_3-a_3b_2)(a_2^2+b_2^2) \\ &\quad +3\cdot 15\cdot 9(a_2a_3+b_2b_3)(a_3^2+b_3^2)(a_2b_3-a_3b_2)(a_2^2+b_2^2) \\ &\quad -3^3\cdot 35(a_2a_3+b_2b_3)^3(a_2b_3-a_3b_2) + 3\cdot 4^2(a_2^2+b_2^2)^4(a_2b_4-a_4b_2) \\ &\quad -9\cdot 4^2(a_2a_4+b_2b_4)(a_2b_4-a_4b_2)(a_2^2+b_2^2)^2 \\ &\quad -9\cdot 3\cdot 4(a_3^2+b_3^2)(a_2b_4-a_4b_2)(a_2^2+b_2^2)^2 \\ &\quad +45\cdot 12(a_2a_3+b_2b_3)^2(a_2b_4-a_4b_2)(a_2^2+b_2^2) \\ &\quad -9\cdot 12(a_2a_3+b_2b_3)(a_3b_4-a_4b_3)(a_2^2+b_2^2)^2 \\ &\quad -9\cdot 20(a_2a_3+b_2b_3)(a_2b_5-a_5b_2)(a_2^2+b_2^2)^2 + 2\cdot 5\cdot 4(a_3b_5-a_5b_3)(a_2^2+b_2^2)^3 \\ &\quad +10\cdot 4(a_2b_6-a_6b_2)(a_2^2+b_2^2)^3. \end{aligned}$$

Here ξ is the coordinate along the $\mathbf{t}$*-direction,* $f_t(\xi)$ *along the* $\mathbf{n}$*-direction and* $g_t(\xi)$ *along the* $\mathbf{b}$*-direction.*

LEMMA 6.3. *Under the same notations as in Lemma 6.2, we have the following assertions:*

(1) *The rank of the Jacobi matrix* $\dfrac{\partial(f_1, f_2)}{\partial(a_i, b_i; i = 1, \dots, 5)}$ *is two.*

(2) *The rank of the Jacobi matrix* $\dfrac{\partial(f_2, f_3)}{\partial(a_i, a_i; i = 1, \dots, 6)}$ *is two.*

The proof of Theorem 2.1 follows from Theorem 6.1 and Lemmas 6.2,6.3.

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