



Title	The focal developable and the binormal indicatrix of a nonlightlike curve in Minkowski 3-space
Author(s)	Pei, D.; Sano, T.
Citation	Hokkaido University Preprint Series in Mathematics, 421, 1-14
Issue Date	1998-8-1
DOI	10.14943/83567
Doc URL	http://hdl.handle.net/2115/69171
Type	bulletin (article)
File Information	pre421.pdf



[Instructions for use](#)

**THE FOCAL DEVELOPABLE AND
THE BINORMAL INDICATRIX
OF A NONLIGHTLIKE CURVE
IN MINKOWSKI 3-SPACE**

D. Pei and T. Sano

Series #421. August 1998

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

- #396 Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, 27 pages. 1997.
- #397 K. Hirata and K. Sugano, On semisimple extensions of serial rings, 6 pages. 1997.
- #398 D. Pei, Singularities of $\mathbb{R}P^2$ -valued Gauss maps of surfaces in Minkowski 3-space, 15 pages. 1997.
- #399 T. Mikami, Markov marginal problems and their applications to Markov optimal control, 28 pages. 1997.
- #400 M. Tsujii, A simple proof for monotonicity of entropy in the quadratic family, 8 pages. 1998.
- #401 M. Nakamura and T. Ozawa, Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth, 9 pages. 1998.
- #402 D. Lehmann and T. Suwa, Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties, 19 pages. 1998.
- #403 T. Nakazi and K. Okubo, ρ -contraction and 2×2 matrix, 6 pages. 1998.
- #404 Y. Kohsaka, Free boundary problem for quasilinear parabolic equation with fixed angle of contact to a boundary, 28 pages. 1998.
- #405 K. Yokoyama, Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions, 25 pages. 1998.
- #406 F. Hiroshima, Ground states and spectrum of quantum electrodynamics of non-relativistic particles, 58 pages. 1998.
- #407 N. Kawazumi and T. Uemura, Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries, 9 pages. 1998.
- #408 T. Nakazi and K. Okubo, Generalized Numerical Radius And Unitary ρ -Dilation, 12 pages. 1998.
- #409 Y. Giga and K. Ito, Loss of convexity of simple closed curves moved by surface diffusion, 16 pages. 1998.
- #410 Y. Giga, K. Inui and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, 34 pages. 1998.
- #411 S. Izumiya, D. Pei and T. Sano, The lightcone gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, 16 pages. 1998.
- #412 M. Tsujii, Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane, 18 pages. 1998.
- #413 J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, 40 pages. 1998.
- #414 T. Suwa, Dual class of a subvariety, 19 pages. 1998.
- #415 M. Tsujii, Piecewise expanding maps on the plane with singular ergodic properties, 9 pages. 1998.
- #416 T. Yoshida, Categorical aspects of generating functions(I): Exponential formulas and Krull-Schmidt categories, 44 pages. 1998.
- #417 K. Yamaguchi, G_2 -Geometry of overdetermined systems of second order, 22 pages. 1998.
- #418 M. Ishikawa and S. Matsui, Existence of a forward self-similar stagnation flow of the Navier-Stokes equations, 8 pages. 1998.
- #419 S. Izumiya, H. Katsumi and T. Yamasaki, The rectifying developable and the spherical Darboux image of a space curve, 16 pages. 1998.
- #420 R. Kobayashi and Y. Giga, Equations with singular diffusivity, 45 pages. 1998.

THE FOCAL DEVELOPABLE AND THE BINORMAL INDICATRIX OF A NONLIGHTLIKE CURVE IN MINKOWSKI 3-SPACE

DONGHE PEI* AND TAKASHI SANO

Department of Mathematics, Faculty of Science,
Hokkaido University, Sapporo 060 JAPAN

ABSTRACT. We define the binormal indicatrix and focal developable of a nonlightlike curve in Minkowski 3-space. We establish the relationships between singularities of these subjects and geometric invariants of curves under the action of Lorentzian group.

1. INTRODUCTION

There are several articles concerning 'generic differential geometry' in the Euclidean space [1-5,7,9,13,etc.]. The main tools in these articles are the distance-squared functions and the height functions on submanifolds. In this paper we introduce the notion of Lorentzian height functions and Lorentzian distance-squared functions on timelike curves or spacelike curves in Minkowski 3-space. We also define the notion of binormal indicatrices and focal developables of timelike curves or spacelike curves in Minkowski 3-space, and establish the relationships between singularities of these subjects and geometric invariants of the curve under the action of Lorentzian group as applications of standard techniques of singularity theory for the above functions. For the basic notions in Lorentzian geometry, see [11].

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be two vectors in \mathbb{R}^3 , the *pseudo scalar product* of \mathbf{x} and \mathbf{y} is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$. $(\mathbb{R}^3, \langle, \rangle)$ is called a 3-dimensional *pseudo Euclidean space*, or *Minkowski 3-space*. We denote \mathbb{R}_1^3 as $(\mathbb{R}^3, \langle, \rangle)$. For any $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}_1^3$, the *pseudo vector product* of \mathbf{x} and \mathbf{y} is defined by

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Moreover, \mathbf{x} in \mathbb{R}_1^3 is called a *spacelike vector*, a *lightlike vector* or a *timelike vector* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. For $\mathbf{x} \in \mathbb{R}_1^3$, the *norm* of \mathbf{x}

1991 *Mathematics Subject Classification*. 53B30, 58C28, 57R70.

*On leave from Department of Mathematics, North East Normal University, Chang Chun 130024, P.R.China

is defined by $\|\mathbf{x}\| = \sqrt{\text{sign}(\mathbf{x}) \langle \mathbf{x}, \mathbf{x} \rangle}$, and \mathbf{x} is called a *unit vector* if $\|\mathbf{x}\| = 1$, where $\text{sign}(\mathbf{x})$ denotes the *signature* of \mathbf{x} which is given by

$$\text{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} : \text{spacelike} \\ 0 & \mathbf{x} : \text{lightlike} \\ -1 & \mathbf{x} : \text{timelike} . \end{cases}$$

Let $\gamma : I \rightarrow \mathbb{R}_1^3$; $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a smooth regular curve in \mathbb{R}_1^3 (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve γ is called a *space-like curve*, a *lightlike curve* or a *timelike curve* if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$ respectively. We call γ a *nonlightlike curve* if γ is a timelike curve or a spacelike curve. The *arc-length* of a nonlightlike curve γ , measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^t \|\dot{\gamma}(t)\| dt.$$

Then the parameter s is determined such that $\|\gamma'(s)\| = 1$ for the nonlightlike curve, where $\gamma'(s) = d\gamma/ds(s)$. So we say that a nonlightlike curve γ is *parameterized by arc-length* if it satisfies that $\|\gamma'(s)\| = 1$. Throughout the reminder in this paper we denote the parameter s of γ as the arc-length parameter. Let us denote $\mathbf{t}(s) = \gamma'(s)$, and we call $\mathbf{t}(s)$ a *unit tangent vector* of γ at s . We define the *curvature* by $k(s) = \sqrt{|\langle \gamma''(s), \gamma''(s) \rangle|}$. If $k(s) \neq 0$ then the *unit principal normal vector* \mathbf{n} of the curve γ at s is given by $\gamma''(s) = k(s) \cdot \mathbf{n}(s)$. We denote that $\varepsilon(\gamma(s)) = \text{sign}(\mathbf{t}(s))$ and $\delta(\gamma(s)) = \text{sign}(\mathbf{n}(s))$. The unit vector $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$ is called a *unit binormal vector* of the curve γ at s . Then the following Frenet-Serret type formula holds:

$$\begin{cases} \mathbf{t}'(s) = k(s) \cdot \mathbf{n}(s) \\ \mathbf{n}'(s) = -\varepsilon(\gamma(s)) \cdot \delta(\gamma(s)) \cdot k(s) \cdot \mathbf{t}(s) + \varepsilon(\gamma(s)) \cdot \tau(s) \cdot \mathbf{b}(s) \\ \mathbf{b}'(s) = \tau(s) \cdot \mathbf{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve γ at s (cf.,[7]). This is the fundamental formula for the study of nonlightlike curves in \mathbb{R}_1^3 .

We now define

$$H_1^2(r)_p = \{x = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \mid \langle x - p, x - p \rangle = -r^2\},$$

$$S_1^2(r)_p = \{x = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \mid \langle x - p, x - p \rangle = r^2\},$$

where $p = (p_1, p_2, p_3)$ and $r \in \mathbb{R} - \{0\}$. We respectively call $H_1^2(r) := H_1^2(r)_p - \{p\}$, $S_1^2(r) := S_1^2(r)_p - \{p\}$ a *hyperbola*, a *pseudo sphere* with *radius* r at the *center* p . We denote that $H_1^2 = H_1^2(1)_0$ and $S_1^2 = S_1^2(1)_0$.

Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $\varepsilon(\gamma)\delta(\gamma(s)) = -1$ and $k(s) \neq 0$. In this case we always have $\mathbf{b}(s) \in S_1^2$. On the other hand, if $\varepsilon(\gamma)\delta(\gamma(s)) = 1$ and $k(s) \neq 0$, then γ is spacelike. In this case, we have $\mathbf{b}(s) \in H_1^2$.

We now define a curve $BN_\gamma : I \longrightarrow S_1^2$ for $\varepsilon(\gamma)\delta(\gamma(s)) = -1$ (respectively, $BN_\gamma : I \longrightarrow H_1^2$ for γ with $\varepsilon(\gamma)\delta(\gamma(s)) = 1$) by

$$BN_\gamma(s) = \mathbf{b}(s),$$

and a ruled surface $E_\gamma : I \times \mathbb{R} \longrightarrow \mathbb{R}_1^3$ by

$$E_\gamma(s, u) = \left(\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + u\mathbf{b}\right)(s).$$

We call BN_γ the *binormal indicatrix* and E_γ the *focal surface* of γ .

In §3, under the assumption that $k(s) \neq 0$, $\tau(s) \neq 0$ and $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) > 0$, we show that there exist a unique pseudo sphere which has four-point contact with nonlightlike curve γ at $\gamma(s_0)$ which is called the *osculating pseudo sphere* of γ at s_0 and its center $v = \left(\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau} \mathbf{b}\right)(s_0)$ is called the *center* of the osculating pseudo sphere at s_0 . Under the assumption that $k(s) \neq 0$, $\tau(s) \neq 0$, $\langle \mathbf{b}(s_0), \mathbf{b}(s_0) \rangle = -\delta(\gamma(s_0))$ and $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) < 0$, we also show that there exist a unique hyperbola which has four-point contact with spacelike curve γ at $\gamma(s_0)$. which is called the *osculating hyperbola* of γ at s_0 and its center $v = \left(\gamma + \frac{1}{\delta(\gamma)k} \mathbf{n} + \frac{k'}{\delta(\gamma)k^2\tau} \mathbf{b}\right)(s_0)$ is called the *center* of the osculating hyperbola at s_0 .

We now consider the following conditions:

(A 1) The number of points p of $\gamma(I)$ where the osculating pseudo sphere at p having five-point contact with the curve γ is finite.

(A 2) There is no point p of $\gamma(I)$ where the osculating pseudo sphere at p having greater than six-point contact with the curve γ .

(A 3) The number of points p of $\gamma(I)$ where the osculating hyperbola at p having five-point contact with the curve γ is finite.

(A 4) There is no point p of $\gamma(I)$ where the osculating hyperbola at p having greater than six-point contact with the curve γ .

(A 5) The number of points at where γ has at least four-point contact with the osculating plane is finite.

(A 6) There are no osculating plane with which γ has five-point contact at a point.

Remark. If the curve γ satisfies the condition $\delta(\gamma(s))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s) > 0$ (respectively, < 0), then we can show that there exists the osculating pseudo sphere (respectively, osculating hyperbola) at each point of γ (cf., §3).

Our main results are formulated as follows:

Theorem A. (1) Let $n\text{-Imm}_+(I, \mathbb{R}_1^3)$ be the space of proper nonlightlike curves with $k(s) \neq 0$ and $\delta(\gamma(s))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s) > 0$ equipped with Whitney C^∞ -topology. Then the set of curves which satisfy (A 1) and (A 2) is a residual set in $n\text{-Imm}_+(I, \mathbb{R}_1^3)$.

(2) Let $n\text{-Imm}_-(I, \mathbb{R}_1^3)$ be the space of proper nonlightlike curves with $k(s) \neq 0$, $\langle \mathbf{b}(s), \mathbf{b}(s) \rangle = -\delta(\gamma(s))$ and $\delta(\gamma(s))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s) < 0$ equipped with Whitney C^∞ -topology. Then the set of curves which satisfy (A 3) and (A 4) is a residual set in $n\text{-Imm}_-(I, \mathbb{R}_1^3)$.

(3) Let $n\text{-Imm}(I, \mathbb{R}_1^3)$ be the space of proper nonlightlike curves with $k(s) \neq 0$ equipped with Whitney C^∞ -topology. Then the set of curves which satisfy (A 5) and (A 6) is a residual set in $n\text{-Imm}(I, \mathbb{R}_1^3)$.

Theorem B. (1) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$ and $\tau(s) \neq 0$. Under the assumption of (A 1) and (A 2) (respectively, (A 3) and (A 4)), if p is a point of the focal surface E_γ of γ at s_0 , then locally at p , the focal surface is

a) diffeomorphic to $C \times \mathbb{R}$ at $(\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + u\mathbf{b})(s_0)$ in \mathbb{R}_1^3 if and only if $k'(s_0) \neq 0$ and $u_0 = (\frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b})(s_0)$. Moreover, if $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) > 0$, then the locus of the center of the osculating pseudo sphere $(\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + u\mathbf{b})(s_0)$ is the cuspidal edge. If $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) < 0$ and $\langle \mathbf{b}(s_0), \mathbf{b}(s_0) \rangle = -\delta(\gamma(s_0))$, then the locus of the center of the osculating hyperbola $(\gamma + \frac{1}{\delta(\gamma)k}\mathbf{n} + u\mathbf{b})(s_0)$ is the cuspidal edge.

b) diffeomorphic to a SW at $(\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b})(s_0)$ if and only if $k'(s_0) \neq 0$ and $(kk'\tau' - kk''\tau + k^2\tau^3 + 2k'^2\tau)(s_0) = 0$.

(2) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$. Under the assumption of (A 3) and (A 4), if p is a point of the binormal indicatrix of γ at s_0 , then locally at p , the binormal indicatrix BN_γ is

a) diffeomorphic to a line if $\tau(s_0) \neq 0$.

b) diffeomorphic to the ordinary cusp C if $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$.

Here, $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ is called the ordinary cusp and $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called the swallow tail.

2. LORENTZIAN INVARIANT FUNCTIONS ON NONLIGHTLIKE CURVES

In this section we introduce two different families of functions on a nonlightlike curve, which are useful to the study of Lorentzian invariants of nonlightlike curves.

Lorentzian height functions

For a nonlightlike curve $\gamma : I \rightarrow \mathbb{R}_1^3$, we now define a function $H : I \times S_1^2 \rightarrow \mathbb{R}$ (respectively, $H : I \times H_1^2 \rightarrow \mathbb{R}$) by $H(s, v) = \langle \gamma, v \rangle$. We call H a *Lorentzian height function*. We denote that $h(s) = Hv_0(s) = H(s, v_0)$ for any fixed $v_0 \in S_1^2$ (respectively, $v_0 \in H_1^2$). We have the following proposition.

Proposition 2.1. Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed nonlightlike curve with $k(s) \neq 0$. If $v \in S_1^2$ (respectively, $v \in H_1^2$), then

- (1) $h'(s_0) = 0$ if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that $v = \lambda\mathbf{n}(s_0) + \mu\mathbf{b}(s_0)$ and $\delta(\gamma)(\lambda^2 - \varepsilon(\gamma)\mu^2) = 1$ (respectively, $\delta(\gamma)(\lambda^2 - \varepsilon(\gamma)\mu^2) = -1$).
- (2) $h'(s_0) = h''(s_0) = 0$ if and only if $v = \pm\mathbf{b}(s_0)$.
- (3) $h'(s_0) = h''(s_0) = h^{(3)}(s_0) = 0$ if and only if $v = \pm\mathbf{b}(s_0)$ and $\tau(s_0) = 0$.
- (4) $h'(s_0) = h''(s_0) = h^{(3)}(s_0) = h^{(4)}(s_0) = 0$ if and only if $v = \pm\mathbf{b}(s_0)$ and $\tau(s_0) = \tau'(s_0) = 0$.

Proof. Let $H : I \times S_1^2 \rightarrow \mathbb{R}$ (respectively, $H : I \times H_1^2 \rightarrow \mathbb{R}$) be the Lorentzian height function on the nonlightlike curve $\gamma : I \rightarrow \mathbb{R}_1^3$. Then we have $\frac{\partial H}{\partial s} = \langle \gamma'(s), v \rangle = \langle \mathbf{t}(s), v \rangle$ and $v \in S_1^2$ (respectively, $v \in H_1^2$). It follows that $\frac{\partial H}{\partial s} = 0$ if and only if $\langle v, \mathbf{t} \rangle = 0$. Then there exist $\lambda, \mu \in \mathbb{R}$ such that $v = \lambda\mathbf{n} + \mu\mathbf{b}$ and $\delta(\gamma) \cdot (\lambda^2 - \varepsilon(\gamma)\mu^2) = 1$ (respectively, $\delta(\gamma)(\lambda^2 - \varepsilon(\gamma)\mu^2) = -1$). Where $\langle \mathbf{b}, \mathbf{b} \rangle = -\varepsilon(\gamma)\delta(\gamma)$ by $\langle \mathbf{t}, \mathbf{t} \rangle = \varepsilon(\gamma)$ and $\langle \mathbf{n}, \mathbf{n} \rangle = \delta(\gamma)$.

On the other hand, we have $\frac{\partial^2 H}{\partial s^2} = \langle k \cdot \mathbf{n}, v \rangle = 0$. It follows that $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = 0$ if and only if $v = \lambda \mathbf{n} + \mu \mathbf{b}$, $\langle k \cdot \mathbf{n}, v \rangle = 0$ and $\delta(\gamma) \cdot (\lambda^2 - \varepsilon(\gamma)\mu^2) = 1$ (respectively, $\delta(\gamma)(\lambda^2 - \varepsilon(\gamma)\mu^2) = -1$). This is equivalent to the condition that $v = \pm \mathbf{b}$.

Since $\frac{\partial^3 H}{\partial s^3} = \langle k' \mathbf{n} + k \mathbf{n}', v \rangle$, $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 H}{\partial s^3} = 0$ if and only if $v = \pm \mathbf{b}$, $\tau = 0$.

Moreover, we have $\frac{\partial^4 H}{\partial s^4} = \langle \mathbf{t}^{(3)}, v \rangle$. Then $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 H}{\partial s^3} = \frac{\partial^4 H}{\partial s^4} = 0$ if and only if $v = \pm \mathbf{b}$, $\tau = \tau' = 0$. \square

Lorentzian distance-squared functions

For a nonlightlike curve $\gamma : I \rightarrow \mathbb{R}_1^3$, we now define a function

$$G : I \times \mathbb{R}_1^3 \rightarrow \mathbb{R}$$

by

$$G(s, v) = \langle \gamma - v, \gamma - v \rangle.$$

We call G a *Lorentzian distance-squared function*. We denote that $g(s) = G_{v_0}(s) = G(s, v_0)$ for any fixed $v_0 \in \mathbb{R}_1^3$. We also have the following proposition.

Proposition 2.2. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed nonlightlike curve with $k(s) \neq 0$ and $\tau(s) \neq 0$. Then*

- (1) $g'(s_0) = 0$ if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that $\gamma(s_0) - v(s_0) = \lambda \mathbf{n}(s_0) + \mu \mathbf{b}(s_0)$.
- (2) $g'(s_0) = g''(s_0) = 0$ if and only if $v = (\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} - \mu \mathbf{b})(s_0)$ for some $\mu \in \mathbb{R}$.
- (3) $g'(s_0) = g''(s_0) = g^{(3)}(s_0) = 0$ if and only if $v = (\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau} \mathbf{b})(s_0)$.
- (4) $g'(s_0) = g''(s_0) = g^{(3)}(s_0) = g^{(4)}(s_0) = 0$ if and only if $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s_0) = 0$ and $v = (\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau} \mathbf{b})(s_0)$.
- (5) $g'(s_0) = g''(s_0) = g^{(3)}(s_0) = g^{(4)}(s_0) = g^{(5)}(s_0) = 0$ if and only if $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s_0) = (2kk'\tau^3 - \varepsilon(\gamma)kk'\tau'' + \varepsilon(\gamma)kk^{(3)}\tau + 3k^2\tau^2\tau' - 3\varepsilon(\gamma)k'k''\tau - 3\varepsilon(\gamma)k'^2\tau')(s_0) = 0$ and $v = (\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau} \mathbf{b})(s_0)$.

Proof. Let $G : I \times \mathbb{R}_1^3 \rightarrow \mathbb{R}$ be a Lorentzian distance-squared function on the nonlightlike curve $\gamma : I \rightarrow \mathbb{R}_1^3$. Then we have $\frac{\partial G}{\partial s} = 2 \langle \gamma', \gamma - v \rangle$. It follows that $\frac{\partial G}{\partial s} = 0$ if and only if $\gamma - v = \lambda \cdot \mathbf{n} + \mu \cdot \mathbf{b}$.

On the other hand, since $\frac{\partial^2 G}{\partial s^2} = 2 \langle \mathbf{t}', \gamma - v \rangle + 2 \langle \mathbf{t}, \mathbf{t} \rangle = 2\{\langle k \cdot \mathbf{n}, \gamma - v \rangle + \varepsilon(\gamma)\}$, $\frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = 0$ if and only if $\gamma - v = \lambda \mathbf{n} + \mu \mathbf{b}$ and $\langle k \cdot \mathbf{n}, \lambda \mathbf{n} + \mu \mathbf{b} \rangle = -\varepsilon(\gamma)$. The last equality is equivalent to the relation that $k\lambda\delta(\gamma) = -\varepsilon(\gamma)$. This means that $v = \gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} - \mu \mathbf{b}$ by $k \neq 0$.

Since $\frac{\partial^3 G}{\partial s^3} = 2 \langle (k \cdot \mathbf{n})', \gamma - v \rangle$, $\frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = 0$ if and only if $k \neq 0$ and $\langle k' \cdot \mathbf{n} + k \cdot \mathbf{n}', -\frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + \mu \mathbf{b} \rangle = 0$, which is equivalent to the condition that $k \neq 0$ and $k' + \varepsilon(\gamma)\delta(\gamma)k^2\tau\mu = 0$. This means that $v = \gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k} \mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau} \mathbf{b}$ by k and τ are not zero.

Moreover, we have $\frac{\partial^4 G}{\partial s^4} = 2 \langle k'' \mathbf{n} + 2k' \mathbf{n}' + k \mathbf{n}'', \gamma - v \rangle + 2 \langle k' \mathbf{n} + k \mathbf{n}', \mathbf{t} \rangle$. Then $\frac{\partial^4 G}{\partial s^4} = 0$ if and only if $\langle (k'' - \varepsilon(\gamma)\delta(\gamma)k^3 + \varepsilon(\gamma)k\tau^2) \mathbf{n} - 3\varepsilon(\gamma)\delta(\gamma)kk'\mathbf{t} + (2\varepsilon(\gamma)k'\tau +$

$\varepsilon(\gamma)k\tau')\mathbf{b}, \gamma - v \rangle = \delta(\gamma)k^2$. Therefore $\frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = \frac{\partial^4 G}{\partial s^4} = 0$ if and only if $\langle (k'' - \varepsilon(\gamma)\delta(\gamma)k^3 + \varepsilon(\gamma)k\tau^2)\mathbf{n} - 3\varepsilon(\gamma)\delta(\gamma)kk'\mathbf{t} + (2\varepsilon(\gamma)k'\tau + \varepsilon(\gamma)k\tau')\mathbf{b}, -\frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} - \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b} \rangle = \delta(\gamma)k^2$ and $v = \gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b}$, which is equivalent to the condition that $kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau = 0$ and $v = \gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b}$.

Finally, we have $\frac{\partial^5 G}{\partial s^5} = 0$ if and only if $\langle \{(k'' - \varepsilon(\gamma)\delta(\gamma)k^3 + \varepsilon(\gamma)k\tau^2)\mathbf{n} - 3\varepsilon(\gamma)\delta(\gamma)kk'\mathbf{t} + (2\varepsilon(\gamma)k'\tau + \varepsilon(\gamma)k\tau')\mathbf{b}\}', \gamma - v \rangle = 5\delta(\gamma)kk'$. This is equivalent to the condition that $\langle (k^{(3)} - 3\varepsilon(\gamma)\delta(\gamma)k^2k' + \varepsilon(\gamma)k'\tau^2 + 3\varepsilon(\gamma)k\tau\tau' - 3\varepsilon(\gamma)\delta(\gamma)k^2k' + 2\varepsilon(\gamma)k'\tau^2)\mathbf{n} + (k^4 - \delta(\gamma)k^2\tau^2 - 3\varepsilon(\gamma)\delta(\gamma)k'^2 - 4\varepsilon(\gamma)\delta(\gamma)kk''\mathbf{t} + (3\varepsilon(\gamma)k''\tau + 2\varepsilon(\gamma)k'\tau' + \varepsilon(\gamma)k'\tau' + \varepsilon(\gamma)k\tau'' - \delta(\gamma)k^3\tau + k\tau^3)\mathbf{b}, \gamma - v \rangle = 5\delta(\gamma)kk'$. So $\frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = \frac{\partial^4 G}{\partial s^4} = \frac{\partial^5 G}{\partial s^5} = 0$ if and only if $kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau = 2kk'\tau^3 - \varepsilon(\gamma)kk'\tau'' + \varepsilon(\gamma)kk^{(3)}\tau + 3k^2\tau^2\tau' - 3\varepsilon(\gamma)k'k''\tau - 3\varepsilon(\gamma)k'^2\tau' = 0$ and $v = \gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b}$. \square

3. LORENTZIAN INVARIANTS OF NONLIGHTLIKE CURVES

In this section we study the geometric properties of the binormal indicatrix and focal developables of the nonlightlike curves. By the propositions in the last section, we can recognize the function $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s)$ has special meanings. Firstly we have the following proposition.

Proposition 3.1. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$, $k'(s) \neq 0$ and $\tau(s) \neq 0$. Then $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s) \equiv 0$ if and only if $p = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}\mathbf{n}(s) + \frac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)}\mathbf{b}(s)$ are constant vectors.*

Proof. We put

$$P(s) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}\mathbf{n}(s) + \frac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)}\mathbf{b}(s),$$

then we have

$$P'(s) = -\frac{(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^3(s)\tau^2(s)}\mathbf{b}(s).$$

It follows that $P'(s) \equiv 0$ if and only if $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s) \equiv 0$. \square

Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$, $\tau(s) \neq 0$ and $\delta(\gamma(s))(k^2\tau^2 - k'^2)(s) \neq 0$ (the last condition is always satisfied for timelike curves), and $G : I \times \mathbb{R}_1^3 \rightarrow \mathbb{R}$ a Lorentzian distance-squared function on the curve γ . Under the conditions $g'(s_0) = g''(s_0) = g^{(3)}(s_0) = 0$ at $s = s_0$, we can show that $\langle \gamma(s_0) - v, \gamma(s_0) - v \rangle = \left(\frac{\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)}{k^4\tau^2}\right)(s_0) > 0$ for $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) > 0$ and $\langle \gamma(s_0) - v, \gamma(s_0) - v \rangle < 0$ for $\delta(\gamma(s_0))(k^2\tau^2 - k'^2)(s_0) < 0$ by Proposition 2.2. (For the case $\delta(\gamma(s_0))(k^2\tau^2 - k'^2)(s_0) = 0$ on spacelike curves, see [6].)

So we have the following corollaries.

Corollary 3.2. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$, $k'(s) \neq 0$, $\tau(s) \neq 0$ and $\delta(\gamma(s))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s) > 0$. Then singularities of focal surface are the locus of*

centers of osculating pseudo spheres of γ which is given by $p(s) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}\mathbf{n}(s) + \frac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)}\mathbf{b}(s)$. Moreover, if $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s) \equiv 0$, then $p(s)$ is a constant point and γ is a curve on the pseudo sphere whose center is $p(s)$ and radius is $r = \sqrt{\frac{\delta(\gamma)(k^2\tau^2 - \varepsilon(\gamma)k'^2)}{k^4\tau^2}}(s)$.

Corollary 3.3. Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a spacelike curve with $k(s) \neq 0$, $k'(s) \neq 0$, $\tau(s) \neq 0$ and $\delta(\gamma(s))(k^2\tau^2 - k'^2)(s) < 0$. Then singularities of focal surface are the locus of centers of osculating pseudo hyperbolas of γ which is given by $p(s) = \gamma(s) + \frac{1}{\delta(\gamma(s))k(s)}\mathbf{n}(s) + \frac{k'(s)}{\delta(\gamma(s))k^2(s)\tau(s)}\mathbf{b}(s)$. Moreover, if $(kk'\tau' - kk''\tau - k^2\tau^3 + 2k'^2\tau)(s) \equiv 0$, then $p(s)$ is a constant point and γ is a curve on the osculating hyperbola whose center is $p(s)$ and radius is $r = \sqrt{-\frac{\delta(\gamma)(k^2\tau^2 - k'^2)}{k^4\tau^2}}(s)$.

Let $F : \mathbb{R}_1^3 \longrightarrow \mathbb{R}$ be a submersion and $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a nonlightlike curve. We say that γ and $F^{-1}(0)$ have k -point contact for $t = t_0$ if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \dots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. By Propositions 2.1, 2.2 and 3.1, we have the following proposition.

Proposition 3.4. (1) Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$. Then the binormal \mathbf{b} and the osculating plane of γ have 3-point contact for $s = s_0$ if and only if $\tau(s_0) \equiv 0$ and $\tau'(s_0) \neq 0$.

(2) Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a nonlightlike curve with $k(s) \neq 0$, $k'(s) \neq 0$ and $\tau(s) \neq 0$. Then γ and the osculating pseudo sphere have 4-point contact for $s = s_0$ if and only if $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) > 0$, $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s_0) = 0$ and $(2kk'\tau^3 - \varepsilon(\gamma)kk'\tau'' + \varepsilon(\gamma)kk^{(3)}\tau + 3k^2\tau^2\tau' - 3\varepsilon(\gamma)k'k''\tau - 3\varepsilon(\gamma)k'^2\tau')(s_0) \neq 0$.

(3) Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a spacelike curve with $k(s) \neq 0$, $k'(s) \neq 0$ and $\tau(s) \neq 0$. Then γ and the osculating hyperbola have 4-point contact for $s = s_0$ if and only if $\delta(\gamma(s_0))(k^2\tau^2 - \varepsilon(\gamma)k'^2)(s_0) < 0$, $(kk'\tau' - kk''\tau - k^2\tau^3 + 2k'^2\tau)(s_0) = 0$ and $(2kk'\tau^3 - kk'\tau'' + kk^{(3)}\tau + 3k^2\tau^2\tau' - 3k'k''\tau - 3k'^2\tau')(s_0) \neq 0$.

4. UNFOLDINGS OF FUNCTIONS BY ONE-VARIABLE

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [3]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$ be a function germ. We call F an r -parameter unfolding of f , where $f(s) = F_{x_0}(s, x_0)$. We say that f has A_k -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has $A_{\geq k}$ -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has A_k -singularity ($k \geq 1$) at s_0 . We denote the $(k-1)$ -jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}s^j$ for $i = 1, \dots, r$. Then F is called a (p) versal unfolding if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ ($k-1 \leq r$). There is an important set concerning the unfoldings relative to the above notions. The bifurcation set of F is the set

$$\mathfrak{B}_F = \{x \in \mathbb{R}^r \mid \text{there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, x)\}.$$

Then we have the following well-known theorem (cf., [4]).

Theorem 4.1. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of $f(s)$ which has the A_k singularity at s_0 .

Suppose that F is an (p) versal unfolding.

- (a) If $k = 2$, then \mathfrak{B}_F is diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.
- (b) If $k = 3$, then \mathfrak{B}_F is diffeomorphic to $C \times \mathbb{R}^{r-2}$.
- (c) If $k = 4$, then \mathfrak{B}_F is diffeomorphic to $SW \times \mathbb{R}^{r-3}$.

Where, $C = \{(x_1, x_2) | x_1^2 = x_2^3\}$ and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

For the proof of Theorem B we have the following fundamental theorem.

Theorem 4.2.

- (1) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed nonlightlike curve with $k(s) \neq 0$ and $H : I \times S_1^2 \rightarrow \mathbb{R}$ be the height function on γ .
If $h(s) = H_{v_0}(s)$ has the A_k -singularity ($k = 2, 3$) at s_0 , then H is the (p) versal unfolding of h .
- (2) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed spacelike curve with $k(s) \neq 0$ and $H : I \times H_1^2 \rightarrow \mathbb{R}$ be the height function on γ .
If $h(s) = H_{v_0}(s)$ has the A_k -singularity ($k = 2, 3$) at s_0 , then H is the (p) versal unfolding of h .
- (3) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed nonlightlike curve with $k(s) \neq 0$ and $G : I \times \mathbb{R}_1^3 \rightarrow \mathbb{R}$ be the distance-squared function. We consider the point $(s_0, v_0) \in I \times \mathbb{R}_1^3$.
If $g(s) = G_{v_0}(s)$ has the A_k -singularity ($k = 2, 3, 4$) at s_0 , then G is the (p) versal unfolding of $g := G_{v_0}$.

Proof. (1) Let $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ be a nonlightlike curve and $v = (v_1, v_2, v_3) \in S_1^2$. By definition, in this case we have

$$H(s, v) = -v_1 x_1(s) + v_2 x_2(s) + v_3 x_3(s) = -v_1 x_1(s) \pm \sqrt{1 + v_1^2 - v_3^2} x_2(s) + v_3 x_3(s).$$

So that $\frac{\partial H}{\partial v_1} = -x_1(s) \pm \frac{v_1 x_2(s)}{\sqrt{1 + v_1^2 - v_3^2}} = -x_1(s) + \frac{v_1}{v_2} x_2(s)$, (since v is spacelike, v_2 and v_3 are not both zero, so we let $v_2 \neq 0$) so the 2-jet at s_0 is $-(sx_1'(s) + \frac{1}{2}s^2 x_1''(s)) + \frac{v_1}{v_2}(sx_1'(s) + \frac{1}{2}s^2 x_1''(s))$. We also have $\frac{\partial H}{\partial v_3} = \mp \frac{v_3 x_2(s)}{\sqrt{1 + v_1^2 - v_3^2}} + x_3 = -\frac{v_3}{v_2} x_2 + x_3$, so the 2-jet at s_0 is $(sx_3'(s_0) + \frac{1}{2}s^2 x_3''(s_0)) - \frac{v_3}{v_2}(sx_2'(s_0) + \frac{1}{2}s^2 x_2''(s_0))$.

- (i) By Proposition 2.1, h has the A_2 -singularity at s_0 if and only if $v(s_0) = \pm \mathbf{b}(s_0)$ and $\tau(s_0) \neq 0$. When h has A_2 at s_0 , we require the 1×2 matrix $(-x_1'(s) + \frac{v_1}{v_2} x_2'(s), -\frac{v_3}{v_2} x_2' + x_3')$ to have rank 1, which follows from the proof of the case (ii).
- (ii) By Proposition 2.1, h has A_3 at s_0 if and only if $v = \pm \mathbf{b}(s_0)$, $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$. When h has A_3 at s , we require the 2×2 matrix

$$\begin{pmatrix} -x_1'(s) + \frac{v_1}{v_2} x_2'(s) & -\frac{v_3}{v_2} x_2' + x_3' \\ -\frac{1}{2} x_1''(s) + \frac{v_1}{2v_2} x_2''(s) & -\frac{v_3}{2v_2} x_2'' + \frac{1}{2} x_3'' \end{pmatrix}$$

to be nonsingular, which it always does. In fact, by $v = \pm \mathbf{b}(s_0)$, $v(s_0) = \pm \mathbf{t}(s_0) \wedge \mathbf{n}(s_0) = \pm \frac{1}{k(s_0)}(\gamma'(s_0) \wedge \gamma''(s_0)) = (v_1, v_2, v_3) \in S_1^2$. So that

$$\begin{cases} v_1 = \pm \frac{1}{k}(-x_2'x_3'' + x_3'x_2'') \\ v_2 = \pm \frac{1}{k}(x_3'x_1'' - x_1'x_3'') \\ v_3 = \pm \frac{1}{k}(x_1'x_2'' - x_2'x_1''). \end{cases}$$

Hence, the determinant of the above matrix is $\frac{1}{2}\{(x_3'x_1'' - x_1'x_3'') + \frac{v_3}{v_2}(x_1'x_2'' - x_2'x_1'') - \frac{v_1}{v_2}(-x_2'x_3'' + x_3'x_2'')\} = \pm \frac{k}{2}(v_2 + \frac{v_3^2}{v_2} - \frac{v_1^2}{v_2}) = \pm \frac{k}{2v_2} \neq 0$ by $k(s_0) \neq 0$ and $v_2 \neq 0$. This means that H is (p)versal when $h(s)$ has the A_k -singularity ($k=2,3$) at s_0 .

(2) Let $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ be a spacelike curve and $v = (v_1, v_2, v_3) \in H_1^2$. By definition, in this case we have

$$H(s, v) = -v_1x_1(s) + v_2x_2(s) + v_3x_3(s) = \mp \sqrt{1 + v_2^2 + v_3^2}x_1(s) + v_2x_2(s) + v_3x_3(s).$$

So that $\frac{\partial H}{\partial v_2} = +x_2(s) \mp \frac{v_2x_1(s)}{\sqrt{1+v_2^2+v_3^2}} = -\frac{v_2}{v_1}x_1(s) + x_2(s)$, so the 2-jet at s_0 is $(sx_2'(s) + \frac{1}{2}s^2x_2''(s)) - \frac{v_2}{v_1}(sx_1'(s) + \frac{1}{2}s^2x_1''(s))$. We also have $\frac{\partial H}{\partial v_3} = \mp \frac{v_3x_1(s)}{\sqrt{1+v_2^2+v_3^2}} + x_3 = -\frac{v_3}{v_1}x_1 + x_3$, so the 2-jet at s_0 is $(sx_3'(s_0) + \frac{1}{2}s^2x_3''(s_0)) - \frac{v_3}{v_1}(sx_1'(s_0) + \frac{1}{2}s^2x_1''(s_0))$.

- (i) By Proposition 2.1, h has the A_2 -singularity at s_0 if and only if $v(s_0) = \pm \mathbf{b}(s_0)$ and $\tau(s_0) \neq 0$. When h has A_2 at s_0 , we require the 1×2 matrix $(-\frac{v_2}{v_1}x_1'(s) + x_2'(s), -\frac{v_3}{v_1}x_1' + x_3')$ to have rank 1, which follows from the proof of the case (ii).
- (ii) By Proposition 2.1, h has A_3 at s_0 if and only if $v = \pm \mathbf{b}(s_0)$, $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$. When h has A_3 at s , we require the 2×2 matrix

$$\begin{pmatrix} -\frac{v_2}{v_1}x_1'(s) + x_2'(s) & -\frac{v_3}{v_1}x_1' + x_3' \\ -\frac{v_2}{2v_1}x_1''(s) + \frac{1}{2}x_2''(s) & -\frac{v_3}{2v_1}x_1'' + \frac{1}{2}x_3'' \end{pmatrix}$$

to be nonsingular, which it always does. By the same reason as the case 1), the determinant of the above matrix is $\frac{1}{2}\{\frac{v_2}{v_1}(x_3'x_1'' - x_1'x_3'') + \frac{v_3}{v_1}(x_1'x_2'' - x_2'x_1'') - (-x_2'x_3'' + x_3'x_2'')\} = \pm \frac{k}{2}(\frac{v_2^2}{v_1} + \frac{v_3^2}{v_1} - v_1) = \mp \frac{k}{2v_1} \neq 0$ by $k(s_0) \neq 0$ and $v \in H_1^2$. This means that H is (p)versal when $h(s)$ has the A_k -singularity ($k=2,3$) at s_0 .

(3) In this case we have

$$G(s, v) = -(x_1(s) - v_1)^2 + (x_2(s) - v_2)^2 + (x_3(s) - v_3)^2,$$

where $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ and $v = (v_1, v_2, v_3)$.

Thus we have $\frac{\partial G}{\partial v_1}(s) = 2(x_1(s) - v_1)$, so the 2-jet at s_0 is $2(sx_1'(s_0) + \frac{1}{2}s^2x_1''(s_0))$ and the 3-jet at s_0 is $2(sx_1'(s_0) + \frac{1}{2}s^2x_1''(s_0) + \frac{1}{3}s^3x_1^{(3)}(s_0))$. We also have $\frac{\partial G}{\partial v_i}(s) = -2(x_i(s) - v_i)$ ($i = 2, 3$), so the 2-jet at s_0 is $-2(sx_i'(s_0) + \frac{1}{2}s^2x_i''(s_0))$ and the 3-jet at s_0 is $-2(sx_i'(s_0) + \frac{1}{2}s^2x_i''(s_0) + \frac{1}{3}s^3x_i^{(3)}(s_0))$. The condition for (p)versality can be checked as follows:

- (i) By Proposition 2.2, g has the $A_{\geq 2}$ -singularity at s_0 if and only if $v = \gamma(s_0) + \frac{\varepsilon(\gamma(s_0))}{\delta(\gamma(s_0))k(s_0)}\mathbf{n}(s_0) - \mu\mathbf{b}(s_0)$ and $k(s_0), \tau(s_0) \neq 0$. When g has the $A_{\geq 2}$ -singularity

at s_0 , we require the 1×3 matrix $(2x'_1(s_0), -2x'_2(s_0), -2x'_3(s_0))$ to have rank 1, which it always does since γ is regular.

- (ii) It also follows from Proposition 2.2 that g has the $A_{\geq 3}$ -singularity at s_0 if and only if $v = (\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b})(s_0)$ and $k(s_0), \tau(s_0) \neq 0$. When g has A_2 at s_0 , we require the 2×3 matrix

$$\begin{pmatrix} 2x'_1(s_0) & -2x'_2(s_0) & -2x'_3(s_0) \\ x_1''(s_0) & -x_2''(s_0) & -x_3''(s_0) \end{pmatrix}$$

to have rank 2, which follows from the proof of the case (iii).

- (iii) By Proposition 2.3, g has the $A_{\geq 4}$ -singularity at s_0 if and only if $(kk'\tau' - kk''\tau - \varepsilon(\gamma)k^2\tau^3 + 2k'^2\tau)(s_0) = 0$ and $v = (\gamma + \frac{\varepsilon(\gamma)}{\delta(\gamma)k}\mathbf{n} + \frac{k'}{\varepsilon(\gamma)\delta(\gamma)k^2\tau}\mathbf{b})(s_0)$. Where $k(s_0), \tau \neq 0$. When g has $A_{\geq 4}$ at s_0 , we require the 3×3 matrix

$$\begin{pmatrix} 2x'_1(s_0) & -2x'_2(s_0) & -2x'_3(s_0) \\ x_1''(s_0) & -x_2''(s_0) & -x_3''(s_0) \\ \frac{2}{3}x_1^{(3)}(s_0) & -\frac{2}{3}x_2^{(3)}(s_0) & -\frac{2}{3}x_3^{(3)}(s_0) \end{pmatrix}$$

to be nonsingular, which it always does since the determinant of the above matrix is $\frac{4}{3}\varepsilon(\gamma(s_0))k^2(s_0)\tau(s_0) \neq 0$ by $k(s_0) \neq 0$ and $\tau(s_0) \neq 0$. This completes the proof. \square

For the proof of Theorem B, we consider the bifurcation set of H and G . By Propositions 2.1 and 2.2, the bifurcation set of H is

$$\mathfrak{B}_H = \{v \in \mathbb{R}_1^3 \mid v = \mathbf{b}(s)\},$$

and the bifurcation set of G is

$$\mathfrak{B}_G = \{v \in \mathbb{R}_1^3 \mid v = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}\mathbf{n}(s) + u\mathbf{b}(s) \text{ with the real number } u \text{ is not zero}\}.$$

The assertions of Theorem B follow from Propositions 2.1, 2.2 and Theorems 4.1, 4.2. \square

5. GENERIC PROPERTIES OF NONLIGHTLIKE CURVES

We now consider the notion of Lorentzian Monge-Taylor maps for timelike curves analogous to the ordinary notion of Monge-Taylor maps for space curves in Euclidean space (cf., [4]). (For spacelike curves, see [6].) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a (regular) timelike curve, with I an open connected subset of the unit circle S^1 , increasing t corresponding the anticlockwise orientation of S^1 . We now choose a smooth family of unit vectors $\mathbf{n}(t)$, with $\mathbf{n}(t)$ pseudonormal to γ at t , so $\|\mathbf{n}(t)\| = 1$ and $\langle \mathbf{n}(t), \mathbf{t}(t) \rangle = 0$ for all $t \in I$. Such $\mathbf{n}(t)$ can be obtained as follows: consider the smooth map $\mathbf{t} : I \rightarrow S_1^2$ which takes t to the unit tangent vector $\mathbf{t}(t)$. If V is any vector in S_1^2 we can obtain the vector field $\mathbf{n}(t)$ by pseudo-orthogonally projecting V onto each of the normal planes and normalizing. Thus $\mathbf{n}(t) = \frac{V + \langle V, \mathbf{t}(t) \rangle \mathbf{t}(t)}{\|V + \langle V, \mathbf{t}(t) \rangle \mathbf{t}(t)\|}$, then we have $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ and $\langle \mathbf{n}, \mathbf{t} \rangle = 0$. We can obtain a second smooth family of unit vectors $\mathbf{b}(t) = \mathbf{t}(t) \wedge \mathbf{n}(t)$ normal to γ at t . We remark that the triple $\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)$ is the Lorentzian frame along γ . We now use the

psuedoperpendicular lines spanned by $\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)$ as axes at $\gamma(t)$ with the unit points on the axes corresponding to the three given vectors. We remark that the Minkowski metric is invariant under the Lorentz transformation. Note the curve $\gamma(t)$ not necessarily unit speed, with $\gamma(t_0) = 0$. Then the coordinates η, ζ and χ of $\gamma(t)$ relative to axes \mathbf{t}, \mathbf{n} and \mathbf{b} are functions of t : $\zeta(t) = \gamma(t) \cdot \mathbf{t}(t_0)$, $\eta(t) = \gamma(t) \cdot \mathbf{n}(t_0)$, $\chi(t) = \gamma(t) \cdot \mathbf{b}(t_0)$, $\eta(t) = f(\zeta(t))$, $\chi(t) = g(\zeta(t))$ where $f = f_0$, $g = g_0$. So $j^k f_t(0) = a_2(t)\zeta^2 + a_3(t)\zeta^3 + \dots + a_k(t)\zeta^k$, $j^k g_t(0) = b_2(t)\zeta^2 + b_3(t)\zeta^3 + \dots + b_k(t)\zeta^k$ in the neighbourhood $(0, 0, 0)$. Locally then $\gamma(I)$ can be written in the form $\{\zeta, (f_t(\zeta), g_t(\zeta))\}$, with $f(0) = g(0) = j^1 f_t(0) = j^1 g_t(0) = 0$. If V_k denotes the space of polynomials in ζ of degree ≥ 2 and $\leq k$ we have a map, the *Lorentzian Monge-Taylor map* for the space curve γ , $\mu_\gamma : I \rightarrow V_k \times V_k$ given by $\mu_\gamma(t) = (j^k f_t(0), j^k g_t(0))$. ($V_k \times V_k$ can be identified with $\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} = \mathbb{R}^{2(k-1)}$ via the coordinates $(a_2, \dots, a_k, b_2, \dots, b_k)$.) Of course μ_γ depends rather heavily on our choice of unit normals $\mathbf{n}(t)$. Where, $a_i(t) = \frac{f_t(0)^{(i)}}{i!}$, $b_i(t) = \frac{g_t(0)^{(i)}}{i!}$ ($2 \leq i \leq k$), that is $V_k \times V_k =$

$$\{(a_2\zeta^2 + b_3\zeta^3 + \dots + b_k\zeta^k), (b_2\zeta^2 + b_3\zeta^3 + \dots + a_k\zeta^k)\}.$$

Let P_k denote the set of maps $\psi : \mathbb{R}_1^3 \rightarrow \mathbb{R}_1^3$ of the form $\psi(x, y, z) = (\psi_1(x, y, z), \psi_2(x, y, z), \psi_3(x, y, z))$ where $\psi_i(x, y, z)$ is a polynomial in x, y and z of degree $\leq k$. So an element $\psi \in P_k$ is determined by the coefficients of the various monomials $x^l y^i z^j$ occurring in ψ_1, ψ_2 and ψ_3 . There are altogether $1 + 3 + \dots + (2k - 1) + 2k + 1 = (k + 1)^2$ monomials of degree $\leq k$, so P_k can be thought of as a Euclidean space $\mathbb{R}^{(k+1)^2}$. It is this space which will provide the required deformations of the curve.

To simplify matters we now assume that the curve γ is proper (i.e., γ satisfies the condition that the inverse image of a compact set is compact) and $\gamma(I)$ is bounded. The identity map $1_{\mathbb{R}_1^3} : \mathbb{R}_1^3 \rightarrow \mathbb{R}_1^3$, is of course an element of P_k (provided $k \geq 1$), and using the above assumption of γ it easily follows that there is an open neighbourhood U of $1_{\mathbb{R}_1^3}$ in P_k with the property that if $\psi \in U$ then the linear map $T\psi(\gamma(t)) : \mathbb{R}_1^3 \rightarrow \mathbb{R}_1^3$; $v \mapsto D\psi(\gamma(t)) \cdot v$ satisfy $D\psi(\gamma(t)) \cdot \mathbf{n}(t)$ is spacelike vector and $D\psi(\gamma(t)) \cdot \mathbf{t}(t)$ is timelike vector. Where, $D\psi(\gamma(t))$ denotes the derivative of ψ at $\gamma(t)$. (In fact, in this case the two conditions can be written two conditions of algebraic inequality in the open neighbourhood of $\gamma(t)$, so by the compactness of $\gamma(S^1)$ the set which satisfy the two conditions is intersection of finite sets. Hence, it is open.) If we deform the original curve by the map ψ , then we can also obtain the required new smooth family of normal vectors $\mathbf{n}_\psi(t)$ as follows: since the map $\psi : \mathbb{R}_1^3 \rightarrow \mathbb{R}_1^3$ is a diffeomorphism on some open set containing $\gamma(I)$, the vector $\mathbf{n}(t)$ will be sent to some new spacelike vector $D\psi(\gamma(t))\mathbf{n}(t)$ which will be neither zero nor tangent to $\psi \circ \gamma$ at t . Orthogonally projecting this vector onto the normal plane to $\psi \circ \gamma$ at t and normalizing, that is $\mathbf{n}_\psi(t) = \frac{D\psi(\gamma(t))\mathbf{n}(t) + \langle D\psi(\gamma(t))\mathbf{n}(t), \mathbf{t}_\psi \rangle \mathbf{t}_\psi}{\|D\psi(\gamma(t))\mathbf{n}(t) + \langle D\psi(\gamma(t))\mathbf{n}(t), \mathbf{t}_\psi \rangle \mathbf{t}_\psi\|}$, $\langle \mathbf{n}_\psi(t), \mathbf{n}_\psi(t) \rangle = 1$. Where, \mathbf{t}_ψ denotes the tangent vector of the curve $\psi \circ \gamma$ at t . So assuming as the above, we choose an open neighbourhood U of $1_I \in P_k$ consisting of polynomial maps which map an open set containing $\gamma(I)$ diffeomorphically to its image. We have now shown that there is a smooth map

$$\mu : I \times U \rightarrow V_k \times V_k$$

defined by $\mu(-, \psi) =$ Monge-Taylor map for the curve $\psi \circ \gamma$ using the family of normal vectors $\mathbf{n}_\psi(t)$. By exactly the same arguments as those in the proof of Theorem 9.9 in [4], we have the following theorem.

Theorem 5.1. Let γ be a nonlightlike curve and Q a manifold in $V_k \times V_k = \mathbb{R}_1^{2k-2}$. For some open set $U_1 \subset U$ containing the identity map, the map $\mu : I \times U_1 \rightarrow V_k$ defined by $\mu(t, \psi) = \mu_{\psi \circ \gamma}(t)$ is transverse to Q . (In fact we can prove that μ is a submersion so Q does not enter the argument at all.)

In order to give a proof of Theorem A, we prepare some lemmas.

Lemma 5.2. (1) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a timelike curve defined by $\gamma(t) = (\zeta, f_t(\zeta), g_t(\zeta)) = (a_2\zeta^2 + a_3\zeta^3 + \dots, \zeta, b_2\zeta^2 + b_3\zeta^3 + \dots)$ with $\zeta(t_0) = 0$ and $k(s_0) \neq 0$. We denote that $M_{ij} = a_i b_j - a_j b_i$, $N_{ij} = a_i a_j - \varepsilon(\gamma) b_i b_j$. Then:

(a) $\tau(s_0) = \tau'(s_0) = 0$ at t_0 if and only if

$$\begin{cases} M_{32} = 0 \\ 2M_{42}N_{22} - 3M_{32}N_{23} = 0. \end{cases}$$

(b) $(kk'\tau' - kk''\tau + k^2\tau^3 + 2k'^2\tau)(s_0) = (kk'\tau' - kk''\tau + k^2\tau^3 + 2k'^2\tau)'(s_0) = 0$ at t_0 with $k'(s_0) \neq 0$ and $\tau(s_0) \neq 0$. if and only if

$$\begin{cases} A_1 = 4M_{42}N_{23}N_{22} + 9M_{23}N_{23}^2 - 4M_{32}N_{24}N_{22} - 3M_{32}N_{33}N_{22} - 4M_{32}N_{22}^3 + 3M_{32}^3 = 0 \\ A_2 = N_{23}(18M_{32}^3 + 12M_{43}N_{22}^2 + 20M_{52}N_{22}^2 + 48M_{24}N_{23}N_{22} - 72M_{23}N_{23}^2 \\ + 24M_{23}N_{24}N_{22} + 18M_{23}N_{22}N_{33}) - M_{32}(27N_{23}^3 + 72N_{23}N_{22}^2 - 36N_{23}N_{24}N_{22} \\ - 27N_{23}N_{22}N_{33} + 36N_{34}N_{22}^2 + 20N_{25}N_{22}^2 + 3N_{22}^2N_{33}) + 3(M_{32}^2 + N_{23}^2)(12M_{42}N_{22} \\ + 18M_{23}N_{23}) + 3M_{32}N_{23}(9N_{23}^2 + 12N_{24}N_{22} + 9N_{22}N_{33} + 12N_{22}^3) = 0. \end{cases}$$

(2) Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a spacelike curve defined by $\gamma(t) = (f_t(\zeta), \zeta, g_t(\zeta)) = (a_2\zeta^2 + a_3\zeta^3 + \dots, \zeta, b_2\zeta^2 + b_3\zeta^3 + \dots)$ with $\zeta(t_0) = 0$ and $k(s_0) \neq 0$. And we denote that $M_{ij} = a_i b_j - a_j b_i$, $N_{ij} = a_i a_j - \varepsilon(\gamma) b_i b_j$.

Then:

(a) $\tau = 0$ and $\tau' = 0$ at t_0 if and only if

$$\begin{cases} M_{32} = 0 \\ 2M_{42}|N_{22}| + 3M_{32}N_{32} = 0. \end{cases}$$

(b) By the direct calculation, $(kk'\tau' - kk''\tau - k^2\tau^3 + 2k'^2\tau)(s_0) = (2kk'\tau^3 - kk'\tau'' + kk^{(3)}\tau + 3k^2\tau^2\tau' - 3k'k''\tau - 3k'^2\tau')(s_0) = 0$ at t_0 with $k'(s_0) \neq 0$ and $\tau(s_0) \neq 0$ if and only if

$$\begin{cases} B_1 = \pm 4M_{42}N_{23}|N_{22}| + 3M_{23}N_{23}^2 \mp 4M_{32}N_{24}|N_{22}| \mp 3M_{32}N_{33}|N_{22}| + 4M_{32}|N_{22}|^2N_{22} \\ - 3M_{32}^3 = 0 \\ B_2 = \pm N_{23}(18M_{32}^3 + 12M_{42}|N_{22}|^2 + 20M_{52}|N_{22}|^2 \pm 48M_{24}N_{23}|N_{22}| - 72M_{23}N_{23}^2 \\ \pm 24M_{23}N_{24}|N_{22}| \pm 18M_{23}|N_{22}|N_{33}) + M_{32}(27N_{23}^3 + 72N_{23}N_{22}|N_{22}|^2 \\ - 36N_{23}N_{24}|N_{22}| - 27N_{23}|N_{22}|N_{33} \pm 36N_{34}|N_{22}|^2 \pm 20N_{25}|N_{22}|^2 \pm 3|N_{22}|^2N_{33}) \\ - 3M_{32}N_{23}(-9N_{23}^2 + 12N_{24}|N_{22}| + 9|N_{22}|N_{33} - 12N_{22}|N_{22}|^2) \\ - 3(M_{32}^2 - N_{23}^2)(12M_{42}|N_{22}| \pm 18M_{23}N_{23}) = 0. \end{cases}$$

Here ζ is the coordinate along the t -direction, $f_t(\zeta)$ along the n -direction and $g_t(\zeta)$ along the b -direction.

Since the proof of Lemma 5.2 is analogous to that of Lemma 5.2 in [6], we omit it.

Lemma 5.3. We consider smooth maps $\rho_i : V_4 \times V_4 = \mathbb{R}^6 \longrightarrow \mathbb{R}$ ($i = 1, 2$) given by

$$\begin{cases} \rho_1 = M_{32} \\ \rho_2 = 2M_{42}N_{22} - 3M_{32}N_{23}, \end{cases}$$

$\theta_i : V_4 \times V_4 = \mathbb{R}^6 \longrightarrow \mathbb{R}$ ($i = 1, 2$) given by

$$\begin{cases} \theta_1 = M_{32} \\ \theta_2 = 2M_{42}|N_{22}| + 3M_{32}N_{32}, \end{cases}$$

$\phi_i : V_5 \times V_5 = \mathbb{R}^8 \longrightarrow \mathbb{R}$ ($i = 1, 2$) given by

$$\begin{cases} \phi_1 = A_1 \\ \phi_2 = A_2, \end{cases}$$

and $\mu_i : V_5 \times V_5 = \mathbb{R}^8 \longrightarrow \mathbb{R}$ ($i = 1, 2$) given by

$$\begin{cases} \mu_1 = B_1 \\ \mu_2 = B_2. \end{cases}$$

Here, M_{ij} , N_{ij} , P_{ij} , A_i and B_i are given as the above. Then:

- (1) The set $Q_1 = \{(a_2, a_3, a_4, b_2, b_3, b_4) \in \mathbb{R}^6 \mid \rho_1 = \rho_2 = 0\}$ is a codimension two submanifold in \mathbb{R}^8 .
- (2) The set $Q_2 = \{(a_2, a_3, a_4, b_2, b_3, b_4) \in \mathbb{R}^6 \mid \theta_1 = \theta_2 = 0\}$ is a codimension two submanifold in \mathbb{R}^8 .
- (3) The set $Q_3 = \{(a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5) \in \mathbb{R}^8 \mid \phi_1 = \phi_2 = 0\}$ is a codimension two submanifold in \mathbb{R}^8 .
- (4) The set $Q_4 = \{(a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5) \in \mathbb{R}^8 \mid \mu_1 = \mu_2 = 0\}$ is a codimension two submanifold in \mathbb{R}^8 .

Proof. (1) The Jacobian matrix of the map (ρ_1, ρ_2) is calculated as follows:

$$J(\rho_1, \rho_2) = \begin{pmatrix} \frac{\partial \rho_1}{\partial a_2} & b_2 & 0 & \frac{\partial \rho_1}{\partial b_2} & -a_2 & 0 \\ \frac{\partial \rho_2}{\partial a_2} & \frac{\partial \rho_2}{\partial a_3} & 2b_2(\frac{k}{2})^2 & \frac{\partial \rho_2}{\partial b_2} & \frac{\partial \rho_2}{\partial b_3} & -2a_2(\frac{k}{2})^2 \end{pmatrix}$$

Since $k(s_0) = 2(a_2^2 + b_2^2)^{\frac{1}{2}} \neq 0$, the numbers a_2 and b_2 are not both zero. That is $\text{rank} J(\rho_1, \rho_2) = 2$. This means that Q_1 is a submanifold in \mathbb{R}^8 with codimension two.

(4) The Jacobian matrix of the map (μ_1, μ_2) is also calculated as follows:

$J(\mu_1, \mu_2) =$

$$\begin{pmatrix} \frac{\partial \mu_1}{\partial a_2} & \frac{\partial \mu_1}{\partial a_3} & \pm 4b_3(\frac{k}{2})^4 & 0 & \frac{\partial \mu_1}{\partial b_2} & \frac{\partial \mu_1}{\partial b_3} & \mp 4a_3(\frac{k}{2})^4 & 0 \\ \frac{\partial \mu_2}{\partial a_2} & \frac{\partial \mu_2}{\partial a_3} & \frac{\partial \mu_2}{\partial a_4} & \pm 5(a_2 + b_2)k^2 & \frac{\partial \mu_2}{\partial b_2} & \frac{\partial \mu_2}{\partial b_3} & \frac{\partial \mu_2}{\partial b_4} & \mp 5(a_2 + b_2)k^2 \end{pmatrix}$$

where $k = 2|a_2^2 - b_2^2|^{\frac{1}{2}} \neq 0$. Then $b_2 \pm a_2 \neq 0$. And either $a_3 \neq 0$ or $b_3 \neq 0$ by $\tau(s_0) \neq 0$. Hence $\text{rank}J(\phi_1, \phi_2) = 2$. So that Q_4 is a submanifold in \mathbb{R}^8 with codimension two.

The assertions (2) and (3) follows similar arguments as in the proofs of cases (1) and (4). This completes the proof. \square

We can use Theorem 5.1, Lemmas 5.2 and 5.3 for the proof of Theorem A exactly the same way as the proof of Corollary 9.7 in [4]. So we omit the detail here.

Acknowledgment. We would like to thank Professor S.Izumiya for proposing the problems in this paper and for helpful advice.

REFERENCES

1. J. W. Bruce, *On singularities, envelopes and elementary differential geometry*, Math.Proc.Camb.Phil.Soc. **89** (1981), 43–48.
2. J. W. Bruce and P. J. Giblin, *Generic curves and surfaces*, J. London Math. Soc. **24** (1981), 555–561.
3. J. W. Bruce and P. J. Giblin, *Generic Geometry*, Amer. Math. Monthly **90** (1983), 529–545.
4. J. W. Bruce and P. J. Giblin, *Curves and singularities (second edition)*, Cambridge University press, 1992.
5. D. L. Fidal and P. J. Giblin, *Generic 1-parameter families of caustics by reflexion in the plane*, Math. Proc. Camb. Phil. Soc. **96** (1984), 425–432.
6. S. Izumiya, D. Pei and T. Sano, *The lightcone Gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space*, Hokkaido University preprint series in Math. **411** (1998).
7. S. Izumiya and A. Takiyama, *A time-like surface in Minkowski 3-space which contains pseudocircles*, Proceedings of the Edinburgh Mathematical society **40** (1997), 127–136.
8. M. Kossowski, *The Null Blow-Up of a Surface in Minkowski 3-space and Intersection in the Spacelike Grassman*, Michigan Math. J. **38** (1991), 401–415.
9. D.K.H. Mochida, R.C. Romero-Fuster, and M.A. Ruas, *The geometry of surfaces in 4-space from a contact viewpoint*, Geometriae Dedicata **54** (1995), 323–332.
10. B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
11. D. Pei, *Singularities of $\mathbb{R}P^2$ -valued Gauss maps of surfaces in Minkowski 3-space* (to appear in Hokkaido Mathematical Journal).
12. I. Porteous, *The normal singularities of submanifold*, J. Diff. Geom. **5** (1971), 543–564.