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Strichartz estimates for wave equations in the homogeneous Besov space

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Abstract

We prove Strichartz estimates for wave equations in the homogeneous Besov space. The main purpose in this paper is to present a unified way to derive Strichartz estimates given by Bak-McMichael-Oberlin [1, Theorem 6'], Ginibre-Velo [3, Proposition 3.1], Harmse [5, Theorem 2.3] and Oberlin [10, Theorem 3]. Our argument proceeds under the abstract setting once we have used the stationary phase estimate for wave equations, and the main tool is the complex interpolation method, by which we shall obtain new Strichartz estimates.

1 Introduction

We consider the inhomogeneous wave equations

$$\begin{aligned} &\partial_t^2 u(t,x) - \Delta u(t,x) = f(t,x), \quad t \in \mathbf{R},\ x \in \mathbf{R}^n,\ n \geq 2, \\ &u(0,x) = \partial_t u(0,x) = 0, \end{aligned} \tag{1.1}$$

where n denotes the space dimension, f is a complex valued function on $\mathbf{R} \times \mathbf{R}^n$, and Δ denotes the Laplacian in space variables.

We shall prove Strichartz estimates of the following type

$$\|u; L^q(I, \dot{B}^{\rho}_{r,2})\| \leq C \|f; L^{\tilde{q}}(I, \dot{B}^{\tilde{\rho}}_{\tilde{r},2})\|, \tag{1.2}$$

where I denotes an interval in $\mathbf{R}$, $\dot{B}^{\rho}_{r,2}$ denotes the homogeneous Besov space defined later and the constant C is independent of f and I. For any $1 \leq q \leq \infty$ and a Banach space X, we write the mixed norm of a function $g : I \to X$ by

$$\begin{aligned} &\|g; L^q(I,X)\| = \{\textstyle\int_{t\in I} \|g(t); X\|^q dt\}^{1/q} \quad \text{for } 1 \leq q < \infty, \\ &\|g; L^\infty(I,X)\| = \sup_{t\in I} \|g(t); X\|. \end{aligned} \tag{1.3}$$

On the estimate (1.2), Ginibre and Velo in [3] have shown some generalization of almost all Strichartz-type estimates obtained up to that point, in which one of conditions necessary for (1.2) is given by

$$\rho + \delta(r) - 1/q = 2 + \tilde{\rho} + \delta(\tilde{r}) - 1/\tilde{q}, \quad \rho, \tilde{\rho} \in \mathbf{R}, \tag{1.4}$$

where $\delta(r) = n(1/2 - 1/r)$ (see [3, Proposition 3.1]). On the other hand, Harmse in [5], Oberlin in [10], Bak,McMichael and Oberlin in [1] have already shown the "off duality" estimates, namely (1.2) for $(n+1)/2n - 2/(n+1) < 1/r < (n-1)/2n$ with $q = r$, $\tilde{q} = \tilde{r}$ and $\rho = \tilde{\rho} = 0$ in (1.4). The above two results meet only on the original Strichartz estimate [11], otherwise they are independent.

We will present Strichartz estimates which involve the above results and have new ones. In our proof, once we have used the estimate derived from the stationary phase, we could proceed our argument under the abstract setting applying the unitarity of the operator $\exp(it\sqrt{-\Delta})$ $(i = \sqrt{-1})$, the duality argument, the Hardy-Littlewood-Sobolev inequality and the complex interpolation method. The key method is complex interpolation(see [2, Chapter 4] or Proposition 2. 1 below), by which we could loosen the conditions restricted by the Hardy-Littlewood-Sobolev inequality, therefore our results could involve [1, Theorem 6'], [5, Theorem 2.3] and [10, Theorem 3].

Our main result is Proposition 2. 2 . Locating the results by [1],[3],[5] and [10] to our results will be done in Remark 3 and Remark 4.

Recently Keel and Tao [7] have obtained the estimate at the "endpoint" by the real interpolation method. We shall introduce their results in somewhat extended direction as Lemma 2. 1 , and use that to supplement our methods in the critical cases.

2 Notation and propositions

As usually done, we will rewrite (1.1) to the integral equation. For that purpose, we introduce some operators defined on the tempered distributions $\mathcal{S}'(\mathbf{R}^n)$ or $\mathcal{S}'(\mathbf{R} \times \mathbf{R}^n)$. We denotes by ω^λ, $U(t)$ the operators on $\mathcal{S}'(\mathbf{R}^n)$ defined by $\omega^\lambda = (-\Delta)^{\lambda/2}$, $U(t) = \exp(it\sqrt{-\Delta})$, and by $G_0, G_\pm$ the integral operators defined by

$$G_0 f(t) = \int_0^t U(t-s) f(s) ds, \quad G_\pm f(t) = \int_{\pm\infty}^t U(t-s) f(s) ds, \tag{2.5}$$

for any function f in $\mathcal{S}'(\mathbf{R}^{n+1})$. We denote by G any of $G_0, G_\pm$, and by H the operator $\omega^{-1}G$. To show the required inequality (1.2), it suffices to show the boundedness of the operator H from $L^{\tilde{q}}(\mathbf{R}, \dot{B}^{\tilde{\rho}}_{\tilde{r},2}(\mathbf{R}^n))$ to $L^q(\mathbf{R}, \dot{B}^{\rho}_{r,2}(\mathbf{R}^n))$.

Here we shall introduce the homogeneous Besov space $\dot{B}^{\rho}_{r,s}(\mathbf{R}^n)$ for any $\rho \in \mathbf{R}$ and $1 \le r, s \le \infty$ (see also [2], [3] and [12]). For $1 \le q \le \infty$ and a normed space X, we denotes by $\ell^q_j(X)$ the space of $\{a_j\}_{j\in\mathbf{Z}}$, $a_j \in X$, with the norm given by

$$\begin{aligned} \|a_j; \ell^q_j(X)\| &= \{\textstyle\sum_{j\in\mathbf{Z}} \|a_j; X\|^q\}^{1/q} \quad \text{for } 1 \le q < \infty, \\ \|a_j; \ell^\infty_j(X)\| &= \sup_{j\in\mathbf{Z}} \|a_j; X\|. \end{aligned} \tag{2.6}$$

We denote by F the Fourier transform in $\mathbf{R}^n$, and by $*$ the convolution in space. Let $\{\varphi_j\}_{j\in\mathbf{Z}} \subset C^\infty(\mathbf{R}^n)$ such that

$$\mathrm{supp} F\varphi_j \subset \{x \mid 2^{j-1} < |x| < 2^{j+1}\}, \qquad \sum_{j\in\mathbf{Z}} F\varphi_j(x) = 1 \quad \text{for } |x| \neq 0. \tag{2.7}$$

We denote by $\dot{B}^{\rho}_{r,s}(\mathbf{R}^n)$ the space given by

$$\{u \in \mathcal{S}'(\mathbf{R}^n) \mid \|u; \dot{B}^{\rho}_{r,s}(\mathbf{R}^n)\| \equiv \|2^{\rho j}\varphi_j * u; \ell^s_j(L^r(\mathbf{R}^n))\| < \infty\}. \tag{2.8}$$

We make abbreviation such as $\dot{B}^{\rho}_{r} = \dot{B}^{\rho}_{r,2}(\mathbf{R}^n)$ and $L^q\dot{B}^{\rho}_{r} = L^q(\mathbf{R}, \dot{B}^{\rho}_{r})$.

The main tools in our paper are embeddings(see [2, Theorem 6.5.1])

$$\dot{B}^0_r \hookrightarrow L^r \quad \text{for } 2 \le r < \infty, \quad L^r \hookrightarrow \dot{B}^0_r \quad \text{for } 1 < r \le 2, \tag{2.9}$$

$$\dot{B}^{\rho}_r \hookrightarrow \dot{B}^{\rho_1}_{r_1} \quad \text{for } \rho \ge \rho_1 \text{ with } \rho - n/r = \rho_1 - n/r_1, \tag{2.10}$$

and the following complex interpolation method (see [2, Th 5.1.2, Th 6.4.5]). Let μ be a positive measure on $\mathbf{R}$, and for any Banach space X, let $L^q(\mathbf{R}, \mu; X)$ be the space of a function $f : \mathbf{R} \to X$ with the norm

$$\begin{aligned} &\{\textstyle\int_{\mathbf{R}} \|f; X\|^q d\mu\}^{1/q} && \text{for } 1 \le q < \infty, \\ &\sup_{t\in\mathbf{R}} \|f(t); X\| && \text{for } q = \infty. \end{aligned} \tag{2.11}$$

Proposition 2. 1 *Let $n \ge 1$. Let $1 \le s_0, s_1, r_0, r_1 \le \infty$, $1 \le q_0, q_1 < \infty$ and $\rho_0, \rho_1 \in \mathbf{R}$. Let K be an bounded operator from $L^{q_0}(\mathbf{R}, \mu; \dot{B}^{\rho_0}_{r_0})$ to $\dot{B}^0_{s_0}$, and from $L^{q_1}(\mathbf{R}, \mu; \dot{B}^{\rho_1}_{r_1})$ to $\dot{B}^0_{s_1}$. Then K is a bounded operator from $L^q(\mathbf{R}, \mu; \dot{B}^{\rho}_{r})$ to $\dot{B}^0_{s}$, where s, r, q, ρ are given by*

$$\begin{aligned} 1/s &= (1-\theta)/s_0 + \theta/s_1, \quad 1/r = (1-\theta)/r_0 + \theta/r_1, \\ 1/q &= (1-\theta)/q_0 + \theta/q_1, \quad \rho = (1-\theta)\rho_0 + \theta\rho_1, \end{aligned} \tag{2.12}$$

for any $0 \le \theta \le 1$.

In order to describe our statement in concise form, following Kato [6], it is convenient to use the following geometric notation. We denote by $\square$ the closed unit square in $\mathbf{R}^2$, defined by $0 \le x, y \le 1$. Throughout this paper we denote by Q and $\tilde{Q}$ the points $(1/q, 1/r)$ and $(1/\tilde{q}, 1/\tilde{r})$ in $\square$ respectively, and we write $x(Q) = 1/q$, $y(Q) = 1/r$. For $P, Q \in \square$, $[PQ]$ and (PQ) represent the closed and open segment connecting P and Q respectively. And $[PQ)$ denotes $[PQ] \backslash \{Q\}$. We denotes by q' the conjugate of q, namely $q' = q/(q-1)$ for $1 < q \le \infty$ and $q' = \infty$ for $q = 1$. And for $Q \in \square$, Q' denotes $(1/q', 1/r')$. We introduce some special points and sets in $\square$, by which it is convenient to state our propositions.

$$
\begin{aligned}
&O = (0,0), \quad A = (1,1), \quad B = (0,1/2), \quad C = (1/2, (n-3)/2(n-1)), \\
&(\ C = (1/4, 0) \text{ if } n = 2\ ), \quad D = (1/2, 0), \\
&E = (1, (n-3)/2(n-1)), \quad F = (0, (n-3)/2(n-1)), \\
&(\ E = D,\ F = O \text{ if } n = 2\ ), \\
&T_0 = [OBCD]\ (T_0 = [OBC] \text{ if } n = 2,\ T_0 = [OBC]\backslash\{C\} \text{ if } n = 3\ ), \\
&T = \{B\} \cup (BEF),
\end{aligned}
\tag{2.13}
$$

where $[OBCD]$ denotes the closure of the square defined by O, B, C, D, and (BEF) denotes the interior domain of the triangle defined by B, E, F. For a set S in $\square$, we denote by S' the set of the point Q' with $Q \in S$.

If we introduce the linear functionals

$$
\pi(Q) = 1/r + 2/(n-1)q, \quad \pi_1(Q) = 1/r + 1/(n-1)q, \tag{2.14}
$$

for Q in $\square$, then B and C are on the line defined by $\pi(Q) = 1/2$, B and E are on $\pi_1(Q) = 1/2$, B' and C' are on $\pi(Q) = (n+3)/2(n-1)$, B' and E' are on $\pi_1(Q) = (n+1)/2(n-1)$. The pair $(Q, \tilde{Q})$ will be called a *conjugate* pair if Q and $\tilde{Q}$ in $\square$ satisfy

$$
\pi(\tilde{Q}) = \pi(Q) + 2/(n-1). \tag{2.15}
$$

In particular, for $Q \in [BC]$ and $\tilde{Q} \in [B'C']$, $(Q, \tilde{Q})$ is a *conjugate* pair. We now refer to the following two properties. Let (Q, Q') be a *conjugate* pair. If $x(Q) = 0$ and $x(\tilde{Q}) = 1$, then $y(Q) = y(\tilde{Q})$. If Q is on $[BE]$ and $x(\tilde{Q}) = 1$, then $y(Q') = y(\tilde{Q})$.

We call the pair $(Q, \tilde{Q})$ *admissible* if the linear operator H is bounded from $L^{\tilde{q}}\dot{B}^{\tilde{\rho}}_{\tilde{r}}$ to $L^q\dot{B}^{\rho}_{r}$ for any ρ and $\tilde{\rho}$ in $\mathbf{R}$ such that

$$
\rho + \delta(r) - 1/q = 2 + \tilde{\rho} + \delta(\tilde{r}) - 1/\tilde{q}. \tag{2.16}
$$

Since ω^λ ($\lambda \in \mathbf{R}$) is an isomorphism from $\dot{B}^\rho_r$ to $\dot{B}^{\rho-\lambda}_r$, if the linear operator G is bounded from $L^{\tilde{q}}\dot{B}^{\tilde{\rho}}_{\tilde{r}}$ to $L^q\dot{B}^\rho_r$ for any ρ and $\tilde{\rho}$ in $\mathbf{R}$ such that

$$\rho + \delta(r) - 1/q = 1 + \tilde{\rho} + \delta(\tilde{r}) - 1/\tilde{q}, \tag{2.17}$$

then $(Q, \tilde{Q})$ is admissible.

We are now in a position to state our main proposition.

Proposition 2. 2 *Let $n \geq 2$. Let $(Q, \tilde{Q})$ be a conjugate pair with $x(Q) < x(\tilde{Q})$. And let Q and $\tilde{Q}$ satisfy one of the following conditions.*
(1) $\tilde{Q} \in T'$, $(n-3)/(n-1)\tilde{r}' \leq 1/r$ ($(n-3)/(n-1)\tilde{r}' < 1/r$ *for* $n = 3$). *Moreover* $\pi_1(Q) < 1/2$ *and* $0 < x(Q)$ *if* $\tilde{Q} \notin [B'C')$.

(2) $Q \in T$, $1/\tilde{r} \leq 1-(n-3)/(n-1)r$ ($1/\tilde{r} < 1-(n-3)/(n-1)r$ *for* $n = 3$). *Moreover* $\pi_1(\tilde{Q}) > (n+1)/2(n-1)$ *and* $x(\tilde{Q}) < 1$ *if* $Q \notin [BC)$.
Then the pair $(Q, \tilde{Q})$ is admissible.

Remark 1. Let $(Q, \tilde{Q})$ be an admissible pair with $\tilde{q} \neq \infty$ and $\tilde{r} \neq \infty$. Then $(\tilde{Q}', Q')$ is also an admissible pair. Indeed, H', the dual operator of H, is a bounded operator from $L^{q'}\dot{B}^{-\rho}_{r'}$ to $L^{\tilde{q}'}\dot{B}^{-\tilde{\rho}}_{\tilde{r}'}$, and (2.16) could be written as

$$-\tilde{\rho} + \delta(\tilde{r}') - 1/\tilde{q}' = 2 - \rho + \delta(r') - 1/q'. \tag{2.18}$$

Since H is written as a linear combination of H_0', $H_\pm'$, therefore $(\tilde{Q}', Q')$ is also an admissible pair. In this sense, the proof for the case (2) in Proposition 2. 2 follows from that of (1) immediately.

Remark 2. In Proposition 2. 2 , applying the Sobolev embedding theorem, we could take Q and $\tilde{Q}$ in $\Box$ more widely. For example, let $(Q, \tilde{Q})$ be an admissible pair, then for any $r_1, \tilde{r}_1$ with $0 \leq 1/r_1 \leq 1/r$ and $1/\tilde{r} \leq 1/\tilde{r}_1 \leq 1$, $((1/q, 1/r_1), (1/\tilde{q}, 1/\tilde{r}_1))$ is also an admissible pair (note that the embeddings $\dot{B}^\rho_r \hookrightarrow \dot{B}^{\rho_1}_{r_1}$ and $\dot{B}^{\tilde{\rho}_1}_{\tilde{r}_1} \hookrightarrow \dot{B}^{\tilde{\rho}}_{\tilde{r}}$ imply $\rho+\delta(r) = \rho_1+\delta(r_1)$ and $\tilde{\rho}+\delta(\tilde{r}) = \tilde{\rho}_1+\delta(\tilde{r}_1)$ in (2.16) respectively).

To show some typical examples the Sobolev embedding theorem applied to Proposition 2. 2 , we introduce a set S in $\Box$. For $\tilde{Q} \in T'$, let ν be the supremum of $x(Q)$ with $(Q, \tilde{Q})$ in Proposition 2. 2 with (1). Let now S be a set given by

$$
\begin{aligned}
S \equiv\ & \{Q \in \square \mid \pi(\tilde{Q}) \geq \pi(Q) + 2/(n-1),\ x(Q) < x(\tilde{Q}), \\
& \quad 0 < x(Q) \leq \nu\ (0 < x(Q) < \nu \ \text{ for } n = 3)\} \qquad \text{if } \tilde{Q} \in (B'C'E'), \\
S \equiv\ & B \cup \{Q \in \square \mid \pi_1(Q) < 1/2, \pi(\tilde{Q}) \geq \pi(Q) + 2/(n-1), \\
& \quad x(Q) \leq \nu\ (x(Q) < \nu \ \text{ for } n = 3), x(Q) < x(\tilde{Q})\} \ \text{ if } \tilde{Q} \in T' \backslash (B'C'E').
\end{aligned} \tag{2.19}
$$

For $Q \in T$, let S be the set defined by Q' as above, and let S' be the set of the point Q_1' with $Q_1 \in S$.

Corollary 2. 1 *Let $\tilde{Q} \in T'$ and $Q \in S$. Or let $Q \in T$ and $\tilde{Q} \in S'$. Then $(Q, \tilde{Q})$ is admissible.*

Proof) It suffices to prove the first case only. The other case would follow from its duality(see Remark 1).

For $\tilde{Q} \in (B'C'E')$, there exists a point Q_1 on the vertical line involving Q such that $(Q_1, \tilde{Q})$ is conjugate. Since $(Q_1, \tilde{Q})$ is admissible by Proposition 2. 2 , by the argument in Remark 2, the conclusion holds. The same argument also holds for $\tilde{Q} \in T' \backslash (B'C'E')$ if we take $\tilde{Q}_1$ in $T' \backslash (B'C'E')$ with $x(\tilde{Q}_1) = x(\tilde{Q})$ and $y(\tilde{Q}_1) \leq y(\tilde{Q})$, and take $Q_1 \in \square$ such that $(Q_1, \tilde{Q}_1)$ is conjugate with $x(Q_1) = x(Q)$ and $y(Q_1) \geq y(Q)$. $\square$

Remark 3. The most familiar Strichartz-type estimates are the mixed space-time estimates in the Lebesgue space. If the conjugate pair $(Q, \tilde{Q})$ satisfies (2.16) with $\rho = \tilde{\rho} = 0$, then it holds

$$
1/\tilde{r} - 1/r = 1/\tilde{q} - 1/q = 2/(n+1). \tag{2.20}
$$

Therefore if Q and $\tilde{Q}$ satisfy (2.20) and (1) or (2) in Proposition 2. 2 , then we have

$$
\|Hf; L^q L^r\| \leq C\|f; L^{\tilde{q}} L^{\tilde{r}}\|, \tag{2.21}
$$

for any $f \in L^{\tilde{q}} L^{\tilde{r}}$, where we have used the embedding (2.9). Especially for the diagonal case, namely $r = q$ and $\tilde{r} = \tilde{q}$, we obtain the estimate given by [1, theorem 6'], [5, Theorem 2.3], [10, Theorem 3]. Indeed for $(n+1)/2n - 2/(n+1) < 1/r \leq (n-1)/2(n+1)$, the above Q and $\tilde{Q}$ satisfy (1) in Proposition 2. 2 , and for $(n-1)/2(n+1) < 1/r < (n-1)/2n$, (2) in Proposition 2. 2 . In the above argument, Q is uniquely determined by $\tilde{Q}$ as (2.20). But we should note that if $(Q, \tilde{Q})$ in Corollary 2. 1 satisfies (2.16) with $\rho = \tilde{\rho} = 0$, then (2.21) also holds.

In Proposition 2. 2 , we must assume $x(Q) < x(\tilde{Q})$ and $x(Q) > 0$, or $x(\tilde{Q}) < 1$. The following proposition could give some supplements for the cases $x(Q) = x(\tilde{Q}) = 1/2$, $x(Q) = 0$ and $x(\tilde{Q}) = 1$.

Proposition 2. 3 *Let $2 \le r \le \infty$, $1 \le \tilde{q} \le 2 \le q \le \infty$. Let $\tilde{r} = r'$, and let $\pi(\tilde{Q}) > \pi(Q) + 2/(n-1)$. Then $(Q, \tilde{Q})$ is admissible.*

The results in Proposition 2. 3 for the case $1 < \tilde{q} < 2 < q < \infty$ are also obtained by Corollary 2. 1 and Remark 2.

Let now $\tilde{Q}$ be fixed with $x(\tilde{Q}) = 1$, and let Q_c be the point such that

$$\pi(\tilde{Q}) = \pi(Q_c) + 2/(n-1) \quad \text{and} \quad \pi_1(Q_c) = 1/2. \tag{2.22}$$

And let T_1, S_1 be the sets given by

$$T_1 \equiv \{Q \in \square \mid \pi(Q) \le 1/2, x(Q) \le 1/2 \ (x(Q) < 1/2 \text{ for } n = 3)\}, \tag{2.23}$$

$$S_1 \equiv T_1 \cup \{Q \in \square \mid \pi_1(Q) < 1/2, x(Q) < x(Q_c), x(Q) \le 1/2\}. \tag{2.24}$$

For Q with $x(Q) = 0$, let S_1 be the set given by Q' as above, and let S_1' be the set of the point Q_1' with $Q_1 \in S_1$.

Corollary 2. 2 *Let $1 \le \tilde{r} \le 2$, $\tilde{Q} = (1, 1/\tilde{r})$ and $Q \in S_1$. Or let $2 \le r \le \infty$, $Q = (0, 1/r)$ and $\tilde{Q} \in S_1'$. Then $(Q, \tilde{Q})$ is admissible.*

Proof) By Remark 1, it suffices to prove the first case only. For $Q \in S_1$ with $y(Q) \ge 1/\tilde{r}'$, $(Q, (1, y(Q')))$ is admissible by Proposition 2. 3 . Since $\tilde{Q}$ is above $(1, y(Q'))$, therefore by Remark 2, $(Q, \tilde{Q})$ is also admissible. From this, the case $Q \in S_1 \backslash T_1$ with $y(Q) < 1/\tilde{r}'$ also follows from by Remark 2. For the case $Q \in T_1$, since (Q, B') is admissible by Corollary 2. 1 , therefore $(Q, \tilde{Q})$ is admissible by Remark 2. □

Next we consider the case $q = \tilde{q} = 2$ with $n \ge 4$. In this case, applying Proposition 2. 3 and Remark 2, we were able to show the admissibility of $(Q, \tilde{Q})$ for any $Q \in (CD]$ and $\tilde{Q} \in (C'D']$. However the real interpolation method described in [7] could give some extension in this case. Namely with the proof in [7, section 6] slightly modified, we obtain the following lemma.

Lemma 2. 1 *Let $n \ge 4$. Let $(Q, \tilde{Q})$ be a conjugate pair. If $x(Q) = x(\tilde{Q}) = 1/2$ and*

$$(n-1)/2(n-2) < y(\tilde{Q}) < (n^2-5)/2(n-1)(n-2), \tag{2.25}$$

then $(Q, \tilde{Q})$ is admissible.

Since Lemma 2. 1 is obtained quite analogously to [7], we will use it without proof. To proceed our argument, it is convenient to introduce the linear functional

$$\pi_2(Q) = 1/r + 1/(n-2)q. \tag{2.26}$$

For $\tilde{Q}$ with $x(\tilde{Q}) = 1/2$, let T_2, S_2 be sets given by

$$T_2 \equiv \{Q \in \Box \mid \pi_2(Q) \leq 1/2, \pi(Q) \leq \pi(\tilde{Q}) + 2/(n-1), x(Q) \leq 1/2\}, \tag{2.27}$$

$$S_2 \equiv T_2 \;\text{ for }\; \tilde{Q} \in [C'D'], \quad S_2 \equiv T_2 \backslash [OB] \;\text{ for }\; \tilde{Q} \notin [C'D']. \tag{2.28}$$

For Q with $x(Q) = 1/2$, let S_2 be the set given by Q' as above, and let S_2' be the set of the point Q_1' with $Q_1 \in S_2$.

Corollary 2. 3 *Let $n \geq 4$. Let $\tilde{Q} \in \Box$ satisfy $x(\tilde{Q}) = 1/2$ and (2.25), and let $Q \in S_2$. Or let $Q \in \Box$ satisfy $x(Q) = 1/2$ and $(n-3)^2/2(n-1)(n-2) < y(Q) < (n-3)/2(n-2)$, and let $\tilde{Q} \in S_2'$. Then $(Q, \tilde{Q})$ is admissible.*

Proof) We prove the first case only by Remark 1. For $\tilde{Q}$ in Corollary 2. 3 , let Q_0 be the point such that $(Q_0, \tilde{Q})$ is conjugate and $x(Q_0) = 1/2$. Then $(Q, \tilde{Q})$ is admissible for any $Q \in [Q_0 D]$ by Lemma 2. 1 and Remark 2. Especially moreover if $\tilde{Q} \in [C'D']$, then applying the unitarity of the operator $U(t)$, we can immediately show that $(B, \tilde{Q})$ is also admissible. In the following, we prove the case $\tilde{Q} \in [C'D']$ only. The other case would follow analogously.

Let Q in S_2 with $0 < x(Q) < 1/2$. We could find a conjugate pair $(Q_1, \tilde{Q}_1)$ such that $Q_1 \in S_2$, $x(Q_1) = x(Q)$, $\pi_2(Q_1)$ is sufficiently close to $1/2$, and $x(\tilde{Q}_1) = x(\tilde{Q})$, $y(\tilde{Q}_1) \leq y(\tilde{Q})$, and $(Q_1, \tilde{Q}_1)$ satisfies (2) in Proposition 2. 2 , where we should note that the condition in Proposition 2. 2 ,

$$y(\tilde{Q}_1) \leq 1 - (n-3)y(Q_1)/(n-1), \tag{2.29}$$

is guaranteed since $(Q_1, \tilde{Q}_1)$ is conjugate with $x(\tilde{Q}_1) = 1/2$ and satisfies $\pi_2(Q_1) \leq 1/2$. Therefore $(Q_1, \tilde{Q}_1)$ is admissible, so that $(Q, \tilde{Q})$ is also admissible by Remark 2. $\Box$

3 Proof of the propositions

In this section, we will prove Proposition 2. 2 , and 2. 3 . By the stationary phase method and the scaling in the space variables, we have the estimate

$$\|\varphi_j * U(t)\psi; L^\infty\| \leq C \min\{2^{nj}, |t|^{-(n-1)/2} 2^{j(n+1)/2}\} \|\tilde{\varphi}_j * \psi; L^1\|, \tag{3.30}$$

for any $\psi \in \mathcal{S}'(\mathbf{R}^n)$ (see [GV, (3.12)]), where the constant C is independent of j, t, and $\tilde{\varphi}_j$ is given by $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. By the interpolation between (3.30) and the unitarity of $U(t)$ in $L^2(\mathbf{R}^n)$, we have

$$\|\varphi_j * U(t)\psi; L^r\| \le C \min\{2^{2j\delta(r)}, |t|^{-\gamma(r)} 2^{2j\beta(r)}\} \|\tilde{\varphi}_j * \psi; L^{r'}\|, \tag{3.31}$$

for any $2 \le r \le \infty$, where $\gamma(r) = (n-1)(1/2 - 1/r)$, and $\beta(r) = (n+1)/(1/4 - 1/2r)$. For any $\rho \in \mathbf{R}$, if we multiply (3.31) by $2^{j\rho}$ and take the norm in ℓ^2_j, then we have

$$\|U(t)\psi; \dot{B}^{\rho}_{r}\| \le C|t|^{-\gamma(r)} \|\psi; \dot{B}^{\rho+2\beta(r)}_{r'}\|, \tag{3.32}$$

where we have discarded the first term in the minimum and used the definition of the Besov space. We should note that the constant C in (3.32) is independent of ρ. We could rewrite (3.32) as

$$\|U(t-s)f(s); \dot{B}^{-\beta(r)}_{r}\| \le C|t-s|^{-\gamma(r)} \|f(s); \dot{B}^{-\beta(r')}_{r'}\|, \tag{3.33}$$

for any $f \in \mathcal{S}'(\mathbf{R}^{n+1})$. Let now $Q \in [BC)$ ($Q \in [BC]$ for $n = 2$). Integrating (3.33) over s with the L^q norm in time, and applying the Hardy-Littlewood-Sobolev inequality [4, p290], we obtain

$$\|Gf; L^q(\mathbf{R}, \dot{B}^{-\beta(r)}_{r})\| \le C\|f; L^{q'}(\mathbf{R}, \dot{B}^{-\beta(r')}_{r'})\|, \tag{3.34}$$

where by Lemma 2. 1 we may take Q for $Q \in [BC]$ for $n \ge 4$.

The following lemma follows from (3.34) immediately.

Lemma 3. 1 *Let $n \ge 2$. Let $Q \in T_0$, $\tilde{Q} \in T_0'$. Then $(Q, \tilde{Q})$ is admissible.*

Proof) By Remark 2, it suffices to consider the case $Q \in [BC]$ and $\tilde{Q} \in [B'C']$ ($Q \in [BC)$ and $\tilde{Q} \in [B'C')$ for $n = 3$). Let $\tilde{Q}$ be fixed. Applying the unitarity of $U(t)$ in L^2, and the Hölder inequality in space and time, we have

$$\|Gf(t); \dot{B}^0_2\|^2 \le C\|Gf; L^{\tilde{q}'} \dot{B}^{-\beta(\tilde{r}')}_{\tilde{r}'}\| \|f; L^{\tilde{q}} \dot{B}^{\beta(\tilde{r}')}_{\tilde{r}}\|, \tag{3.35}$$

therefore by (3.34) replaced Q with $\tilde{Q}'$, we have

$$\|Gf; L^\infty \dot{B}^0_2\| \le C\|f; L^{\tilde{q}} \dot{B}^{-\beta(\tilde{r})}_{\tilde{r}}\|. \tag{3.36}$$

Interpolating (3.34) replaced Q with $\tilde{Q}'$ and (3.36), we have

$$\|Gf; L^q \dot{B}^{-\beta(r)}_{r}\| \le C\|f; L^{\tilde{q}} \dot{B}^{-\beta(\tilde{r})}_{\tilde{r}}\|. \tag{3.37}$$

for any $Q \in [B\tilde{Q}']$, which means that $(Q, \tilde{Q})$ is admissible for any $Q \in [B\tilde{Q}']$. In the case $Q \notin [B\tilde{Q}']$, the same argument holds replaced Q with $\tilde{Q}'$ and $\tilde{Q}$ with Q'. Since $\tilde{Q}' \in [BQ]$, therefore $(\tilde{Q}', Q')$ is admissible. So that by Remark 1, $(Q, \tilde{Q})$ is also admissible. □

Remark 4. Lemma 3. 1 restricted with $Q \in T_0 \backslash \{C\}$ and $\tilde{Q} \in T_0' \backslash \{C'\}$ when $n \geq 4$ equals to that given by Ginibre and Velo [3, Proposition 3.1]. This result could be also obtained Corollary 2. 1 and 2. 3 combined.

Proof of Proposition 2. 2) If we denote by U' the dual operator of $U(t) : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^{n+1})$ then U' is written by the form

$$U'f \equiv \int_{\mathbf{R}} U(-s)f(s)ds \tag{3.38}$$

for any $f \in \mathcal{S}'(\mathbf{R}^{n+1})$. Applying the same argument on (3.36) to U', we have for any $Q \in [BC]$ ($Q \in [BC)$ for $n = 3$)

$$\|U'f; \dot{B}_2^0\| \leq C\|f; L^{q'}\dot{B}_{r'}^{-\beta(r')}\|. \tag{3.39}$$

By the duality argument, this is equivalent to

$$\|U(t)\psi; L^q\dot{B}_r^{-\beta(r)}\| \leq C\|\psi; \dot{B}_2^0\|. \tag{3.40}$$

Let q_1 and r_1 satisfy $(1/q_1, 1/r_1) \in [BC]$ ($(1/q_1, 1/r_1) \in [BC)$ for $n = 3$). We rewrite (3.32) and (3.40) as

$$\sup_{t\in\mathbf{R}} \||t|^{(n-1)/2}U(t)\psi; \dot{B}_\infty^{-(n+1)/2}\| \leq C\|\psi; \dot{B}_1^0\|, \tag{3.41}$$

$$\{\int \||t|^{(n-1)/2}U(t)\psi; \dot{B}_{r_1}^{-\beta(r_1)}\|^{q_1}|t|^{-q_1(n-1)/2}dt\}^{1/q_1} \leq C\|\psi; \dot{B}_2^0\|. \tag{3.42}$$

Therefore applying Proposition 2. 1 to the dual operator of $|t|^{(n-1)/2}U(t)$ with the weighted measure $|t|^{-q_1(n-1)/2}dt$, and then taking its duality, we obtain

$$\||t|^{\theta(n-1)/2}U(t)\psi; L^\ell\dot{B}_r^\alpha\| \leq C\|\psi; \dot{B}_{\tilde{r}}^0\| \tag{3.43}$$

for any $0 \leq \theta \leq 1$, where

$$1/\tilde{r} = (1+\theta)/2, \quad 1/r = (1-\theta)/r_1, \tag{3.44}$$

$$1/\ell = (1-\theta)/q_1, \quad \alpha = -\theta(n+1)/2 - (1-\theta)\beta(r_1). \tag{3.45}$$

Starting from (3.43), we have the mixed norm estimate for Gf as follows. Let g be a function in $\mathcal{S}'(\mathbf{R}^{n+1})$. Then we have

$$\langle Gf, g\rangle = \iint dsdt\langle U(t-s)f(s), g(t)\rangle. \tag{3.46}$$

Applying the Hölder inequality in space, and changing the variable t to τ as $\tau = t - s$, we have

$$|\langle Gf, g\rangle| \le C\iint ds d\tau \| |\tau|^{\theta(n-1)/2}U(\tau)f(s); \dot{B}^{\alpha}_{r}\|\cdot\| |\tau|^{-\theta(n-1)/2}g(s+\tau); \dot{B}^{-\alpha}_{r'}\|, \tag{3.47}$$

hence by the Hölder inequality in time and (3.43), we have for $1 \le \tilde{q} \le \infty$,

$$\begin{aligned} & |\langle Gf, g\rangle| \\ \le\ & C\int ds\|f(s); \dot{B}^{0}_{\tilde{r}}\| \cdot \| |\tau|^{-\theta(n-1)/2}g(s+\tau); L^{\ell'}_{\tau}\dot{B}^{-\alpha}_{r'}\|, && (3.48) \\ \le\ & C\|f(s); L^{\tilde{q}}\dot{B}^{0}_{\tilde{r}}\| \cdot \|\| |\tau|^{-\theta(n-1)/2}g(s+\tau); L^{\ell'}_{\tau}\dot{B}^{-\alpha}_{r'}\|; L^{\tilde{q}'}_{s}\|. && (3.49) \end{aligned}$$

On the second term of the last inequality, we have

$$\begin{aligned} & \|\| |\tau|^{-\theta(n-1)/2}g(s+\tau); L^{\ell'}_{\tau}\dot{B}^{-\alpha}_{r'}\|; L^{\tilde{q}'}_{s}\| \\ =\ & \{\textstyle\int\{\int |\tau|^{-\theta\ell'(n-1)/2}\|g(s+\tau); \dot{B}^{-\alpha}_{r'}\|^{\ell'}d\tau\}^{\tilde{q}'/\ell'}ds\}^{\ell'/\tilde{q}'\ell'}. \end{aligned} \tag{3.50}$$

Let $1 < r^* < \infty$ satisfy

$$1/r^* > 1/r^* - \ell'/\tilde{q}' = -\ell'\theta(n-1)/2 + 1 > 0, \tag{3.51}$$

then by the Hardy-Littlewood-Sobolev inequality, the last term in (3.50) could be replaced with the $L^{\ell' r^*}\dot{B}^{-\alpha}_{r'}$ norm of g. Therefore we have

$$|\langle Gf, g\rangle| \le C\|f; L^{\tilde{q}}\dot{B}^{0}_{\tilde{r}}\| \cdot \|g; L^{\ell' r^*}\dot{B}^{-\alpha}_{r'}\|. \tag{3.52}$$

It holds $\ell' r^* \ne \infty$ and $r' \ne \infty$ by the definitions of ℓ, r^* and r, so that we obtain

$$\|Gf; L^{(\ell' r^*)'}\dot{B}^{\alpha}_{r}\| \le C\|f; L^{\tilde{q}}\dot{B}^{0}_{\tilde{r}}\|, \tag{3.53}$$

for any $f \in \mathcal{S}'(\mathbf{R}^{n+1})$. Or equivalently we obtain the following lemma.

Lemma 3. 2 *Let $(Q, \tilde{Q})$ be a conjugate pair with $0 \le x(Q) < x(\tilde{Q}) < 1$, $y(Q) \le y(\tilde{Q}')$ and*

$$(n-3)/(n-1)\tilde{r}' \le 1/r \quad (\ (n-3)/(n-1)\tilde{r}' < 1/r \text{ for } n = 3). \tag{3.54}$$

And let $\pi_1(\tilde{Q}) > (n+1)/2(n-1)$. Then $(Q, \tilde{Q})$ is admissible.

Proof of Lemma 3. 2) For r and $\tilde{r}$ in Lemma 3. 2 , let θ, r_1 be $\theta = 2/\tilde{r} - 1$, $1/r_1 = \tilde{r}'/2r$. Then r_1 satisfies $1/r_1 \le 1/2$ and

$$(n-3)/2(n-1) \le 1/r_1 \quad ((n-3)/2(n-1) < 1/r_1 \text{ for } n = 3), \tag{3.55}$$

and θ, r_1 satisfy (3.44). We note that ℓ and α in (3.45) could be written as $1/\ell = (n-1)/(1/2 - 1/2r - 1/2\tilde{r})$ and $\alpha = (n+1)(1/2r - 1/2\tilde{r})$, and $\theta(n-1)/2$ in (3.51) as $(n-1)(1/\tilde{r} - 1/2)$. Therefore if we define r^* by the equality in (3.51), and put $q = (\ell' r^*)'$, then the condition $1 < r^* < \infty$ is written by

$$x(Q) < 1, \quad \pi_1(\tilde{Q}) > (n+1)/2(n-1). \tag{3.56}$$

And the condition (3.51) is satisfied if $(Q, \tilde{Q})$ is a conjugate pair with $x(Q) < x(\tilde{Q}) < 1$. Since for the conjugate pair $(Q, \tilde{Q})$ and the above α, the gap condition (2.17) is satisfied by (3.53), therefore $(Q, \tilde{Q})$ is admissible. □

Let Q and $\tilde{Q}$ be those in Proposition 2. 2 with (1). If $y(Q) \le y(\tilde{Q}')$ and $\tilde{Q} \ne B'$, then $(Q, \tilde{Q})$ is admissible by Lemma 3. 2 . In the case $y(Q) > y(\tilde{Q}')$ and $Q \ne B$, since $(\tilde{Q}', Q')$ is admissible by Lemma 3. 2 , therefore $(Q, \tilde{Q})$ is also admissible by Remark 1. For the cases $\tilde{Q} = B'$ or $Q = B$, the proofs follow from Lemma 3. 1 . By Remark 1, the proof of Proposition 2. 2 is completed. □

Proof of Proposition 2. 3) From (3.31), we have

$$\begin{aligned} &\|\varphi_j * Gf; L^r\| \\ \le\ & C \int \min\{2^{2j\delta(r)}, |t-s|^{-\gamma(r)} 2^{2j\beta(r)}\} \|\tilde{\varphi}_j * f(s); L^{r'}\| ds. \end{aligned} \tag{3.57}$$

Let $1/q^* \equiv 1/q + 1/\tilde{q}'$. Then it holds $0 \le 1/q^* \le 1$ and $\gamma(r) q^* > 1$ by the assumption in Proposition 2. 3 . Taking the L^q norm in time in (3.57) and applying the Young inequality in time, we have

$$\begin{aligned} &\|\varphi_j * Gf; L^q L^r\| \\ \le\ & C \|\min\{2^{2j\delta(r)}, |t|^{-\gamma(r)} 2^{2j\beta(r)}\}; L_t^{q^*}\| \cdot \|\tilde{\varphi}_j * f; L^{\tilde{q}} L^{r'}\| \\ \le\ & C 2^{1/q^*} (\gamma(r) q^* / (\gamma(r) q^* - 1))^{1/q^*} 2^{(2\delta(r) - 1/q^*) j} \|\tilde{\varphi}_j * f; L^{\tilde{q}} L^{r'}\| \end{aligned} \tag{3.58}$$

by explicit calculation. Since $\tilde{q} \le 2 \le q$, applying the Minkowski inequality when taking the ℓ_j^2 norm in (3.58) yields

$$\|Gf; L^q \dot{B}_r^0\| \le C \|f; L^{\tilde{q}} \dot{B}_{r'}^{2\delta(r) - 1/q^*}\|, \tag{3.59}$$

for any $f \in L^{\tilde{q}}\dot{B}_{r'}^{2\delta(r)-1/q^*}$. Since $2\delta(r) - 1/q^*$ could be written as

$$\delta(r) - 1/q = 1 + 2\delta(r) - 1/q^* + \delta(r') - 1/\tilde{q}, \tag{3.60}$$

therefore (3.59) shows that $(Q, \tilde{Q})$ is admissible. So that the proof of Proposition 2. 3 is completed. □

Remark 5. We have some comments on the inhomogeneous Schrödinger equations,

$$i\partial_t u(t,x) - \Delta u(t,x) = f(t,x), \quad (t,x) \in \mathbf{R}^{n+1}. \tag{3.61}$$

As referred in [1], [7], the results for wave equations are mostly true for Schrödinger equations with a slight modification. Indeed, since it holds for Schrödinger equations

$$\| \exp(-it\Delta)\psi; L^r(\mathbf{R}^n)\| \leq C|t|^{-n(1/2-1/r)}\|\psi; L^{r'}(\mathbf{R}^n)\|, \tag{3.62}$$

for any $2 \leq r \leq \infty$ and $\psi \in L^{r'}(\mathbf{R}^n)$ instead of (3.32), therefore our results not applied the Sobolev embedding theorem are also true for Schrödinger equations if we replace $\gamma(r)$ with $n(1/2 - 1/r)$, $\dot{B}_r^\rho$ with L^r, and $\dot{B}_{\tilde{r}}^{\tilde{\rho}}$ with $L^{\tilde{r}}$. For example, let now replace n with $n+1$ in (2.13), (2.14). Let us call the pair $(Q, \tilde{Q})$ *s-conjugate* if it satisfies (2.15) replaced n with $n+1$, and *s-admissible* if the following inequality holds

$$\| \int_J \exp(-i(t-s)\Delta)f(s)ds; L^q(\mathbf{R}, L^r(\mathbf{R}^n))\| \leq C\|f; L^{\tilde{q}}(\mathbf{R}, L^{\tilde{r}}(\mathbf{R}^n))\|, \tag{3.63}$$

for any $J = [0,t]$, $(\pm\infty, t]$, $\mathbf{R}$, and $f \in L^{\tilde{q}}(\mathbf{R}, L^{\tilde{r}}(\mathbf{R}^n))$. Then Proposition 2. 2 , Lemma 2. 1 , Lemma 3. 1 with $Q \in [BC]$ and $\tilde{Q} \in [B'C']$, are also true replaced n with $n+1$, *conjugate* with *s-conjugate* and *admissible* with *s-admissible.*

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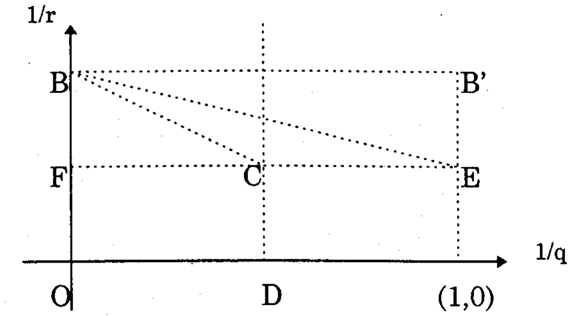
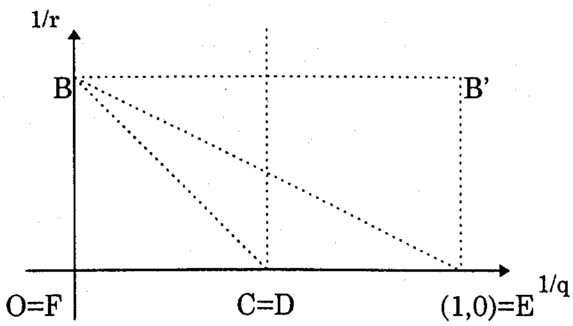
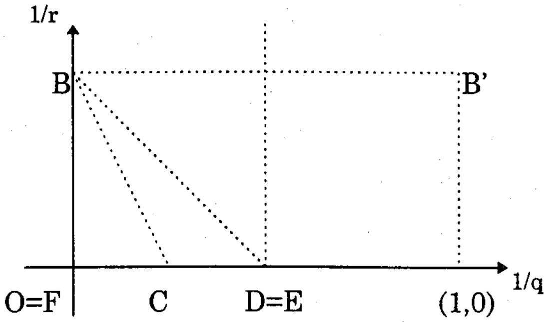


Fifure 1. n>3.



Fifure 2. n=3.



Fifure 3. n=2.

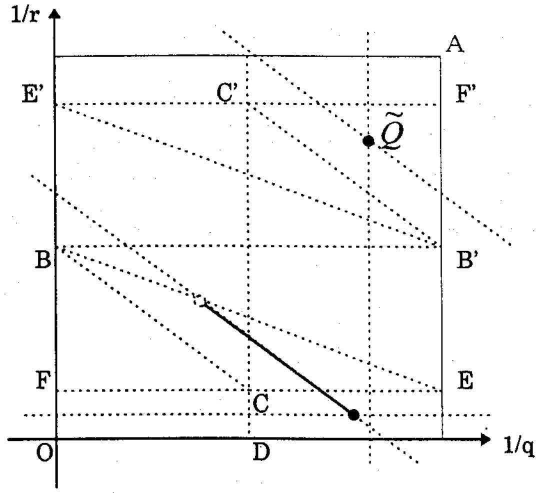


Figure 4. Proposition 2.2 with (1) (n>3).

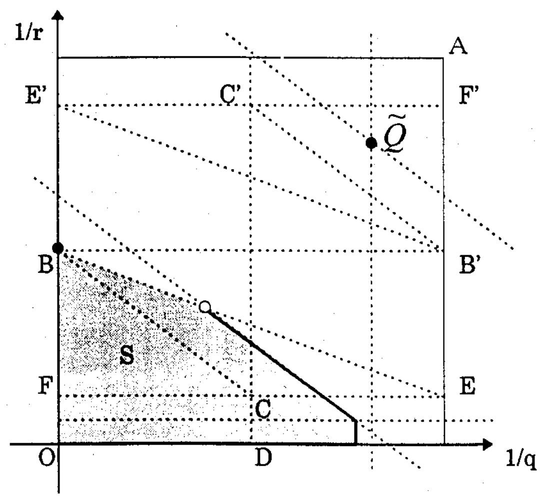


Figure 5. Corollary 2.1 (n>3).

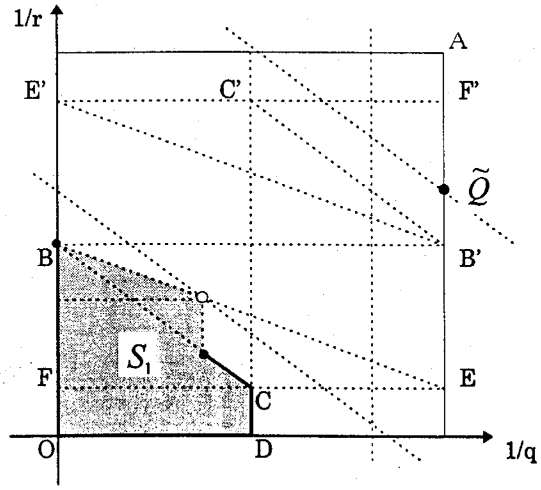


Figure 6. Corollary 2.2 (n>3).

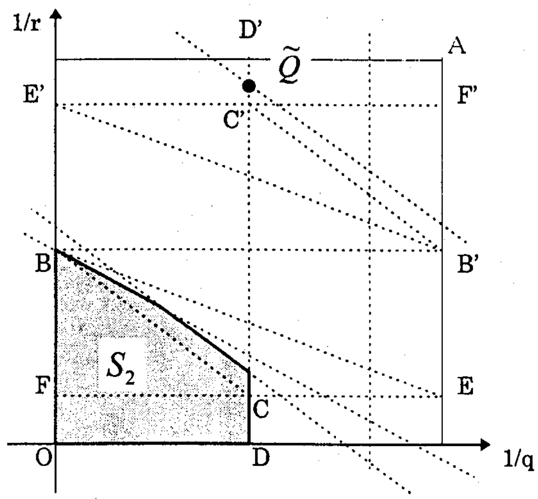


Figure 7. Corollary 2.3.