On the Essential Spectra of Quantum Field Hamiltonians

Asao Arai

Series #426. September 1998

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#401 M. Nakamura and T. Ozawa, Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth, 9 pages. 1998.

#402 D. Lehmann and T. Suwa, Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties, 19 pages. 1998.

#403 T. Nakazi and K. Okubo, ρ-contraction and 2×2 matrix, 6 pages. 1998.

#404 Y. Kohsaka, Free boundary problem for quasilinear parabolic equation with fixed angle of contact to a boundary, 28 pages. 1998.

#405 K. Yokoyama, Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions, 25 pages. 1998.

#406 F. Hiroshima, Ground states and spectrum of quantum electrodynamics of non-relativistic particles, 58 pages. 1998.

#407 N. Kawazumi and T. Uemura, Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries, 9 pages. 1998.

#408 T. Nakazi and K. Okubo, Generalized Numerical Radius And Unitary ρ-Dilation, 12 pages. 1998.

#409 Y. Giga and K. Ito, Loss of convexity of simple closed curves moved by surface diffusion, 16 pages. 1998.

#410 Y. Giga, K. Inui and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, 34 pages. 1998.

#411 S. Izumiya, D. Pei and T. Sano, The lightcone gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, 16 pages. 1998.

#412 M. Tsujii, Absolutely continuous invarient measures for piecewise real-analytic expanding maps on the plane, 18 pages. 1998.

#413 J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, 40 pages. 1998.

#414 T. Suwa, Dual class of a subvariety, 19 pages. 1998.

#415 M. Tsujii, Piecewise expanding maps on the plane with singular ergodic properties, 9 pages. 1998.

#416 T. Yoshida, Categorical aspects of generating functions(I): Exponential formulas and Krull-Schmidt categories, 44 pages. 1998.

#417 K. Yamaguchi, G_2-Geometry of overdetermined systems of second order, 22 pages. 1998.

#418 M. Ishikawa and S. Matsui, Existence of a forward self-similar stagnation flow of the Navier-Stokes equations, 8 pages. 1998.

#419 S. Izumiya, H. Katsumi and T. Yamasaki, The rectifying developable and the spherical Darboux image of a space curve, 16 pages. 1998.

#420 R. Kobayashi and Y. Giga, Equations with singular diffusivity, 45 pages. 1998.

#421 D. Pei and T. Sano, The focal developable and the binormal indicatrix of a nonlightlike curve in Minkowski 3-space, 14 pages. 1998.

#422 R. Kobayashi, J. A. Warren and W. C. Carter, Modeling grain boundaries using a phase field technique, 12 pages. 1998.

#423 T. Tsukada, Reticular Legendrian Singularities, 28 pages. 1998.

#424 A. N. Kirillov and M. Shimozono, A generalization of the Kostka-Foulkes polynomials, 37 pages. 1998.

#425 M. Nakamura, Strichartz estimates for wave equations in the homogeneous Besov space, 17 pages. 1998.

On the Essential Spectra of Quantum Field Hamiltonians

Asao Arai
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
e-mail: arai@math.sci.hokudai.ac.jp

September 20, 1998

Abstract

We present a method to locate the essential spectrum of a self-adjoint operator on the tensor product of a Hilbert space and the abstract boson Fock space, and discuss its application to Hamiltonians of quantum field models.

1 Introduction

The standard method to locate (partly) the spectrum of the Hamiltonian H of a quantum field model mainly consists of two parts: (i) proving the existence of a ground state of H and (ii) constructing asymptotic fields or wave operators ([22, 27], [17, Chapter 13] and references therein). In this paper, we concentrate our attention on the essential spectra of quantum field Hamiltonians and present a new method to locate them. The method is based on commutation properties of the Hamiltonian under consideration with the creation operator of the relevant Fock space. These properties enable one to apply Weyl's criterion on essential spectrum.

We first establish in Section 2 fundamental results on the essential spectrum of a self-adjoint operator acting on the tensor product of a Hilbert space and the abstract boson Fock space. This self-adjoint operator is an abstract version of the Hamiltonians of the $P(\phi)_2$ model (e.g., [16, 17, 34]) and models of quantum particles coupled to a Bose field. Recently mathematical studies of the latter models have made important progress in some respects, e.g., existence of ground states and of resonances, including the case where the Bose field is massless, and asymptotic completeness (e.g., [9, 10, 11, 13, 15, 18, 19, 20, 21, 23, 24, 25, 26, 29, 35, 36]). In Section 3 we discuss the application of the main results of Section 2 to three types of quantum field models: the generalized spin-boson model[9], a model of quantum particles coupled to a Bose field in a translation-invariant way (an abstract version of Nelson's model [28] with particles in an external scalar potential) and the Pauli-Fierz model [30].

2 Results in an Abstract Framework

Let $\mathcal{K}$ be a Hilbert space and

$$\mathcal{F}_{\mathrm{b}}(\mathcal{K}) := \oplus_{n=0}^{\infty}(\otimes_{\mathrm{s}}^{n}\mathcal{K}) \tag{2.1}$$

be the boson Fock space over $\mathcal{K}$, where $\otimes_{\mathrm{s}}^{n}\mathcal{K}$ is the n-fold symmetric tensor product of $\mathcal{K}$ with convention $\otimes_{\mathrm{s}}^{0}\mathcal{K} := \mathbf{C}$ [31, §II.4, Example 2]. Let $\mathcal{H}$ be a Hilbert space. Then one can make a Hilbert space

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K}). \tag{2.2}$$

Let S be a nonnegative self-adjoint operator on $\mathcal{K}$ with $\ker S = \{0\}$ and A be a self-adjoint operator on $\mathcal{H}$ bounded below. We denote by $d\Gamma(S)$ the second quantization of S [31, §VIII.10, Example 2]. Then

$$H_0 := A \otimes I + I \otimes d\Gamma(S) \tag{2.3}$$

is self-adjoint on its natural domain $D(H_0) := D(A\otimes I) \cap D(I \otimes d\Gamma(S))$ and bounded below, where I denotes identity operator and $D(\cdot)$ operator domain. Let H_I be a symmetric operator on $\mathcal{F}$ such that $D(H_0) \cap D(H_I)$ is dense in $\mathcal{F}$. Then we are concerned with the symmetric operator

$$H := H_0 + H_I \tag{2.4}$$

on $\mathcal{F}$ $[D(H) := D(H_0) \cap D(H_I)]$.

We denote by $a(f)$ the annihilation operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ with "test vector" $f \in \mathcal{K}$ [32, §X.7]. By definition, $a(f)$ is densely defined, closed and antilinear in f.

We denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $(\cdot\,,\cdot)_{\mathcal{X}}$ (complex linear in the second variable) and $\|\cdot\|_{\mathcal{X}}$ respectively, but, if there is no danger of confusion, then we write them simply as $(\cdot\,,\cdot)$ and $\|\cdot\|$.

The following fact is well known [6, Lemma 2.1]: For all $f \in D(S^{-1/2})$, $D(d\Gamma(S)^{1/2}) \subset D(a(f)^*) \cap D(a(f))$ and, for all $\Psi \in D(d\Gamma(S)^{1/2})$,

$$\|a(f)\Psi\| \le \|S^{-1/2}f\|\|d\Gamma(S)^{1/2}\Psi\|, \tag{2.5}$$
$$\|a(f)^*\Psi\| \le \|S^{-1/2}f\|\|d\Gamma(S)^{1/2}\Psi\| + \|f\|\|\Psi\|. \tag{2.6}$$

For convenience, we introduce a notion of weak commutator of two operators [7].

Definition 2.1 Let $\mathcal{X}$ be a Hilbert space. Let B and C be densely defined linear operators on $\mathcal{X}$ with the following property: there exist a dense subspace $\mathcal{W}$ of $\mathcal{X}$ and a linear operator L such that $\mathcal{W} \subset D(L) \cap D(B) \cap D(B^*) \cap D(C) \cap D(C^*)$ and

$$(B^*\psi, C\phi) - (C^*\psi, B\phi) = (\psi, L\phi) \tag{2.7}$$

for all $\psi, \phi \in \mathcal{W}$. Then we say that the couple $\langle B, C\rangle$ has the weak commutator on $\mathcal{W}$, defined by

$$[B, C]_{\mathrm{w},\mathcal{W}} := L|\mathcal{W},$$

where, for a linear operator T and a subspace $D \subset D(T)$, $T|D$ denotes the restriction of T to D.

Remark 2.1 (i) A linear operator L satisfying (2.7) is uniquely determined on $\mathcal{W}$, since $\mathcal{W}$ is dense.

(ii) If $\langle B, C\rangle$ has the weak commutator on $\mathcal{W}$, then B and C are closable, since $D(B^*)$ and $D(C^*)$ are dense.

(iii) For linear operators X and Y on a Hilbert space $\mathcal{X}$, their commutator $[X, Y]$ is defined by

$$D([X,Y]) := D(XY) \cap D(YX), \quad [X,Y] := XY - YX. \tag{2.8}$$

If X and Y are bounded with $D(X) = D(Y) = \mathcal{X}$, then, for every dense subspace $\mathcal{W}$, the couple $\langle X, Y\rangle$ has the weak commutator on $\mathcal{W}$ and $[X,Y]_{\mathrm{w},\mathcal{W}} = [X,Y]|\mathcal{W}$.

The case where B and C has the weak commutator on $\mathcal{W}$ as in Definition 2.1 is equivalent to the case where the commutator $[B, C]$ *in the sense of sesquilinear form* on $\mathcal{W} \times \mathcal{W}$ defines a linear operator.

We now state our hypothesis.

Hypothesis (H)

(H.1) H is essentially self-adjoint.

(H.2) The operator H has an eigenvalue $E \in \mathbf{R}$ with an eigenvector $\Psi_E \in D(H)$: $H\Psi_E = E\Psi_E$, $\|\Psi_E\| = 1$.

(H.3) For each $f \in D(S) \cap D(S^{-1/2})$, the couple $\langle H_I, I \otimes a(f)^*\rangle$ has the weak commutator on $D(H)$ and, for all sequences $\{f_n\}_{n=1}^{\infty} \subset D(S) \cap D(S^{-1/2})$ such that $\|f_n\| = 1$, $n \geq 1$ and $\text{w-}\lim_{n\to\infty} f_n = 0$ (w-lim denotes weak limit) and all $\Psi \in D(H)$,

$$\lim_{n\to\infty} [H_I, I \otimes a(f_n)^*]_{\mathrm{w},D(H)} \Psi = 0.$$

For a self-adjoint operator T, we denote by $\sigma(T)$ and by $\sigma_{\mathrm{ess}}(T)$ the spectrum and the essential spectrum of T respectively. If T is bounded below, then we define

$$E_0(T) := \inf \sigma(T). \tag{2.9}$$

A nonzero vector in $\ker(T - E_0(T))$ is called a ground state of T.

For a closable operator T (resp. a set F of $\mathbf{R}$), we denote its closure by $\overline{T}$ (resp. $\overline{F}$).

Theorem 2.2 *Assume (H). Then*

$$E + \overline{\sigma_{\mathrm{ess}}(S) \setminus \{0\}} \subset \sigma_{\mathrm{ess}}(\overline{H}). \tag{2.10}$$

As a corollary to this theorem, we have the following.

Corollary 2.3 *Let* $\sigma_{\mathrm{ess}}(S) = [0, \infty)$. *Suppose that* $\overline{H}$ *is bounded below and assume (H) with* $E = E_0(\overline{H})$. *Then*

$$\sigma_{\mathrm{ess}}(\overline{H}) = [E_0(\overline{H}), \infty). \tag{2.11}$$

Proof. By the present assumption, $\overline{\sigma_{\rm ess}(S)\setminus\{0\}} = [0,\infty)$. Hence, by Theorem 2.2, we have $[E_0(\overline{H}),\infty) \subset \sigma_{\rm ess}(\overline{H})$. The converse inclusion relation is obvious. Thus (2.11) follows. ■

Remark 2.2 The condition (H.2) with $E = E_0(\overline{H})$ corresponds to the existence of a ground state of $\overline{H}$ belonging to the domain $D(H)$. In concrete models, the assumption $\sigma_{\rm ess}(S) = [0,\infty)$ is satisfied in the case where S is a one-particle Hamiltonian of a massless boson.

Before going into the proof of Theorem 2.2, we first recall some basic facts. The operator

$$\widetilde{H}_0 := H_0 - E_0(A) \tag{2.12}$$

is nonnegative and self-adjoint. It is easy to see that $D(I\otimes d\Gamma(S)^{1/2}) \supset D(\widetilde{H}_0^{1/2})$ and, for all $\Psi \in D(\widetilde{H}_0^{1/2})$,

$$\|I\otimes d\Gamma(S)^{1/2}\Psi\| \le \|\widetilde{H}_0^{1/2}\Psi\|. \tag{2.13}$$

Combining these facts with (2.5) and (2.6), we obtain the following lemma.

Lemma 2.4 *For all $f \in D(S^{-1/2})$, $D(\widetilde{H}_0^{1/2}) \subset D(I\otimes a(f)^*)\cap D(I\otimes a(f))$ and, for all $\Psi \in D(\widetilde{H}_0^{1/2})$,*

$$\begin{aligned} \|I\otimes a(f)\Psi\| &\le \|S^{-1/2}f\|\|\widetilde{H}_0^{1/2}\Psi\|, & (2.14)\\ \|I\otimes a(f)^*\Psi\| &\le \|S^{-1/2}f\|\|\widetilde{H}_0^{1/2}\Psi\| + \|f\|\|\Psi\|. & (2.15)\end{aligned}$$

Let $\Omega_0 := \{1,0,0,\cdots\} \in \mathcal{F}_{\rm b}(\mathcal{K})$ be the Fock vacuum in $\mathcal{F}_{\rm b}(\mathcal{K})$. For a subspace $M \subset \mathcal{K}$, we denote by $\mathcal{F}_{0,\rm fin}(M)$ the subspace algebraically spanned by Ω_0 and all vectors of the form

$$a(f_1)^*\cdots a(f_n)^*\Omega_0,\ n\ge 1,\ f_j\in M,\ j=1,\cdots,n.$$

If M is dense in $\mathcal{K}$, then $\mathcal{F}_{0,\rm fin}(M)$ is dense in $\mathcal{F}_{\rm b}(\mathcal{K})$.

Lemma 2.5 *Let $f \in D(S^{-1/2})\cap D(S)$. Then the couple $\langle H_0, I\otimes a(f)^*\rangle$ has the weak commutator on $D(H_0)$ and*

$$[H_0, I\otimes a(f)^*]_{{\rm w},D(H_0)} = I\otimes a(Sf)^*|D(H_0). \tag{2.16}$$

Proof. By Lemma 2.4, $D(H_0) \subset D(I\otimes a(f))\cap D(I\otimes a(f)^*)$. Let $\Psi \in D(A)\otimes_{\rm alg} \mathcal{F}_{0,\rm fin}(D(S))$, where $\otimes_{\rm alg}$ means algebraic tensor product. Then $\Psi \in D(H_0 I\otimes a(f)^*)\cap D(I\otimes a(f)^* H_0)$ and $[H_0, I\otimes a(f)^*]\Psi = I\otimes a(Sf)^*\Psi$. Hence, for all $\Phi \in D(H_0)$, we have

$$(H_0\Phi, I\otimes a(f)^*\Psi) - (I\otimes a(f)\Phi, H_0\Psi) = (\Phi, I\otimes a(Sf)^*\Psi). \tag{2.17}$$

Since $D(A)\otimes_{\rm alg}\mathcal{F}_{0,\rm fin}(D(S))$ is a core of H_0 and we have Lemma 2.4, (2.17) extends, via a simple limiting argument, to all $\Psi \in D(H_0)$. Thus the desired result follows. ■

The following criterion on essential spectrum (Weyl's criterion) is well known (e.g., [1, Lemma 5.19]).

Lemma 2.6 *Let T be a self-adjoint operator on a Hilbert space. Then $\lambda \in \sigma_{\text{ess}}(T)$ if and only if there exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T)$ such that $\|\psi_n\| = 1$, $\text{w-}\lim_{n\to\infty} \psi_n = 0$ and $\lim_{n\to\infty}(T-\lambda)\psi_n = 0$.*

We use also the following fact, a strengthened version of the necessary condition for $\lambda \in \sigma_{\text{ess}}(T)$ in Lemma 2.6.

Lemma 2.7 *Let T be a nonnegative self-adjoint operator on a Hilbert space with $\ker T = \{0\}$ and $0 < a \le 1$. Let $\lambda \in \sigma_{\text{ess}}(T) \setminus \{0\}$. Then there exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T) \cap D(T^{-a})$ such that $\|\psi_n\| = 1$, $n \ge 1$, $\text{w-}\lim_{n\to\infty} \psi_n = 0$, $\lim_{n\to\infty}(T-\lambda)\psi_n = 0$ and $\lim_{n\to\infty} T^{-a}(T-\lambda)\psi_n = 0$.*

Proof. There exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T)$ with the properties described in Lemma 2.6. Since $\lambda > 0$, we can choose $\{\psi_n\}_{n=1}^{\infty}$ such that the support of the measure $\|E_T(\cdot)\psi_n\|^2$ ($E_T(\cdot)$ is the spectral measure of T) is included in an interval $[\delta, \infty)$ with $\delta > 0$ a constant independent of n. Hence

$$\int_{\mathbf{R}} \mu^{-2a} d\|E_T(\mu)\psi_n\|^2 = \int_{[\delta,\infty)} \mu^{-2a} d\|E_T(\mu)\psi_n\|^2 \le \frac{1}{\delta^{2a}}\|\psi_n\|^2 < \infty.$$

Hence $\psi_n \in D(T^{-a})$. A similar calculation shows that $\|T^{-a}(T-\lambda)\psi_n\| \le \delta^{-a}\|(T-\lambda)\psi_n\| \to 0$ $(n \to \infty)$. ■

Proof of Theorem 2.2

Let $\Psi \in D(H)$ and $f \in D(S) \cap D(S^{-1/2})$. Then we have by Lemma 2.5 and Hypothesis (H)

$$\begin{aligned}((H-E)\Psi, I\otimes a(f)^*\Psi_E) &= (H\Psi, I\otimes a(f)^*\Psi_E) - (I\otimes a(f)\Psi, H\Psi_E) \\ &= \left(\Psi, \left\{I\otimes a(Sf)^* + [H_I, I\otimes a(f)^*]_{\text{w},D(H)}\right\}\Psi_E\right).\end{aligned}$$

Since H is essentially self-adjoint, it follows that $I\otimes a(f)^*\Psi_E \in D(\overline{H})$ and

$$(\overline{H}-E)I\otimes a(f)^*\Psi_E = \left\{I\otimes a(Sf)^* + [H_I, I\otimes a(f)^*]_{\text{w},D(H)}\right\}\Psi_E. \tag{2.18}$$

Now let $\lambda \in \sigma_{\text{ess}}(S) \setminus \{0\}$. Then, by Lemma 2.7, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset D(S) \cap D(S^{-1/2})$ such that $\|f_n\| = 1$, $\text{w-}\lim_{n\to\infty} f_n = 0$, $\lim_{n\to\infty}(S-\lambda)f_n = 0$ and $\lim_{n\to\infty} S^{-1/2}(S-\lambda)f_n = 0$. In general, we can show by using the canonical commutation relations of $a(f)$ and $a(f)^*$ ($f \in \mathcal{K}$) and a limiting argument that every $\Psi \in D(I\otimes a(f))$ is in $D(I\otimes a(f)^*)$ and

$$\|I\otimes a(f)^*\Psi\|^2 = \|f\|^2\|\Psi\|^2 + \|I\otimes a(f)\Psi\|^2. \tag{2.19}$$

Hence

$$\|I\otimes a(f)^*\Psi\|^2 \ge \|f\|^2\|\Psi\|^2. \tag{2.20}$$

Using this inequality, we see that the vector $I\otimes a(f_n)^*\Psi_E$ is not zero with

$$\|I\otimes a(f_n)^*\Psi_E\| \ge 1. \tag{2.21}$$

Hence we can define a unit vector

$$\Psi_n := \frac{I \otimes a(f_n)^* \Psi_E}{\|I \otimes a(f_n)^* \Psi_E\|}.$$

Then, by (2.18), we have

$$(\overline{H} - E - \lambda)\Psi_n = \frac{I \otimes a((S-\lambda)f_n)^* \Psi_E}{\|I \otimes a(f_n)^* \Psi_E\|} + \frac{[H_I, I \otimes a(f_n)^*]_{\mathrm{w},D(H)} \Psi_E}{\|I \otimes a(f_n)^* \Psi_E\|}.$$

Hence

$$\begin{aligned} \|(\overline{H} - E - \lambda)\Psi_n\| &\leq \|S^{-1/2}(S-\lambda)f_n\| \|I \otimes d\Gamma(S)^{1/2}\Psi_E\| \\ &\quad + \|(S-\lambda)f_n\| + \|[H_I, I \otimes a(f_n)^*]_{\mathrm{w},D(H)}\Psi_E\| \\ &\to 0 \ (n \to \infty). \end{aligned}$$

Hence $\|(\overline{H} - E - \lambda)\Psi_n\| \to 0$ $(n \to \infty)$. To show that $\text{w-}\lim_{n\to\infty} \Psi_n = 0$, it is sufficient to prove that, for all $\Psi \in \mathcal{H} \otimes_{\mathrm{alg}} \mathcal{F}_{0,\mathrm{fin}}(\mathcal{K})$, $\lim_{n\to\infty}(\Psi, \Psi_n) = 0$. So let $\Psi = \psi \otimes a(g_1)^* \cdots a(g_m)^* \Omega_0$ ($\psi \in \mathcal{H}$, $g_j \in \mathcal{K}, j = 1, \cdots, m$). Then we have

$$\begin{aligned} |(\Psi, \Psi_n)| &\leq \left| \sum_{j=1}^{m} (g_j, f_n)(\psi \otimes a(g_1)^* \cdots a(g_{j-1})^* a(g_{j+1})^* \cdots a(g_m)^* \Omega_0, \Psi_E) \right| \\ &\to 0 \ (n \to \infty). \end{aligned}$$

These facts and Lemma 2.6 imply that $\lambda + E \in \sigma_{\mathrm{ess}}(\overline{H})$. Hence $E + \sigma_{\mathrm{ess}}(S) \setminus \{0\} \subset \sigma_{\mathrm{ess}}(\overline{H})$. Since the essential spectrum of a self-adjoint operator is closed, (2.10) follows. ■

3 Application to Quantum Field Models

In this section we apply Theorem 2.2 and Corollary 2.3 to the Hamiltonians of quantum field models.

We first present an abstract fact on weak commutators.

Lemma 3.1 *Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ be a dense subspace of $\mathcal{X}$. Let B and C be densely defined linear operators on $\mathcal{X}$ such that $\mathcal{W} \subset D(B) \cap D(B^*) \cap D(C) \cap D(C^*)$. Suppose that there exist a densely defined closed linear operator T on $\mathcal{X}$ and a core D_c of T with the following properties:*

(i) $D_c \subset \mathcal{W} \subset D(T)$.

(ii) *B and C are T-bounded on D_c.*

(iii) *$D_c \subset D(BC) \cap D(CB)$ (so that $[B, C]$ is defined on D_c) and*

$$L := [B, C]|D_c$$

is T-bounded on D_c.

(iv) $D(B^*C^*) \cap D(C^*B^*)$ *is dense.*

Then L is closable with $D(T) \subset D(\overline{L})$ and the couple $\langle B, C\rangle$ has the weak commutator on $\mathcal{W}$ given by

$$[B, C]_{\mathrm{w},\mathcal{W}} = \overline{L}|\mathcal{W}.$$

Proof. By property (iii), L is densely defined and $L^* \supset [C^*, B^*]$. By (iv), $D([C^*, B^*])$ is dense. Hence $D(L^*)$ is dense. Therefore L is closable. By the T-boundedness of L on D_{c}, there exist constants $a, b \geq 0$ such that, for all $\eta \in D_{\mathrm{c}}$,

$$\|L\eta\| \leq a\|T\eta\| + b\|\eta\|. \tag{3.1}$$

Since D_{c} is a core of T, it follows from a limiting argument that $D(T) \subset D(\overline{L})$ and (3.1) extends to all $\eta \in D(T)$, L being replaced by $\overline{L}$.

Let $\phi \in \mathcal{W}$. Then, by (i), there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset D_{\mathrm{c}}$ such that $\phi_n \to \phi$, $T\phi_n \to T\phi\,(n \to \infty)$. For all $\psi \in \mathcal{W}$, we have $(B^*\psi, C\phi_n) - (C^*\psi, B\phi_n) = (\psi, L\phi_n)$. The T-boundedness and the closability of B and C [Remark 2.1(ii)] imply that $B\phi_n \to \overline{B}\phi = B\phi$ and $C\phi_n \to \overline{C}\phi = C\phi$ as $n \to \infty$. By (3.1) with L replaced by $\overline{L}$ and $\eta \in D(T)$, we have $L\phi_n \to \overline{L}\phi\,(n \to \infty)$. Hence we obtain $(B^*\psi, C\phi) - (C^*\psi, B\phi) = (\psi, \overline{L}\phi)$. Thus the desired result follows. ■

3.1 The generalized spin-boson model

For each $f \in \mathcal{H}$, the operator

$$\phi(f) := \frac{1}{\sqrt{2}}(a(f)^* + a(f)), \tag{3.2}$$

called the Segal field operator, is essentially self-adjoint on $\mathcal{F}_{0,\mathrm{fin}}(\mathcal{H})$ [32, §X.7, Theorem X.41]. We denote its closure by the same symbol.

The first of the quantum field models to which we apply the abstract results established in Section 2 is the generalized spin-boson(GSB) model, which was introduced in [9] as an abstract unification of some nonrelativistic quantum field models. The Hilbert space for this model is taken to be

$$\mathcal{F}_{\mathrm{GSB}} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbf{R}^\nu)) \tag{3.3}$$

(i.e., the case where $\mathcal{K} = L^2(\mathbf{R}^\nu)$, ν being an arbitrarily fixed natural number). The Hamiltonian of the model is given by

$$H_{\mathrm{GSB}} := A \otimes I + I \otimes d\Gamma(\hat{\omega}) + \alpha \sum_{j=1}^{J} B_j \otimes \phi(\lambda_j). \tag{3.4}$$

Here A is as in Section 2, B_j $(j = 1, \cdots, J; J < \infty)$ are symmetric operators on $\mathcal{H}$, $\lambda_j \in L^2(\mathbf{R}^\nu)\,(j = 1, \cdots, J)$, $\hat{\omega}$ is the multiplication self-adjoint operator of a nonnegative function ω on $\mathbf{R}^\nu$ and $\alpha \in \mathbf{R}$ is a coupling parameter. We need the following conditions:

(GSB.1) Each λ_j is continuous on $\mathbf{R}^\nu$ with $\lambda_j/\sqrt{\omega} \in L^2(\mathbf{R}^\nu)$, $j = 1, \cdots, J$.

(GSB.2) Let

$$\widetilde{A} := A - E_0(A) \geq 0.$$

Then $D(\widetilde{A}^{1/2}) \subset D(B_j), j = 1, \cdots, J$, and there exist constants $a_j \geq 0,\ b_j \geq 0,\ j = 1, \cdots, J$, such that, for all $u \in D(\widetilde{A}^{1/2})$,

$$\|B_j u\| \leq a_j \|\widetilde{A}^{1/2} u\| + b_j \|u\|, \quad j = 1, \cdots, J.$$

and

$$|\alpha| \left(\sum_{j=1}^{J} a_j \left\| \frac{\lambda_j}{\sqrt{\omega}} \right\|_{L^2(\mathbf{R}^\nu)} \right) < 1.$$

(GSB.3) The function $\omega(k)$ $(k \in \mathbf{R}^\nu)$ is continuous on $\mathbf{R}^\nu$ with

$$\lim_{|k| \to \infty} \omega(k) = \infty$$

and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma (1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbf{R}^\nu.$$

We write

$$H_{\mathrm{GSB}} = H_0^{\mathrm{GSB}} + H_I^{\mathrm{GSB}} \tag{3.5}$$

with

$$H_0^{\mathrm{GSB}} := A \otimes I + I \otimes d\Gamma(\hat{\omega}), \quad H_I^{\mathrm{GSB}} := \alpha \sum_{j=1}^{J} B_j \otimes \phi(\lambda_j). \tag{3.6}$$

It is shown that, under conditions (GSB.1) and (GSB.2), H_{GSB} is self-adjoint with $D(H_{\mathrm{GSB}}) = D(H_0^{\mathrm{GSB}})$ $[D(H_0^{\mathrm{GSB}}) \subset D(H_I^{\mathrm{GSB}})]$ and bounded below.

For each $f \in L^2(\mathbf{R}^\nu)$, we define

$$L(f) := \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{J} (\lambda_j, f)_{L^2(\mathbf{R}^\nu)} B_j \otimes I. \tag{3.7}$$

Let

$$\mathcal{D}_{\mathrm{GSB}} := D(H_0^{\mathrm{GSB}}) \cap D(H_I^{\mathrm{GSB}}) = D(H_0^{\mathrm{GSB}}) = D(A \otimes I) \cap D(I \otimes d\Gamma(\hat{\omega})). \tag{3.8}$$

Lemma 3.2 *Assume (GSB.1) and (GSB.2). Then, for all $f \in D(\hat{\omega}^{-1/2})$, the couple $\langle H_I^{\mathrm{GSB}}, I \otimes a(f)^* \rangle$ has the weak commutator on $\mathcal{D}_{\mathrm{GSB}}$ and*

$$[H_I^{\mathrm{GSB}}, I \otimes a(f)^*]_{\mathrm{w}, \mathcal{D}_{\mathrm{GSB}}} = L(f) | \mathcal{D}_{\mathrm{GSB}}. \tag{3.9}$$

In particular, for all sequences $\{f_n\}_{n=1}^\infty \subset D(\hat{\omega}) \cap D(\hat{\omega}^{-1/2})$ such that $\|f_n\| = 1, n \geq 1$ and $\mathrm{w}\text{-}\lim_{n \to \infty} f_n = 0$ and all $\Psi \in \mathcal{D}_{\mathrm{GSB}}$,

$$\lim_{n \to \infty} [H_I^{\mathrm{GSB}}, I \otimes a(f_n)^*]_{\mathrm{w}, \mathcal{D}_{\mathrm{GSB}}} \Psi = 0. \tag{3.10}$$

Proof. The dense subspace $D_c := D(A) \otimes_{\text{alg}} \mathcal{F}_{0,\text{fin}}(D(\hat{\omega}))$ is a core of H_0^{GSB}. It is easy to see that

$$D_c \subset D(H_I^{\text{GSB}} I \otimes a(f)^*) \cap D(I \otimes a(f)^* H_I^{\text{GSB}}) \cap D(H_I^{\text{GSB}} I \otimes a(f)) \cap D(I \otimes a(f) H_I^{\text{GSB}})$$

and, for all $\Psi \in D_c$,

$$[H_I^{\text{GSB}}, I \otimes a(f)^*]\Psi = L(f)\Psi.$$

The operators $I \otimes a(f)^*$ and H_I^{GSB} are H_0^{GSB}-bounded. We can show also that $L(f)$ is H_0^{GSB}-bounded. Hence we can apply Lemma 3.1 with $B = H_I^{\text{GSB}}$, $C = I \otimes a(f)^*$, $T = H_0^{\text{GSB}}$ and $\mathcal{W} = \mathcal{D}_{GSB}$ to obtain the first half of the lemma.

By (3.7), we have for all $\Psi \in \mathcal{D}_{\text{GSB}}$

$$\|L(f_n)\Psi\| \leq \frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^{J} |(\lambda_j, f_n)_{L^2(\mathbf{R}^\nu)}| \|B_j \otimes I\Psi\|.$$

Since $(\lambda_j, f_n)_{L^2(\mathbf{R}^\nu)} \to 0$ $(n \to \infty)$, we have $\|L(f_n)\Psi\| \to 0$ $(n \to \infty)$. Hence (3.10) follows. ■

Let

$$m := \inf_{k \in \mathbf{R}^\nu} \omega(k) \geq 0. \tag{3.11}$$

Lemma 3.3 [9, Theorem 1.2]. *Let $m > 0$. Assume (GSB.1)–(GSB.3) and that A has compact resolvent. Then H_{GSB} has purely discrete spectrum in $[E_0(H_{\text{GSB}}), E_0(H_{\text{GSB}})+m)$. In particular, H_{GSB} has a ground state.*

Theorem 3.4 *Under the same assumption as in Lemma 3.3,*

$$\sigma_{\text{ess}}(H_{\text{GSB}}) = [m, \infty). \tag{3.12}$$

Proof. By Lemmas 3.2 and 3.3, H_{GSB} satisfies Hypothesis (H) with $H = H_{\text{GSB}}, H_0 = H_0^{\text{GSB}}$ and $H_I = H_I^{\text{GSB}}$. Thus we can apply Theorem 2.2 to obtain $[E_0(H_{\text{GSB}}) + m, \infty) \subset \sigma_{\text{ess}}(H_{\text{GSB}})$. [Note that $\sigma_{\text{ess}}(\hat{\omega}) = [m, \infty)$.] By Lemma 3.3, the converse inclusion relation holds. Thus (3.12) follows. ■

We next consider the case $m = 0$.

Lemma 3.5 [9, Theorem 1.3] *Let $m = 0$ and suppose that (GSB.1)–(GSB.3) hold with additional conditions $\lambda_j/\omega \in L^2(\mathbf{R}^\nu)$, $j = 1, \cdots, J$ and that A has compact resolvent. Then there exists a constant $\alpha_0 > 0$ such that, for all $|\alpha| < \alpha_0$, H_{GSB} has a ground state.*

Remark 3.1 An estimate for α_0 from below is obtained from (1.35) in [9].

Theorem 3.6 *Under the same assumption as in Lemma 3.5,*

$$\sigma_{\text{ess}}(H_{\text{GSB}}) = [E_0(H_{\text{GSB}}), \infty). \tag{3.13}$$

Proof. In this case $\sigma_{\text{ess}}(\hat{\omega}) = [0, \infty)$. We can apply Corollary 2.3 to obtain (3.13). ■

Remark 3.2 Theorems 3.4 and 3.6 can be extended to a model whose Hamiltonian is of the form

$$\widetilde{H}_{\text{GSB}} := H_{\text{GSB}} + \sum_{j=1}^{J} [C_j \otimes \phi(g_j)]^2,$$

where $g_j \in L^2(\mathbf{R}^\nu)$ and C_j is a bounded self-adjoint operator on $\mathcal{H}$. See [8].

3.2 A model of quantum particles coupled to a Bose field

In this subsection we consider a model of N quantum particles interacting with a Bose field ($N \in \mathbf{N}$), where the particles move in $\mathbf{R}^\nu$. To treat such a quantum system rigorously, we introduce some notions.

We take the Hilbert space of the system of N quantum particles to be $L^2(\mathbf{R}^{\nu N})$. A Hilbert space for the coupled system of the N particles and the Bose field whose one-particle Hilbert space is $\mathcal{K}$ is given by

$$\mathcal{F}_{\mathrm{pf}}(\mathcal{K}) := L^2(\mathbf{R}^{\nu N}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K}). \tag{3.14}$$

This Hilbert space has a natural identification with

$$L^2(\mathbf{R}^{\nu N}; \mathcal{F}_{\mathrm{b}}(\mathcal{K})) = \int_{\mathbf{R}^{\nu N}}^{\oplus} \mathcal{F}_{\mathrm{b}}(\mathcal{K}) dx, \tag{3.15}$$

the Hilbert space of $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$-valued Lebesgue square integrable functions on $\mathbf{R}^{\nu N}$ [the constant fibre direct integral with base space $(\mathbf{R}^{\nu N}, dx)$ and fibre $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$]. The identification is given by the unitary correspondence given by $\psi \otimes \Psi \to \psi(x)\Psi$ ($\psi \in L^2(\mathbf{R}^{\nu N}), \Psi \in \mathcal{F}_{\mathrm{b}}(\mathcal{K})$, $x = (x_1, \cdots, x_N) \in \mathbf{R}^{\nu N}$, $x_j \in \mathbf{R}^\nu$, $j = 1, \cdots, N$) [31, §II.4, Theorem II.10(b)]. In what follows we use this identification freely.

Let $g : \mathbf{R}^{\nu N} \to \mathcal{K}$ be a strongly continuous $\mathcal{K}$-valued function on $\mathbf{R}^{\nu N}$. Then, for all $t \in \mathbf{R}$, the unitary operator-valued function $:x \to \exp[it\phi(g(x))]$ on $\mathbf{R}^{\nu N}$ is strongly continuous [32, §X.7, Theorem X.41(d)]. Hence it follows that the operator-valued funciton:$x \to (\phi(g(x)) + i)^{-1}$ on $\mathbf{R}^{\nu N}$ is strongly continuous. Thus we can define a decomposable self-adjoint operator

$$[\phi](g) := \int_{\mathbf{R}^{\nu N}}^{\oplus} \phi(g(x)) dx \tag{3.16}$$

on $\mathcal{F}_{\mathrm{pf}}(\mathcal{K})$[33, §XIII.16, Theorem XIII.85(a)].

We denote by Δ_j the ν-dimensional generalized Laplacian in the variable x_j. Let V be a real-valued measurable function on $\mathbf{R}^{\nu N}$ such that a Hamiltonian of N-particle system

$$H_{\mathrm{p}} := -\sum_{j=1}^{N} \frac{\Delta_j}{2M_j} + V \tag{3.17}$$

is essentially self-adjoint on $D(H_{\mathrm{p}}) = [\cap_{j=1}^{N} D(\Delta_j)] \cap D(V)$ and bounded below, where $M_j > 0$ is a constant denoting the mass of the j-th particle. We then consider a model of particle-field interaction whose Hamiltonian is of the form

$$H_{\mathrm{pf}} := H_0^{\mathrm{pf}} + [\phi](g), \tag{3.18}$$

where

$$H_0^{\mathrm{pf}} := \overline{H}_{\mathrm{p}} + d\Gamma(S) \tag{3.19}$$

and S is as in Section 2.

We say that a $\mathcal{K}$-valued measurable function f on $\mathbf{R}^{\nu N}$ is in $L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$ if

$$\|f\|_\infty := \text{ess.sup}_{x \in \mathbf{R}^{\nu N}} \|f(x)\|_{\mathcal{K}} < \infty, \tag{3.20}$$

where "ess.sup" means essential supremum.

We assume the following on the $\mathcal{K}$-valued function g:

Hypothesis (g) For all $x \in \mathbf{R}^{\nu N}$, $g(x) \in D(S^{-1/2})$ and $S^{-1/2}g, g \in L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$.

Lemma 3.7 (i) $D(H_0^{\rm pf}) \subset D([\phi](g))$.

(ii) *$H_{\rm pf}$ is self-adjoint on $D(H_0^{\rm pf})$ and bounded below.*

Proof. (i) Using (2.5) and (2.6), we can show that, for all $\Psi \in D(d\Gamma(S)^{1/2}) \subset \mathcal{F}_{\rm pf}(\mathcal{K})$,

$$\|\phi(g(x))\Psi(x)\|^2 \leq 4\|S^{-1/2}g(x)\|_{\mathcal{K}}^2 \|d\Gamma(S)^{1/2}\Psi(x)\|^2 + \|g(x)\|_{\mathcal{K}}^2 \|\Psi(x)\|^2, \quad \text{a.e.} x.$$

Integrating the both sides with respect to dx, we see that $\Psi \in D([\phi](g))$ and

$$\|[\phi](g)\Psi\| \leq 2\|S^{-1/2}g\|_\infty \|d\Gamma(S)^{1/2}\Psi\| + \|g\|_\infty \|\Psi\|. \tag{3.21}$$

Hence $D(d\Gamma(S)^{1/2}) \subset D([\phi](g))$. Since $D(H_0^{\rm pf}) = D(\overline{H}_{\rm p}) \cap D(d\Gamma(S)) \subset D(d\Gamma(S)^{1/2})$, the assertion of part (i) follows.

(ii) Since $H_0^{\rm pf}$ is self-adjoint on $D(H_0^{\rm pf}) = D(\overline{H}_{\rm p}) \cap D(d\Gamma(S))$, it follows from the closed graph theorem that there exists a constant C such that, for all $\Psi \in D(H_0^{\rm pf})$,

$$\|\overline{H}_{\rm p}\Psi\| + \|d\Gamma(S)\Psi\| \leq C\|(H_0^{\rm pf} - E_0(\overline{H}_{\rm p}) + 1)\Psi\|.$$

Let $\Psi \in D(d\Gamma(S))$. Then, using the elementary inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$$

holding for all $a, b \geq 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} 2\|S^{-1/2}g\|_\infty \|d\Gamma(S)^{1/2}\Psi\| &\leq 2\|S^{-1/2}g\|_\infty \|\Psi\|^{1/2} \|d\Gamma(S)\Psi\|^{1/2} \\ &\leq \varepsilon\|d\Gamma(S)\Psi\| + \frac{\|S^{-1/2}g\|_\infty^2}{\varepsilon}\|\Psi\|. \end{aligned}$$

Hence we obtain for all $\Psi \in D(H_0^{\rm pf})$

$$\begin{aligned} \|[\phi](g)\Psi\| &\leq \varepsilon\|d\Gamma(S)\Psi\| + \left(\frac{\|S^{-1/2}g\|_\infty^2}{\varepsilon} + \|g\|_\infty\right)\|\Psi\| \\ &\leq \varepsilon C\|(H_0^{\rm pf} - E_0(\overline{H}_{\rm p}) + 1)\Psi\| + \left(\frac{\|S^{-1/2}g\|_\infty^2}{\varepsilon} + \|g\|_\infty\right)\|\Psi\|. \end{aligned}$$

This shows that $[\phi](g)$ is infinitesimally small with respect to $H_0^{\rm pf}$. Hence, by the Kato-Rellich theorem, the assertion of part (ii) follows. ■

Lemma 3.8 *For each $f \in D(S^{-1/2})$, the couple $\langle [\phi](g), a(f)^* \rangle$ has the weak commutator on $D(H_0^{\rm pf})$ and it is given by*

$$[\,[\phi](g), a(f)^*]_{{\rm w}, D(H_0^{\rm pf})} = \left(\int_{\mathbf{R}^{\nu N}}^{\oplus} \frac{1}{\sqrt{2}}(g(x), f)_{\mathcal{K}} dx\right)\Big| D(H_0^{\rm pf}).$$

Proof. Similar to the proof of Lemma 3.2. ∎

Theorem 3.9 *Suppose that H_{pf} has a ground state. Then*

$$E_0(H_{\mathrm{pf}}) + \overline{\sigma_{\mathrm{ess}}(S) \setminus \{0\}} \subset \sigma_{\mathrm{ess}}(H_{\mathrm{pf}}). \tag{3.22}$$

If $\sigma_{\mathrm{ess}}(S) = [0, \infty)$ in addition, then

$$\sigma_{\mathrm{ess}}(H_{\mathrm{pf}}) = [E_0(H_{\mathrm{pf}}), \infty). \tag{3.23}$$

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset D(S) \cap D(S^{-1/2})$ be a sequence such that $\|f_n\| = 1$ and $\mathrm{w}\text{-}\lim_{n\to\infty} f_n = 0$. Then, by Lemma 3.8, we have for all $\Psi \in D(H_0^{\mathrm{pf}})$,

$$\|[\,[\phi](g), a(f_n)^*]_{\mathrm{w},D(H_0^{\mathrm{pf}})}\Psi\|^2 = \frac{1}{2}\int_{\mathbf{R}^{\nu N}} |(g(x), f_n)_{\mathcal{K}}|^2 \|\Psi(x)\|^2 dx. \tag{3.24}$$

We have

$$|(g(x), f_n)_{\mathcal{K}}|^2 \|\Psi(x)\|^2 \leq \|g\|_\infty^2 \|\Psi(x)\|^2,$$

which is integrable, and $|(g(x), f_n)_{\mathcal{K}}|^2\|\Psi(x)\|^2 \to 0$ a.e.x $(n \to \infty)$. Hence, by the Lebesgue dominated convergence theorem, the right hand side of (3.24) converges to 0 as $n \to \infty$. Hence $\lim_{n\to\infty}[\,[\phi](g), a(f_n)^*]_{\mathrm{w},D(H_0^{\mathrm{pf}})}\Psi = 0$. Thus we can apply Theorem 2.2 and Corollary 2.3 to obtain the desired assertions. ∎

Example. A typical example of this kind of model is given as follows: $\mathcal{K} = L^2(\mathbf{R}^\nu)$, $S = \hat{\omega}$ as in the GSB model (Section 3.1), and

$$g(x)(k) = \sum_{j=1}^{N} \lambda_j(k) e^{-ikx_j}, \quad k \in \mathbf{R}^\nu,$$

where $\lambda_j \in L^2(\mathbf{R}^\nu)$. This model with $N = 1$ is discussed in [36] and the existence of a ground state can be shown under some additional conditions. Therefore we can apply Theorem 3.9 to this case.

Remark 3.3 Theorem 3.9 can be extended to a model whose Hamiltonian is of the form

$$\widetilde{H}_{\mathrm{pf}} := H_{\mathrm{pf}} + \sum_{j=1}^{J} [\phi](g_j)^2,$$

where $g_j \in \mathcal{K}$, and the models discussed in [10]. See [8].

3.3 The Pauli-Fierz model

This is a model of nonrelativistic quantum electrodynamics first discussed by Pauli and Fierz [30]. As for mathematical studies of this model made so far, we refer to [2, 3, 4, 5, 12, 18, 19, 20, 21, 29, 35]. In [21], the spectrum of the Pauli-Fierz Hamiltonian with N quantum particles in an external scalar potential is determined under some hypothesises. We give an alternative proof for this result.

The Hilbert space for the Pauli-Fierz model with N quantum particles in $\mathbf{R}^\nu$ is taken to be

$$\mathcal{F}_{\rm pf}(\oplus^s L^2(\mathbf{R}^\nu)) = L^2(\mathbf{R}^{\nu N}) \otimes \mathcal{F}_{\rm b}(\oplus^s L^2(\mathbf{R}^\nu)), \tag{3.25}$$

where s is an arbitrarily fixed natural number (the physical case is $\nu = 3, s = \nu - 1 = 2$).

For $\ell = 1, \cdots, \nu$ and $r = 1, \cdots, s$, let $\lambda_\ell^{(r)}$ be an element of $L^2(\mathbf{R}^\nu)$ and set $\lambda := (\lambda_\ell^{(r)})_{\ell=1,\cdots,\nu; r=1,\cdots,s}$, a $\nu \times s$ matrix with entries being elements of $L^2(\mathbf{R}^\nu)$. We define a $\oplus^s L^2(\mathbf{R}^\nu)$-valued function $\lambda_\ell(\cdot)$ on $\mathbf{R}^\nu$ by

$$\lambda_\ell(x)(k) := (\lambda_\ell^{(1)}(k), \cdots, \lambda_\ell^{(s)}(k))e^{-ikx}, \quad x, k \in \mathbf{R}^\nu. \tag{3.26}$$

Then the quantized radiation field $A(x;\lambda) := (A_1(x;\lambda), \cdots, A_\nu(x;\lambda))$ with cutoff λ is defined by

$$A_\ell(x;\lambda) = \phi(\lambda_\ell(x)), \tag{3.27}$$

acting in $\mathcal{F}_{\rm b}(\oplus^s L^2(\mathbf{R}^\nu))$.

We remark that, physically, $\lambda_\ell^{(r)}$ is of the form $\lambda_\ell^{(r)}(k) = \varrho(k)e_\ell^{(r)}(k)/\sqrt{(2\pi)^\nu|k|}$, where $e^{(r)}(k) = (e_1^{(r)}(k), \cdots, e_\nu^{(r)}(k))$ is the r-th polarization vector of the photon with momentum k and ϱ is a momentum-cutoff function. But, for mathematical generality, we do not specify the form of λ.

The self-adjoint operator $A_\ell(x;\lambda)$ on $\mathcal{F}_{\rm b}(\oplus^s L^2(\mathbf{R}^\nu))$ extends to a decomposable self-adjoint operator on $\mathcal{F}_{\rm pf}(\oplus^s L^2(\mathbf{R}^\nu))$ as

$$[A_{\ell j}](\lambda) := \int_{\mathbf{R}^{\nu N}}^{\oplus} A_\ell(x_j;\lambda)dx, \quad j = 1, \cdots, N. \tag{3.28}$$

We introduce

$$[A]_j(\lambda) := ([A_{1j}](\lambda), \cdots, [A_{\nu j}](\lambda)). \tag{3.29}$$

We denote by $D_{j\ell}$ the generalized partial differential operator in the ℓ-th variable $x_{j\ell}$ of $x_j = (x_{j1}, \cdots, x_{j\nu}) \in \mathbf{R}^\nu$ and set

$$\nabla_j := (D_{j1}, \cdots, D_{j\nu}). \tag{3.30}$$

Let $q_j \in \mathbf{R}$ be a constant physically denoting the charge of the j-th particle. Then the minimal interaction between N charged particles and the quantized radiation field is given by

$$H_I(\lambda) := \sum_{j=1}^N \left(\frac{iq_j}{M_j}(\nabla_j[A]_j(\lambda) + [A]_j(\lambda)\nabla_j) + \frac{q_j^2}{2M_j}[A]_j(\lambda)^2 \right). \tag{3.31}$$

The total Hamiltonian of the Pauli-Fierz model we consider is defined by

$$H_{\rm PF} := L_0 + H_I(\lambda) \tag{3.32}$$

with

$$L_0 = H_{\mathrm{p}} + d\Gamma(\oplus^s\hat{\omega}). \tag{3.33}$$

where H_{p} is given by (3.17) and $\hat{\omega}$ is the multiplication self-adjoint operator by a nonnegative continuous function ω on $\mathbf{R}^\nu$ (cf. Section 3.1). We assume the following:

Hypothesis (PF)

(PF.1) $\omega^{\pm 1/2}\lambda_\ell^{(r)}, |k|\lambda_\ell^{(r)} \in L^2(\mathbf{R}^\nu)$, $\ell = 1,\cdots,\nu$, $r = 1,\cdots,s$.

(PF.2) H_{p} is self-adjoint on $D(H_{\mathrm{p}}) = [\cap_{j=1}^N D(\Delta_j)] \cap D(V)$ and bounded below.

It follows from (PF.2) and the closed graph theorem that there exists a constant $C_{V,M} > 0$ depending on V and $M := (M_1,\cdots,M_N)$ such that, for all $\Psi \in D(H_{\mathrm{p}})$ and $j = 1,\cdots,N$,

$$\|(-\Delta_j)\Psi\| + \|V\Psi\| \le C_{V,M}\|(H_{\mathrm{p}} - E_0(H_{\mathrm{p}}) + 1)\Psi\|. \tag{3.34}$$

Let

$$q = \max_{j=1,\cdots,N} |q_j|. \tag{3.35}$$

Lemma 3.10 *The operator $H_I(\lambda)$ is L_0-bounded with*

$$\|H_I(\lambda)\Psi\| \le (c_1 q + c_2 q^2)\|L_0\Psi\| + (d_1 q + d_2 q^2)\|\Psi\|, \quad \Psi \in D(L_0), \tag{3.36}$$

where c_j and d_j $(j = 1,2)$ are positive constants depending on λ, ω, M and $C_{V,M}$, independent of q_j, $j = 1,\cdots,N$.

Proof. Cf. [21]. ■

Remark 3.4 Estimates for the constants c_j and d_j can be made (cf. [21]).

By Lemma 3.10 and the Kato-Rellich theorem, there exists a constant $q_0 > 0$ such that

$$q_0 \ge \frac{-c_1 + \sqrt{c_1^2 + 4c_2}}{2c_2}$$

and, for all $q \in [0, q_0)$, H_{PF} is self-adjoint on $D(L_0)$.

For each $f \in \oplus^s L^2(\mathbf{R}^\nu)$, we define a decomposable operator

$$\Lambda_j(f) := \frac{1}{\sqrt{2}} \int_{\mathbf{R}^{\nu N}}^{\oplus} \left(\int_{\mathbf{R}^\nu} (\lambda_j(k), f(k))_{\mathbf{C}^s} e^{ikx} dk \right) dx. \tag{3.37}$$

and

$$W(f) := \sum_{j=1}^N \left\{ \frac{iq_j}{M_j}(\nabla_j \Lambda_j(f) + \Lambda_j(f)\nabla_j) + \frac{q_j^2}{2M_j}([A]_j(\lambda)\Lambda_j(f) + \Lambda_j(f)[A]_j(\lambda)) \right\}. \tag{3.38}$$

Lemma 3.11 *For each $f \in D((\oplus^s\hat{\omega})^{-1/2})$, the couple $\langle H_I(\lambda), a(f)^*\rangle$ has the weak commutator on $D(L_0)$ and it is given by*

$$[H_I(\lambda), a(f)^*]_{\mathrm{w},D(L_0)} = W(f)|D(L_0). \tag{3.39}$$

Proof. It is easy to see that $\Lambda_j(f)$ is bounded with

$$\|\Lambda_j(f)\| \leq \frac{1}{\sqrt{2}}|(\lambda_j, f)|.$$

Each $D_{j\ell}$ is $(-\Delta_j)^{1/2}$-bounded and $[A]_j(\lambda)$ is $d\Gamma(\oplus^s\hat{\omega})^{1/2}$-bounded. From these facts and (3.34) it follows that $W(f)$ is L_0-bounded and, for all $\Psi \in D(L_0)$,

$$\|W(f)\Psi\| \leq \sum_{j=1}^{N}(|(|k|\lambda_j, f)| + |(\lambda_j, f)|)(C_1\|L_0\Psi\| + C_2\|\Psi\|), \tag{3.40}$$

$C_1 > 0$ and $C_2 > 0$ being constants. We can apply Lemma 3.1 with $B = H_I(\lambda)$, $C = a(f)^*$, $T = L_0$, $\mathcal{W} = D(L_0)$ and $D_c = D(H_p) \otimes_{\mathrm{alg}} \mathcal{F}_{0,\mathrm{fin}}(D(\oplus^s\hat{\omega}))$ to obtain the desired result. ∎

Theorem 3.12 *Let $q < q_0$. Suppose that H_{PF} has a ground state.*

(i) *If $m > 0$, then*

$$\overline{\{E_0(H_{\mathrm{PF}}) + \omega(k)|k \in \mathbf{R}^\nu\}} \subset \sigma_{\mathrm{ess}}(H_{\mathrm{PF}}). \tag{3.41}$$

(ii) *If $\{\omega(k)|k \in \mathbf{R}^\nu\} = [0, \infty)$, then*

$$[E_0(H_{\mathrm{PF}}), \infty) = \sigma_{\mathrm{ess}}(H_{\mathrm{PF}}). \tag{3.42}$$

Proof. To apply Theorem 2.2 and Corollary 2.3 to $H = H_{\mathrm{PF}}$, we need only check condition (H.3) in Hypothesis (H) with $H_I = H_I(\lambda)$. This easily follows from Lemma 3.11 and (3.40). ∎

Remark 3.5 For the existence of a ground state of H_{PF}, see [20, 21].

Remark 3.6 Theorem 3.12 can be extended to the Pauli-Fierz model with spin [14].

Acknowledgements

Part of this work was done during my research visit to Zentrum Mathematik, Technische Universität München (July–August, 1998). I thank Professor Herbert Spohn of this institute for his hospitality and Deutscher Akademischer Austauschdienst (DAAD) for financial support.

References

[1] Amrein, W. O., Jauch, J. M. and Sinha, K. B.: *Scattering Theory in Quantum Mechamics*, Benjamin, Reading, 1977.

[2] Arai, A.: Self-adjointness and spectrum of Hamiltonians in nonrelativistic quantum electrodynamics, *J. Math. Phys.* **22**(1981), 534–537.

[3] Arai, A.: Rigorous theory of spectra and radiation for a model in quantum electrodynamics, *J. Math. Phys.* **24**(1983), 1896–1910.

[4] Arai, A.: A note on scattering theory in non-relativistic quantum electrodynamics, *J. Phys. A: Math. Gen.* **16**(1983), 49–70.

[5] Arai, A.: Asymptotic analysis and its application to the nonrelativistic limit of the Pauli-Fierz and a spin-boson model, *J. Math. Phys.* **31**(1990), 2653–2663.

[6] Arai, A.: Perturbation of embedded eigenvalues: A general class of exactly soluble models in Fock spaces, *Hokkaido Math. J.* **19**(1990), 1–34.

[7] Arai, A.: An abstract sum formula and its applications to special functions, *J. Math. Anal. Appl.* **167**(1992), 245–265.

[8] Arai, A.: Noncommutative extension of a Fock space theory and a unified treatment of models in nonrelativistic quantum field theory, in preparation.

[9] Arai, A. and Hirokawa, M.: On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151**(1997), 455–503.

[10] Bach, V., Fröhlich, J. and Sigal, I. M.: Quantum electrodynamics of confined non-relativistic particles, preprint, 1996 (*Adv. in Math.*, to appear).

[11] Bach, V., Fröhlich, J. and Sigal, I. M.: Renormalization group analysis of spectral problems in quantum field theory, preprint, 1997.

[12] Blanchard, P.: Discussion mathématique du modéle de Pauli et Fierz relatif á la catastrophe infrarouge, *Commun. Math. Phys.* **15**(1969), 156–172.

[13] Dereziński, J.: Asymptotic completeness in nonrelativistic quantum field theory, *Rep. Math. Phys.* **40**(1997), 465–473.

[14] Fefferman, C., Fröhlich, J. and Graf, G. M.: Stability of ultraviolet-cutoff quantum electrodynamics with non-relativistic matter, *Commun. Math. Phys.* **190** (1997), 309–330.

[15] Gérard, C.: Asymptotic completeness for the spin-boson model with a particle number cutoff, *Rev. Math. Phys.* **8**(1996), 549–589.

[16] Glimm, J. and Jaffe, A.: *Quantum Field Theory and Statistical Mechanics: Expositions*, Birkhäuser, Boston, 1985.

[17] Glimm, J. and Jaffe, A.: *Quantum Physics*, Second Edition, Springer, New York, 1987.

[18] Hiroshima, F.: Diamagnetic inequalities for systems of nonrelativistic particles with a quantized field, *Rev. Math. Phys.* **8** (1996), 185–203.

[19] Hiroshima, F.: Functional integral representation of a model in quantum electrodynamics, *Rev. Math. Phys.* **9** (1997), 489–530.

[20] Hiroshima, F.: Existence of ground states of non-relativistic quantum electrodynamics: one particle case, preprint, 1998.

[21] Hiroshima, F.: Ground states and spectrum of quantum electrodynamics of nonrelativistic particles, preprint, 1998.

[22] Høegh-Krohn, R.: On the spectrum of the space cut-off : $P(\varphi)$: Hamiltonian in two space-time dimensions, *Commun. Math. Phys.* **21**(1971), 256–260.

[23] Hübner, M. and Spohn, H.: Radiative decay: nonperturbative approaches, *Rev. Math. Phys.* **7**(1995), 363–387.

[24] Hübner, M. and Spohn, H.: Spectral properties of the spin-boson model, *Ann. Inst. Henri Poincaré* **62**(1995), 289–323.

[25] Jakšić, V. and Pillet, C.-A.: On a model for a quantum friction. I. Fermi's golden rule and dynamics at zero temperature, *Ann. Inst. Henri Poincaré* **62**(1995), 47–68.

[26] Jakšić, V. and Pillet, C.-A.: On a model for a quantum friction. II. Fermi's golden rule and dynamics at positive temperature, *Commun. Math. Phys.* **176**(1996), 619–644.

[27] Kato, Y. and Mugibayashi, N.: Asymptotic fields in model field theories, *Prog. Theor. Phys.* **45** (1971), 628–639.

[28] Nelson, E.: Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5**(1964), 1190–1197.

[29] Okamoto, T. and Yajima, K.: Complex scaling technique in non-relativistic massive QED, *Ann. Inst. Henri Poincaré* **42**(1985), 311–327.

[30] Pauli, W. and Fierz, M.: Zur Theorie der Emission langwelliger Lichtquanten, Nuovo Cimento **15**(1938), 167–188.

[31] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, 1972.

[32] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[33] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.

[34] Simon, B.: The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, Princeton, 1974.

[35] Spohn, H.: Asymptotic completeness for Rayleigh scattering, *J. Math. Phys.* **38** (1997), 2281–2296.

[36] Spohn, H.: Ground state of a quantum particle coupled to a scalar Bose field, *Lett. Math. Phys.* **44** (1998),9–16.