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Uniqueness of the Ground State of a Model in Quantum Electrodynamics: A Functional Integral Approach

Fumio Hiroshima

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Uniqueness of the Ground State of a Model in Quantum Electrodynamics:A Functional Integral Approach

Fumio Hiroshima*

September 28, 1998

Abstract

The system of a particle minimally coupled to a radiation field with an ultraviolet cut-off is considered. The Hamiltonian of the system is defined with non-perturbative approach; any assumptions on the magnitude of the absolute value of a coupling constant in the Hamiltonian is not required. It is established that the ground state of the system, if exists, is unique and overlapps with the ground state of a decoupled Hamiltonian.

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1 PRELIMINARIES

The subject of this paper is a system consisting of a particle and a radiation field with an ultraviolet cut-off, which is called the Pauli-Fierz model [Ar1, 2, 3, BaFrSi, FeFrGr, FiPa, Gr3, Hi 1, 2, 3, OkYa]. Under some conditions, the existence of the ground state of the system is known [Hi2,3]. Next issue is its uniqueness. The specific problem addressed here is to show that the ground state of the system is unique.

Another point we want to make in this paper is that, dispensing with the usual perturbative approach, we tackle the Hamiltonian of the system with non-perturbative approach, i.e., we do **not** require any assumptions on the magnitude of the absolute value of a coupling constant e [Remark 3.7]. Methods used in [Hi2,3] require that the coupling constant e is sufficiently small. This is a technical condition needed to apply the usual method of perturbation theory [Ka]. Non-perturbative approach is possible by taking Schrödinger representation instead of Fock-Cook representation [FeFrGr,Hi1]. We believe that such non-perturbative approach for the Pauli-Fierz model mentioned in this paper is valuable, and,

to the best of our knowledge, it has never been done.

In the Schrödinger representation, the total Hamiltonian H_S of the system with an ultraviolet cut-off $\hat{\rho}$ is defined as a self-adjoint operator acting in the tensor product Hilbert space,

$$\mathcal{H}_S \equiv L^2(\mathbf{R}^3, dx) \otimes L^2(\mathcal{A}^{(3)}, d\mu_{\mathcal{A}^{(3)}}) \cong L^2(\mathbf{R}^3 \times \mathcal{A}^{(3)}, dx \otimes d\mu_{\mathcal{A}^{(3)}}),$$

where $(\mathcal{A}^{(3)}, \mu_{\mathcal{A}^{(3)}})$ is a probability measure space. The Hamiltonian has the form:

$$H_S = \frac{1}{2}\left(P \otimes I_S - e\mathbf{A}^{(3)}(\hat{\rho}, \cdot)\right)^2 + I_{el} \otimes H_{rad} + V \otimes I_S.$$

Here $P = (-i\nabla_{x_1}, -i\nabla_{x_2}, -i\nabla_{x_3})$, $\mathbf{A}^{(3)}(\hat{\rho}, x) = (\mathbf{A}_1^{(3)}(\hat{\rho}, x), \mathbf{A}_2^{(3)}(\hat{\rho}, x), \mathbf{A}_3^{(3)}(\hat{\rho}, x))$ denotes a radiation field with the ultraviolet cut-off $\hat{\rho}$ in the Schrödinger representation, H_{rad} the free Hamiltonian in $L^2(\mathcal{A}^{(3)}, d\mu_{\mathcal{A}^{(3)}})$, V an external potential, e a real coupling constant, I_{el} and I_S the identity operators in $L^2(\mathbf{R}^3)$ and $L^2(\mathcal{A}^{(3)}, d\mu_{\mathcal{A}^{(3)}})$, respectively. In the case where the coupling constant e is sufficiently small, a concrete core for H_S is well known [OkYa]. In this paper, we do not require any assumptions on the coupling constant e. In the case where the coupling constant e is sufficiently **large**, by a functional integral, we define a self-adjoint extension, $\widetilde{H}_S$, of H_S. See (3.6), (3.8) and (3.9).

For a self-adjoint operator K in a Hilbert space that is bounded from below and has an eigenvalue as the lowest point in its spectrum, the eigenvector corresponding to that eigenvalue is called the ground state of K. The main purpose in this paper is to show that the ground state of $\widetilde{H}_S$ is unique. The problems of the uniqueness (or non-uniqueness) of the ground states of Hamiltonians have long history [An,Gr1,Je,KrRu] beginning with the classical papers [Fro,Pe] and has been widely investigated for several models; for our purpose most relevant results are in [AlHo,AvHeSi,BaFrSi,Fa1,FaSi,GlJa,Gr2,He,HoSi, Se3,4,5]. Recall the following definition:

Definition 1.1 *Let (M, μ) be a σ-finite measure space. A function $f \in L^2(M, d\mu)$ is called "positive" if $f \geq 0$ a.e. and $f \not\equiv 0$ a.e.. f is called "strictly positive" if $f > 0$ a.e..*

Definition 1.2 *Let (M,μ) be a σ-finite measure space. A bounded operator,A, on $L^2(M,d\mu)$ is called "positivity preserving" if, for all positive θ_1 and θ_2 in $L^2(M,d\mu)$, $(\theta_1, A\theta_2)_{L^2(M,d\mu)} \geq 0$. Moreover, A is called "positivity improving" if, for all positive θ_1 and θ_2 in $L^2(M,d\mu)$, $(\theta_1, A\theta_2)_{L^2(M,d\mu)} > 0$. Here $(\cdot,\cdot)_{L^2(M,d\mu)}$ is the scalar product of $L^2(M,d\mu)$.*

Next proposition is well known:

Proposition 1.3 (ReSi2,Theorem XIII44) *Let (M,μ) be a σ-finite measure space and K be a self-adjoint operator that is bounded from below in $L^2(M,d\mu)$. Suppose that $\inf\sigma(K)$, the infimum of the spectrum of K, is eigenvalue and e^{-tK} is positivity improving. Then $\inf\sigma(K)$ is non-degenerate and the eigenfunction corresponding to the eigenvalue $\inf\sigma(K)$ is strictly positive.*

J.Fröhlich explained [Lemma 4.3] that the Plancherel-Wiener transform ϑ, which is a unitary transformation, of $e^{-t\widetilde{H}_S}$ is positivity preserving, i.e., for all positive Ψ_1, Ψ_2 in $\mathcal{H}_S$,

$$(\Psi_1, \vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta\Psi_2)_{\mathcal{H}_S} \geq 0,$$

where $(\cdot,\cdot)_{\mathcal{H}_S}$ is the scalar product of $\mathcal{H}_S$. In this paper, we shall show that $\vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta$ is positivity improving [Theorem 4.8], i.e., for all positive Ψ_1, Ψ_2 in $\mathcal{H}_S$,

$$(\Psi_1, \vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta\Psi_2)_{\mathcal{H}_S} > 0.$$

Then we shall obtain that the ground state of $\vartheta^{-1}\widetilde{H}_S\vartheta$ is unique and strictly positive, which implies that the ground state of $\widetilde{H}_S$ is unique and the Plancherel-Wiener transform of a strictly positive function in $\mathcal{H}_S$. Hence we also see that the ground state, if exists, overlapps with the ground state of the decoupled Hamiltonian defined by $\widetilde{H}_S$ with the coupling constant e replaced by zero. A rough idea lying behind the results proved in this paper is as follows: The operator $\vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta$ has, roughly speaking, the following integral kernel:

$$e^{-\int_0^t V(\omega(s))ds} J_t^* e^{ie\mathbf{E}^{(4)}(\mathcal{W}(t))} J_0. \qquad (1.\ 1)$$

(See section 3 for the precise definition of notations.) We shall show that (1.1) is positivity improving [Theorem 4.7].

This paper is organized as follows: In Section 2, we shall describe the system in Fock-Cook representation. Section 3 specifies isomorphism between Fock-Cook representation and Schrödinger representation as we need in this paper and contains the definition of the Hamiltonian $\widetilde{H}_S$ of the system in the Schrödinger representation with non-perturbative approach. Section 4 is the main section in this paper. This section states that $\vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta$ is positivity improving in Theorem 4.8. Section 5 is devoted to references.

2 FOCK-COOK REPRESENTATION

Let $W^{(3)} = L^2(\mathbf{R}^3)\oplus L^2(\mathbf{R}^3)$ and $W^{(4)} = L^2(\mathbf{R}^3\times\mathbf{R})\oplus L^2(\mathbf{R}^3\times\mathbf{R})$. Throughout this paper d denotes 3 or 4, and $k_0\in\mathbf{R}, k=(k_1,k_2,k_3)\in\mathbf{R}^3$. The symmetric Fock space $\mathcal{F}^{(d)}$ over $W^{(d)}$ is defined by

$$\mathcal{F}^{(d)} = \bigoplus_{N=0}^{\infty}[\otimes_s^N W^{(d)}].$$

Here $\otimes_s^N W^{(d)}$ denotes the N-fold symmetric tensor product of $W^{(d)}$ and $\otimes_s^0 W^{(d)} = \mathbf{C}$, the set of complex numbers. We denote by $\mathcal{F}^{(d)\infty}$ the finite particle subspace which is defined by the set of vectors $\{\Phi_n\}_{n=0}^{\infty}\in\mathcal{F}^{(d)}$ with at most finitely many non zero elements. We denote the creation operator by $a^{\dagger}_{(d)}(F), F\in W^{(d)}$, and the annihilation operator by $a_{(d)}(F), F\in W^{(d)}$. Set $a^{\dagger 1}_{(d)}(f) = a^{\dagger}_{(d)}(f\oplus 0), a^{\dagger 2}_{(d)}(f) = a^{\dagger}_{(d)}(0\oplus f), a^{1}_{(d)}(f) = a_{(d)}(f\oplus 0), a^{2}_{(d)}(f) = a_{(d)}(0\oplus f), f\in L^2(\mathbf{R}^d)$, respectively. The map $f\to a^{\sharp r}_{(d)}(f)$ is linear. They satisfy that

$$a^{\sharp r}_{(d)}(f)\mathcal{F}^{(d)\infty}\subset\mathcal{F}^{(d)\infty},$$

$$[a^{r}_{(d)}(f), a^{\dagger s}_{(d)}(g)]\Psi = \delta_{rs}(\bar{f},g)_{L^2(\mathbf{R}^d)}\Psi,$$

$$[a^{\dagger r}_{(d)}(f), a^{\dagger s}_{(d)}(g)]\Psi = 0, [a^{r}_{(d)}(f), a^{s}_{(d)}(g)]\Psi = 0,$$

$$(\Psi, a^{r}_{(d)}(f)\Phi)_{\mathcal{F}^{(d)}} = (a^{\dagger r}_{(d)}(\bar{f})\Psi, \Phi)_{\mathcal{F}^{(d)}},$$

for all $\Psi, \Phi \in \mathcal{F}^{(d)\infty}$, all $f, g \in L^2(\mathbf{R}^d)$ and $r, s = 1, 2$. Here $[A, B] = AB - BA$, $\delta_{rs} = 1, r = s, \delta_{rs} = 0, r \neq s$, the symbol $\sharp$ stands for the dagger $\dagger$ or its omission, $\bar{f}$ is the complex conjugate of f and $(\cdot, \cdot)_{\mathcal{F}^{(d)}}$ and $(\cdot, \cdot)_{L^2(\mathbf{R}^d)}$ the scalar products of $\mathcal{F}^{(d)}$ and $L^2(\mathbf{R}^d)$,respectively. In what follows we denote the scalar product of a Hilbert space $\mathcal{K}$ by $(\cdot, \cdot)_{\mathcal{K}}$ and the associated norm $\| \cdot \|_{\mathcal{K}}$. Note that the scalar product $(f, g)_{\mathcal{K}}$ is linear in f and antilinear in g. The vacuum vector $\Omega^{(d)} \in \mathcal{F}^{(d)}$ is defined by

$$\Omega^{(d)} = \{1, 0, 0, ...\}.$$

We formally write

$$a^{\sharp r}_{(d)}(f) = \int a^{\sharp r}_{(d)}(l) f(l) dl,$$

where $l = k (d = 3), l = (k, k_0) (d = 4)$. The number operator in $\mathcal{F}^{(d)}$ is defined by

$$N^{(d)} = \sum_{r=1}^{2} \int a^{\dagger r}_{(d)}(l) a^{r}_{(d)}(l) dl.$$

Set

$$\mathcal{M}^{(d)}_n = \left\{ f \left| \| (\sqrt{w_{(d)}})^n f \|_{L^2(\mathbf{R}^d)} < \infty \right. \right\}.$$

Here $w_{(3)}(k) = |k|$ and $w_{(4)}(k, k_0) = \sqrt{|k|^2 + |k_0|^2}$. A quantized radiation field $A^{(3)}_{\mu}(\hat{\rho}, x), \mu = 1, 2, 3, x \in \mathbf{R}^3$, is defined by

$$A^{(3)}_{\mu}(\hat{\rho}, x) = \sum_{r=1}^{2} \int \frac{dk}{\sqrt{2|k|}} \left\{ \hat{\rho}(k) e^{r}_{\mu}(k) e^{-ikx} a^{\dagger r}_{(3)}(k) + \hat{\rho}(-k) e^{r}_{\mu}(k) e^{ikx} a^{r}_{(3)}(k) \right\}, \hat{\rho} \in \mathcal{M}^{(3)}_{-1}.$$

Here $\hat{f}$ denotes the Fourier transform of f in the sense of distribution and $e^r = (e^r_1, e^r_2, e^r_3), r = 1, 2$, are polarization vectors which satisfy

$$e^r(k) e^s(k) = \delta_{rs}, k e^r(k) = 0, \quad a.e. k \in \mathbf{R}^3, \tag{2. 1}$$

and $e^r_\mu \in C^\infty(\mathbf{R}^3 \setminus \mathcal{P}), \mu = 1,2,3, r = 1,2$, where $\mathcal{P} = \{(0,0,k_3) \in \mathbf{R}^3 | k_3 \in \mathbf{R}\}$. We define, for $\mu = 1,2,3, x \in \mathbf{R}^3$,

$$A^{(4)}_\mu(\hat{\rho}, x) = \sum_{r=1}^{2} \int \frac{dkdk_0}{\sqrt{|k_0|^2 + |k|^2}}$$
$$\times \left\{ \hat{\rho}(k,k_0) e^r_\mu(k) e^{-ikx} a^{\dagger r}_{(4)}(k,k_0) + \hat{\rho}(-k,-k_0) e^r_\mu(k) e^{ikx} a^r_{(4)}(k,k_0) \right\}, \hat{\rho} \in \mathcal{M}^{(4)}_{-2}.$$

The operators $A^{(3)}_\mu(\hat{\rho}, x), \hat{\rho} \in \mathcal{M}^{(3)}_{-1}, \mu = 1,2,3, x \in \mathbf{R}^3$, and $A^{(4)}_\mu(\hat{\rho}, x), \hat{\rho} \in \mathcal{M}^{(4)}_{-2}, \mu = 1,2,3, x \in \mathbf{R}^3$, are well defined and closable on $\mathcal{F}^{(d)\infty}$. We denote their closed extensions by the same symbols. We set $A^{(d)}_\mu(\hat{\rho}, 0) = A^{(d)}_\mu(\hat{\rho}), \mu = 1,2,3$. Note that

$$\sum_{\mu=1}^{3} A^{(d)}_\mu(k_\mu \hat{\rho}) \Psi = 0, \Psi \in \mathcal{F}^{(d)\infty},$$
$$\left(A^{(3)}_\mu(\hat{\rho}) \Omega^{(3)}, A^{(3)}_\nu(\hat{\lambda}) \Omega^{(3)} \right)_{\mathcal{F}^{(3)}} = \frac{1}{2} \int \frac{d_{\mu\nu}(k) \overline{\hat{\rho}}(k) \hat{\lambda}(k)}{|k|} dk,$$
$$\left(A^{(4)}_\mu(\hat{\rho}) \Omega^{(4)}, A^{(4)}_\nu(\hat{\lambda}) \Omega^{(4)} \right)_{\mathcal{F}^{(4)}} = \int \frac{d_{\mu\nu}(k) \overline{\hat{\rho}}(k,k_0) \hat{\lambda}(k,k_0)}{|k|^2 + |k_0|^2} dkdk_0,$$

where

$$d_{\mu\nu}(k) = \sum_{r=1}^{3} e^r_\mu(k) e^r_\nu(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}, \mu, \nu = 1,2,3.$$

The Hilbert space $\mathcal{H}$ of the system is given by

$$\mathcal{H} = L^2(\mathbf{R}^3) \otimes \mathcal{F}^{(3)} \cong L^2(\mathbf{R}^3; \mathcal{F}^{(3)}). \tag{2. 2}$$

Here the right hand side of (2.2) means the $\mathcal{F}^{(3)}$-valued L^2-space over $\mathbf{R}^3$. We denote the identity operator on $L^2(\mathbf{R}^3)$ by I_{el} and on $\mathcal{F}^{(3)}$ by I_E. The Hamiltonian H of the system reads

$$H = \frac{1}{2}(P \otimes I_E - eA^{(3)}(\hat{\rho}, \cdot))^2 + V \otimes I_E + I_{el} \otimes H_E.$$

Here $A^{(3)}(\hat{\rho}, x) = (A^{(3)}_1(\hat{\rho}, x), A^{(3)}_2(\hat{\rho}, x), A^{(3)}_3(\hat{\rho}, x))$, V is an external potential and the free Hamiltonian H_E in $\mathcal{F}^{(3)}$ is defined by

$$H_E = \sum_{r=1}^{2} \int |k| a^{\dagger r}_{(3)}(k) a^r_{(3)}(k) dk.$$

Proposition 2.1 ([Hi3]) *We assume that (1) the ultraviolet cut-off $\hat{\rho}$ satisfies that*

$$\overline{\hat{\rho}(k)} = \hat{\rho}(-k) \tag{2. 3}$$

and $\hat{\rho} \in \mathcal{M}_{-2}^{(3)} \cap \mathcal{M}_{-1}^{(3)} \cap \mathcal{M}_{0}^{(3)} \cap \mathcal{M}_{1}^{(3)}$; (2) the external potential V satisfies that $H_{el} = -\frac{1}{2}\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^3)$, where Δ is the Laplacian in $L^2(\mathbf{R}^3)$, and

$$\left\| -\frac{1}{2}\Delta f \right\|_{L^2(\mathbf{R}^3)} \leq \alpha \left\| H_{el} f \right\|_{L^2(\mathbf{R}^3)} + \beta \| f \|_{L^2(\mathbf{R}^3)}, f \in C_0^\infty(\mathbf{R}^3),$$

where α and β are positive constants; (3) the absolute value of the coupling constant,$|e|$, is sufficiently small. Then H is self-adjoint on $D(H_{el} \otimes I_E) \cap D(I_{el} \otimes H_E)$ and bounded from below, where $D(T)$ denotes the domain of an operator T. Moreover it is essentially self-adjoint on any core for $H_{el} \otimes I_E + I_{el} \otimes H_E$. In addition to (1),(2) and (3), we assume that (4) the infimum of the spectrum of H_{el} is discrete; (5) $\hat{\rho} \in \mathcal{M}_{-3}^{(3)}$. Then the ground states of H exist.

Typical examples of V's are $V = -\frac{1}{|x|}$ and $V = x^2$. That ρ is real implies (2.3).

3 SCHRÖDINGER REPRESENTATION

In this section we construct Schrödinger representation. In the Schrödinger representation, we define the Hamiltonian $\widetilde{H}_S$ of the system under consideration with non-perturbative approach and show that, in the case where the coupling constant e is sufficiently small, $\widetilde{H}_S$ is equal to H, in the case where the coupling constant e is sufficiently large, the Hamiltonian H extends to $\widetilde{H}_S$ in some sense.

3.1 Probabilistic Description

Let $\mathcal{S}(\mathbf{R}^d)$ denote the set of rapidly decreasing infinite times differentiable functions on $\mathbf{R}^d$, $\mathcal{S}_r(\mathbf{R}^d)$ the set of real-valued functions in $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}_r'(\mathbf{R}^d)$ the set of real tempered

distributions on $\mathbf{R}^d$.

$$\mathcal{S}_{\frac{1}{2}} = \left\{ F \in \mathcal{S}'_r(\mathbf{R}^3) \middle| \|F\|^2_{\mathcal{S}_{\frac{1}{2}}} = \int_{\mathbf{R}^3} \frac{|\hat{F}(k)|^2}{|k|} dk < \infty \right\}$$

with the scalar product $(F,G)_{\mathcal{S}_{\frac{1}{2}}} = \int_{\mathbf{R}^3} \frac{\overline{\hat{F}(k)}\hat{G}(k)}{|k|} dk$. In what follows, for any set G, we put $G^{\times 3} = G \times G \times G$. We equip with the positive semi-definite bilinear form $q^{(3)}(\cdot,\cdot)$ on $(\mathcal{S}_{\frac{1}{2}})^{\times 3}$:

$$q^{(3)}(F,G) = \sum_{\mu,\nu=1}^{3} \int_{\mathbf{R}^3} \frac{d_{\mu\nu}(k)\overline{\hat{F}_\mu(k)}\hat{G}_\nu(k)}{|k|} dk,$$
$$F = (F_1,F_2,F_3) \in (\mathcal{S}_{\frac{1}{2}})^{\times 3}, G = (G_1,G_2,G_3) \in (\mathcal{S}_{\frac{1}{2}})^{\times 3}.$$

We define the Hilbert space $\mathcal{S}_1$ by

$$\mathcal{S}_1 = \left\{ F \in \mathcal{S}'_r(\mathbf{R}^3 \times \mathbf{R}) \middle| \|F\|^2_{\mathcal{S}_1} = 2\int_{\mathbf{R}^3 \times \mathbf{R}} \frac{|\hat{F}(k,k_0)|^2}{|k|^2 + |k_0|^2} dk dk_0 < \infty \right\}$$

with the scalar product $(F,G)_{\mathcal{S}_1} = 2\int_{\mathbf{R}^3 \times \mathbf{R}} \frac{\overline{\hat{F}(k,k_0)}\hat{G}(k,k_0)}{|k|^2+|k_0|^2} dk dk_0$. On $(\mathcal{S}_1)^{\times 3}$ we equip with the positive semi-definite bilinear form $q^{(4)}(\cdot,\cdot)$:

$$q^{(4)}(F,G) = 2\sum_{\mu,\nu=1}^{3} \int_{\mathbf{R}^3 \times \mathbf{R}} \frac{d_{\mu\nu}(k)\overline{\hat{F}_\mu(k,k_0)}\hat{G}_\nu(k,k_0)}{|k|^2 + |k_0|^2} dk dk_0,$$
$$F = (F_1,F_2,F_3) \in (\mathcal{S}_1)^{\times 3}, G = (G_1,G_2,G_3) \in (\mathcal{S}_1)^{\times 3}.$$

For instance we put $q^{(d)}(F,F) = q^{(d)}(F)$. We denote the dual pairs $\mathcal{S}'_r(\mathbf{R}^d)$ and $\mathcal{S}_r(\mathbf{R}^d)$ by $< \cdot,\cdot >_{(d)}$ and define, for $\mathbf{A}^{(d)} = (\mathbf{A}^{(d)}_1, \mathbf{A}^{(d)}_2, \mathbf{A}^{(d)}_3) \in (\mathcal{S}'_r(\mathbf{R}^d))^{\times 3}$

$$\mathbf{A}^{(d)}(h) \equiv \sum_{\mu=1}^{3} < \mathbf{A}^{(d)}_\mu, h_\mu >_{(d)}, h = (h_1,h_2,h_3) \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3},$$
$$\mathbf{A}^{(d)}_\mu(h) \equiv < \mathbf{A}^{(d)}_\mu, h >_{(d)}, h \in \mathcal{S}_r(\mathbf{R}^d), \mu = 1,2,3.$$

Note that $(\mathcal{S}_r(\mathbf{R}^3))^{\times 3}$ and $(\mathcal{S}_r(\mathbf{R}^3 \times \mathbf{R}))^{\times 3}$ are dense and continuously embedded in $(\mathcal{S}_{\frac{1}{2}})^{\times 3}$ and $(\mathcal{S}_1)^{\times 3}$, respectively. Since the functional $C^{(d)}(F) = e^{-\frac{1}{4}q^{(d)}(F)}, F \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}$, satisfies that (1) $C^{(d)}(0) = 1$, (2) $C^{(d)}(\cdot)$ is continuous in $(\mathcal{S}_r(\mathbf{R}^d))^{\times 3}$, (3) $\sum_{i,j=1}^{n} \bar{z}_i z_j C^{(d)}(F_i - F_j) \geq 0$, for any $z_i \in \mathbf{C}$ and $F_i \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}, i = 1,...,n$, the Minlos theorem [Si2,Theorem I.10] yields

that $C^{(d)}(F)$ is the characteristic function of a Gaussian measure, $\mu_{\mathcal{A}^{(d)}}$, of mean zero and covariance $\frac{1}{2}q^{(d)}$ over $\tilde{\mathcal{A}}^{(d)} = (\mathcal{S}'_r(\mathbf{R}^d))^{\times 3}$ with a σ-algebra, $\Sigma^{(d)}$, constructed by the cylinder sets, i.e.,

$$\int_{\tilde{\mathcal{A}}^{(d)}} e^{i\mathbf{A}^{(d)}(h)} d\mu_{\mathcal{A}^{(d)}} = e^{-\frac{1}{4}q^{(d)}(h)}, h \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}.$$

By the construction, $\{\mathbf{A}^{(d)}(h), h \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}\}$ is "full" [Si2,p.21] with respect to $\Sigma^{(d)}$, i.e., $\Sigma^{(d)}$ is the smallest σ-algebra generated by $\{\mathbf{A}^{(d)}(F), F \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}\}$. It is known that the measure $\mu_{\mathcal{A}^{(d)}}$ has support contained in the set $\mathcal{A}^{(d)}$ [FeFrGr,Proposition 1]:

$$\mathcal{A}^{(d)} = \left\{ (\mathbf{A}_1^{(d)}, \mathbf{A}_2^{(d)}, \mathbf{A}_3^{(d)}) \in \tilde{\mathcal{A}}^{(d)} \middle| \sum_{\mu=1}^{3} \nabla_\mu \mathbf{A}_\mu^{(d)} = 0 \right\}.$$

We put

$$A^{(d)} = L^2(\mathcal{A}^{(d)}, \Sigma^{(d)}, d\mu_{\mathcal{A}^{(d)}}).$$

For $h \in (\mathcal{S}_{\frac{1}{2}})^{\times 3}$, we define $\mathbf{A}^{(3)}(h) = s - \lim_{n\to\infty} \mathbf{A}^{(3)}(h_n)$ in $A^{(3)}$, where $h_n \to h$ in $(\mathcal{S}_{\frac{1}{2}})^{\times 3}$ with $h_n \in (\mathcal{S}_r(\mathbf{R}^3))^{\times 3}$. Note that, since

$$\|\mathbf{A}^{(3)}(h) - \mathbf{A}^{(3)}(f)\|_{A^{(3)}} = \frac{1}{2}q^{(3)}(h-f) \leq \frac{3}{2}\|h-f\|^2_{(\mathcal{S}_{\frac{1}{2}})^{\times 3}},$$

where $\|F\|^2_{(\mathcal{S}_{\frac{1}{2}})^{\times 3}} = \sum_{\mu=1}^{3} \|F_\mu\|^2_{\mathcal{S}_{\frac{1}{2}}}, F = (F_1, F_2, F_3)$, $\mathbf{A}^{(3)}(h)$ does not depend on the choice of sequences $\{h_n\}_{n=1}^{\infty} \subset (\mathcal{S}_r(\mathbf{R}^3))^{\times 3}$. Similarly we define $\mathbf{A}^{(4)}(h) = s - \lim_{n\to\infty} \mathbf{A}^{(4)}(h_n)$ in $A^{(4)}$ for $h \in (\mathcal{S}_1)^{\times 3}$, where $h_n \to h$ in $(\mathcal{S}_1)^{\times 3}$ with $h_n \in (\mathcal{S}_r(\mathbf{R}^3 \times \mathbf{R}))^{\times 3}$. Thus we have constructed the Gaussian random process $\mathbf{A}^{(d)}(\cdot)$ over the probability measure space $(\mathcal{A}^{(d)}, \Sigma^{(d)}, \mu_{\mathcal{A}^{(d)}})$ indexed by $(\mathcal{S}_{\frac{1}{2}})^{\times 3}$ for $d = 3$ and $(\mathcal{S}_1)^{\times 3}$ for $d = 4$ with the covariance $q^{(d)}$ [Si2,p.15]. It is well known [Si2, (I.25)] that

$$A^{(d)} = \bigoplus_{n=0}^{\infty} A_n^{(d)},$$

where

$$A_n^{(d)} = \overline{L\{: \mathbf{A}^{(d)}(h_1)...\mathbf{A}^{(d)}(h_n) :_{A^{(d)}} | h_j \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}, j = 1, ..., n\}}^{A^{(d)}}, A_0^{(d)} = \mathbf{C}.$$

Here $:\ :_{A^{(d)}}$ is the "Wick product" in $A^{(d)}$ [Si2,p.9], $L\{...\}$ the linear hull of the vectors in $\{...\}$ and $\overline{\{...\}}^{A^{(d)}}$ the closure in $A^{(d)}$. Let

$$A^{(d)\infty} = \bigcup_{N=0}^{\infty} [\oplus_{k=0}^{N} A_k^{(d)}].$$

We define a subset $\mathcal{Q}^{(d)}$ in $A^{(d)}$ by

$$\mathcal{Q}^{(d)} = \left\{ G(\mathbf{A}^{(d)}(h_1), ..., \mathbf{A}^{(d)}(h_n)) | G \in \mathcal{S}(\mathbf{R}^n), h_j \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}, j = 1, ..., n, n \geq 1 \right\}.$$

The subset $\mathcal{Q}^{(d)}$ is dense in $A^{(d)}$ [Si2, Lemma I.5]. Define an operator $\tilde{\omega}$ in $\mathcal{S}_{\frac{1}{2}}$ by

$$\widehat{\tilde{\omega} f}(k) = |k| \hat{f}(k)$$

and $\omega = \tilde{\omega} \oplus \tilde{\omega} \oplus \tilde{\omega}$ as the operator in $(\mathcal{S}_{\frac{1}{2}})^{\times 3}$. An operator $\Gamma(e^{-t\omega})$ on $A^{(3)}$ is defined by

$$\Gamma(e^{-t\omega}) : \mathbf{A}^{(3)}(h_1)...\mathbf{A}^{(3)}(h_n) :_{A^{(3)}} =: \mathbf{A}^{(3)}(e^{-t\omega}h_1)...\mathbf{A}^{(3)}(e^{-t\omega}h_n) :_{A^{(3)}}, n \geq 1,$$
$$\Gamma(e^{-t\omega})\Omega_S^{(3)} = \Omega_S^{(3)},$$

where $\Omega_S^{(d)}$ is the identity in $A^{(d)}$, and linearly extended. One sees that $\Gamma(e^{-t\omega})$ is a one parameter strongly continuous symmetric semi-group in t on $A^{(3)}$. It enable us to state that there exists a nonnegative self-adjoint operator H_{rad} in $A^{(3)}$ so that

$$\Gamma(e^{-t\omega}) = e^{-tH_{rad}}.$$

It is shown that $e^{-tH_{rad}}$ is positivity preserving [Si2, I.12] and the self-adjoint operator H_{rad} has the strictly positive unique ground state $\Omega_S^{(3)}$:

$$H_{rad}\Omega_S^{(3)} = 0,$$

which implies that the semigroup $e^{-tH_{rad}}$ is ergodic [Fa1]. Then $e^{-tH_{rad}}$ is not only positivity preserving but also positivity improving [Si1]. Let the family of operators $j_t, t \geq 0$, from $\mathcal{S}_{\frac{1}{2}}$ to $\mathcal{S}_1$ be defined by

$$\widehat{j_t f}(k_0, k) = \frac{1}{\sqrt{2\pi}} e^{-itk_0} \hat{f}(k), f \in \mathcal{S}_{\frac{1}{2}},$$

and $\mathbf{j}_t = j_t \oplus j_t \oplus j_t$ as the operator from $(\mathcal{S}_{\frac{1}{2}})^{\times 3}$ to $(\mathcal{S}_1)^{\times 3}$. The family of operators $J_t, t \geq 0$, from $A^{(3)}$ to $A^{(4)}$ is defined by

$$J_t : \mathbf{A}^{(3)}(h_1)...\mathbf{A}^{(3)}(h_n) :_{A^{(3)}} =: \mathbf{A}^{(4)}(\mathbf{j}_t h_1)...\mathbf{A}^{(4)}(\mathbf{j}_t h_n) :_{A^{(4)}}, n \geq 1,$$
$$J_t \Omega_S^{(3)} = \Omega_S^{(4)}$$

and linearly extended.

Lemma 3.1 *The operators J_s^* and J_t, $s, t \in \mathbf{R}$, are positivity preserving and we have*

$$(1)\mathbf{j}_t^* \mathbf{j}_s = e^{-|t-s|\omega}, t, s \in \mathbf{R}, \tag{3. 1}$$

$$(2)J_t^* J_s = e^{-|t-s|H_{rad}}, t, s \in \mathbf{R}. \tag{3. 2}$$

Proof: It is straightforwardly seen. We omit it. See [Si2, Propositions III.3,III.4 and Theorem III.5]. □

Let $N_S^{(d)}$ denote the number operator in $A^{(d)}$ which is defined by

$$N_S^{(d)} : \mathbf{A}^{(d)}(h_1)...\mathbf{A}^{(d)}(h_n) :_{A^{(d)}} = n : \mathbf{A}^{(d)}(h_1)...\mathbf{A}^{(d)}(h_n) :_{A^{(d)}},$$
$$N_S^{(d)} \Omega_S^{(d)} = 0$$

with

$$D(N_S^{(d)}) = \left\{ \Psi = \{\Psi^n\}_{n=0}^{\infty} \in A^{(d)}, \Psi^n \in A_n^{(d)} \middle| \sum_{n=0}^{\infty} n^2 \|\Psi^n\|_{A^{(d)}}^2 < \infty \right\}.$$

We shall see relations between $\mathcal{F}^{(d)}$ and $A^{(d)}$. We define a map $U_{(d)} : \mathcal{F}^{(d)} \longrightarrow A^{(d)}$ by

$$U_{(d)} : A_{\mu_1}^{(d)}(\hat{\rho}_1)...A_{\mu_n}^{(d)}(\hat{\rho}_n) :_{\mathcal{F}^{(d)}} \Omega^{(d)} \equiv: \mathbf{A}_{\mu_1}^{(d)}(\rho_1)...\mathbf{A}_{\mu_n}^{(d)}(\rho_n) :_{A^{(d)}}, n \geq 1,$$
$$U_{(d)} \Omega^{(d)} = \Omega_S^{(d)},$$

where $\rho_j \in (\mathcal{S}_r(\mathbf{R}^3))^{\times 3}$, $\mu_j = 1, 2, 3, j = 1, ..., n$, and $: :_{\mathcal{F}^{(d)}}$ is the "Wick product" in $\mathcal{F}^{(d)}$. The operator $U_{(d)}$ is extended by linearity.

Lemma 3.2 *The operators $U_{(d)}$ extends a unitary operator from $\mathcal{F}^{(d)}$ to $A^{(d)}$ and satisfies the followings:*

(1) *$U_{(3)}$ maps $D(H_E)$ to $D(H_{rad})$ with*

$$H_E = U_{(3)}^{-1} H_{rad} U_{(3)}.$$

(2) *$U_{(d)}$ maps $D(N^{(d)})$ to $D(N_S^{(d)})$ with*

$$N^{(d)} = U_{(d)}^{-1} N_S^{(d)} U_{(d)}.$$

(3) *Assume that $\rho \in \mathcal{S}_{\frac{1}{2}}$. Then $U_{(d)}$ maps $D(A_\mu^{(d)}(\hat{\rho}))$ to $D(\mathbf{A}_\mu^{(d)}(\rho))$ with*

$$A_\mu^{(d)}(\hat{\rho}) = U_{(d)}^{-1} \mathbf{A}_\mu^{(d)}(\rho) U_{(d)}, \quad \mu = 1, 2, 3.$$

Proof: We put

$$\mathcal{D}^{(d)} = L\Big\{ : A_{\mu_1}^{(d)}(\hat{\rho}_1) \ldots A_{\mu_n}^{(d)}(\hat{\rho}_n) :_{\mathcal{F}^{(d)}} \Omega^{(d)}, \Omega^{(d)} \Big| \\ \hat{\rho}_j \in C_0^\infty(\mathbf{R}^d \setminus \{0\}), \rho_j : \text{real}, \mu_j = 1, 2, 3, j = 1, \ldots, n, n \geq 1 \Big\},$$

$$\mathcal{D}_S^{(d)} = L\Big\{ : \mathbf{A}_{\mu_1}^{(d)}(\rho_1) \ldots \mathbf{A}_{\mu_n}^{(d)}(\rho_n) :_{A^{(d)}}, \Omega_S^{(d)} \Big| \\ \hat{\rho}_j \in C_0^\infty(\mathbf{R}^d \setminus \{0\}), \rho_j : \text{real}, \mu_j = 1, 2, 3, j = 1, \ldots, n, n \geq 1 \Big\}.$$

Since

$$\sum_{\mu=1}^{3} A_\mu^{(3)}(e_\mu^r \sqrt{w_{(3)}} \hat{\rho}) = \frac{1}{\sqrt{2}} (a_{(3)}^{\dagger r}(\hat{\rho}) + a_{(3)}^r(\hat{\rho})), \hat{\rho} \in C_0^\infty(\mathbf{R}^3 \setminus \{0\}),$$

$$\sum_{\mu=1}^{3} A_\mu^{(4)}(e_\mu^r \sqrt{w_{(4)}} \hat{\lambda}) = a_{(4)}^{\dagger r}(\hat{\lambda}) + a_{(4)}^r(\hat{\lambda}), \hat{\lambda} \in C_0^\infty(\mathbf{R}^4 \setminus \{0\}),$$

we conclude that $\mathcal{D}^{(d)}$ is dense in $\mathcal{F}^{(d)}$.

Fundamental calculation yields that

$$
\begin{aligned}
&\| : A^{(3)}_{\mu_1}(\hat{\rho}_1)...A^{(3)}_{\mu_n}(\hat{\rho}_n) :_{\mathcal{F}^{(3)}} \Omega^{(3)}\|^2_{\mathcal{F}^{(3)}} \\
&= \frac{1}{2^n}\sum_{\pi\in\mathcal{P}_n}\left\{\left(\frac{e^1_{\mu_1}\hat{\rho}_1}{\sqrt{w_{(3)}}},\frac{e^1_{\mu_{\pi}(1)}\hat{\rho}_{\pi(1)}}{\sqrt{w_{(3)}}}\right)_{L^2(\mathbf{R}^3)}+\left(\frac{e^2_{\mu_1}\hat{\rho}_1}{\sqrt{w_{(3)}}},\frac{e^2_{\mu_{\pi}(1)}\hat{\rho}_{\pi(1)}}{\sqrt{w_{(3)}}}\right)_{L^2(\mathbf{R}^3)}\right\}\times ... \\
&\qquad ... \times\left\{\left(\frac{e^1_{\mu_n}\hat{\rho}_n}{\sqrt{w_{(3)}}},\frac{e^1_{\mu_{\pi(n)}}\hat{\rho}_{\pi(n)}}{\sqrt{w_{(3)}}}\right)_{L^2(\mathbf{R}^3)}+\left(\frac{e^2_{\mu_n}\hat{\rho}_n}{\sqrt{w_{(3)}}},\frac{e^2_{\mu_{\pi(n)}}\hat{\rho}_{\pi(n)}}{\sqrt{w_{(3)}}}\right)_{L^2(\mathbf{R}^3)}\right\} \\
&= \frac{1}{2^n}\sum_{\pi\in\mathcal{P}_n}\left(\frac{d_{\mu_1\mu_{\pi(1)}}\hat{\rho}_1}{\sqrt{w_{(3)}}},\frac{\hat{\rho}_{\pi(1)}}{\sqrt{w_{(3)}}}\right)_{L^2(\mathbf{R}^3)}\times ... \times\left(\frac{d_{\mu_n\mu_{\pi(n)}}\hat{\rho}_n}{\sqrt{w_{(3)}}},\frac{\hat{\rho}_{\pi(n)}}{\sqrt{w_{(3)}}}\right)_{L^2(\mathbf{R}^3)} \\
&= \sum_{\pi\in\mathcal{P}_n}\left(\mathbf{A}^{(3)}_{\mu_1}(\rho_1),\mathbf{A}^{(3)}_{\mu_{\pi(1)}}(\rho_{\pi(1)})\right)_{A^{(3)}}\times .... \times\left(\mathbf{A}^{(3)}_{\mu_n}(\rho_n),\mathbf{A}^{(3)}_{\mu_{\pi(n)}}(\rho_{\pi(n)})\right)_{A^{(3)}} \\
&= \| : \mathbf{A}^{(3)}_{\mu_1}(\rho_1)...\mathbf{A}^{(3)}_{\mu_n}(\rho_n) :_{A^{(3)}} \|^2_{A^{(3)}}.
\end{aligned}
$$

Here $\mathcal{P}_n$ is the set of the n-degree permutation. Also we see that

$$
\begin{aligned}
&\| : A^{(4)}_{\mu_1}(\hat{\rho}_1)....A^{(4)}_{\mu_n}(\hat{\rho}_n) :_{\mathcal{F}^{(4)}} \Omega^{(4)}\|^2_{\mathcal{F}^{(4)}} \\
&= \sum_{\pi\in\mathcal{P}_n}\left\{\left(\frac{e^1_{\mu_1}\hat{\rho}_1}{w_{(4)}},\frac{e^1_{\mu_{\pi(1)}}\hat{\rho}_{\pi(1)}}{w_{(4)}}\right)_{L^2(\mathbf{R}^3\times\mathbf{R})}+\left(\frac{e^2_{\mu_1}\hat{\rho}_1}{w_{(4)}},\frac{e^2_{\mu_{\pi(1)}}\hat{\rho}_{\pi(1)}}{w_{(4)}}\right)_{L^2(\mathbf{R}^3\times\mathbf{R})}\right\}\times ... \\
&\qquad ... \times\left\{\left(\frac{e^1_{\mu_n}\hat{\rho}_n}{w_{(4)}},\frac{e^1_{\mu_{\pi(n)}}\hat{\rho}_{\pi(n)}}{w_{(4)}}\right)_{L^2(\mathbf{R}^3\times\mathbf{R})}+\left(\frac{e^2_{\mu_n}\hat{\rho}_n}{w_{(4)}},\frac{e^2_{\mu_{\pi(n)}}\hat{\rho}_{\pi(n)}}{w_{(4)}}\right)_{L^2(\mathbf{R}^3\times\mathbf{R})}\right\} \\
&= \sum_{\pi\in\mathcal{P}_n}\left(\frac{d_{\mu_1\mu_{\pi(1)}}\hat{\rho}_1}{w_{(4)}},\frac{\hat{\rho}_{\pi(1)}}{w_{(4)}}\right)_{L^2(\mathbf{R}^3\times\mathbf{R})}\times ... \times\left(\frac{d_{\mu_n\mu_{\pi(n)}}\hat{\rho}_n}{w_{(4)}},\frac{\hat{\rho}_{\pi(n)}}{w_{(4)}}\right)_{L^2(\mathbf{R}^3\times\mathbf{R})} \\
&= \sum_{\pi\in\mathcal{P}_n}\left(\mathbf{A}^{(4)}_{\mu_1}(\rho_1),\mathbf{A}^{(4)}_{\mu_{\pi(1)}}(\rho_{\pi(1)})\right)_{A^{(4)}}\times .... \times\left(\mathbf{A}^{(4)}_{\mu_n}(\rho_n),\mathbf{A}^{(4)}_{\mu_{\pi(n)}}(\rho_{\pi(n)})\right)_{A^{(4)}} \\
&= \| : \mathbf{A}^{(4)}_{\mu_1}(\rho_1)...\mathbf{A}^{(4)}_{\mu_n}(\rho_n) :_{A^{(4)}} \|^2_{A^{(4)}}.
\end{aligned}
$$

Hence it follows that $U_{(d)}$ maps the dense subspace $\mathcal{D}^{(d)}$ to the dense subspace $\mathcal{D}^{(d)}_S$ and is isometry on $\mathcal{D}^{(d)}$. Thus $U_{(d)}$ extends to a unitary operator from $\mathcal{F}^{(d)}$ to $A^{(d)}$. We shall prove (1). Since $e^{-tH_E}\mathcal{D}^{(3)} \subset \mathcal{D}^{(3)}$, $\mathcal{D}^{(3)}$ is a core for H_E [ReSi1,Theorem X.49]. Moreover since $e^{-tH_{rad}}\mathcal{D}^{(3)}_S \subset \mathcal{D}^{(3)}_S$, $\mathcal{D}^{(3)}_S$ is a core for H_{rad}. Then $U_{(3)}$ maps the core $\mathcal{D}^{(3)}$ for H_E to the core $\mathcal{D}^{(3)}_S$ for H_{rad}. It is directly seen that $U^{-1}_{(3)}e^{-tH_{rad}}U_{(3)}\lceil_{\mathcal{D}^{(3)}} = e^{-tH_E}\lceil_{\mathcal{D}^{(3)}}$. Thus we have $U^{-1}_{(3)}H_{rad}U_{(3)}\lceil_{\mathcal{D}^{(3)}} = H_E\lceil_{\mathcal{D}^{(3)}}$. Hence (1) follows. (2) is similarly proved. Next we shall prove

(3). It follows directly that $A_\mu^{(d)}(\hat{\rho}) = U_{(d)}^{-1}\mathbf{A}_\mu^{(d)}(\rho)U_{(d)}$ on $\mathcal{D}^{(d)}$. Since $\mathcal{D}^{(d)}$ is a core for $A_\mu^{(d)}(\hat{\rho})$ and $\mathcal{D}_S^{(d)}$ for $\mathbf{A}_\mu^{(d)}(\rho)$, (3) follows. □

From Lemma 3.2 it follows that $U_{(d)}$ implements the following isomorphism:

$$A^{(d)} \cong \mathcal{F}^{(d)}.$$

Set

$$\mathcal{H}_S = L^2(\mathbf{R}^3, dx) \otimes A^{(3)} \cong L^2(\mathbf{R}^3 \times \mathcal{A}^{(3)}, dx \otimes d\mu_{\mathcal{A}^{(3)}}). \tag{3. 3}$$

Without notice we shall use the identification in (3.3) in what follows. For $\rho \in \mathcal{S}_{\frac{1}{2}}$, we define $\rho(\cdot - x), x \in \mathbf{R}^3$, by

$$\rho(\cdot - x) \equiv \left(e^{-i(\cdot,x)}\hat{\rho}(\cdot)\right)^\vee,$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product and $f^\vee$ denotes the inverse Fourier transform of f in the sense of distribution. An operator H_S is defined by

$$H_S = \frac{1}{2}\left(P \otimes I_S - e\mathbf{A}^{(3)}(\rho,\cdot)\right)^2 + I_{el} \otimes H_{rad} + V \otimes I_S,$$

where $\mathbf{A}^{(3)}(\rho,x) = (\mathbf{A}_1^{(3)}(\rho,x), \mathbf{A}_2^{(3)}(\rho,x), \mathbf{A}_3^{(3)}(\rho,x))$ with $\mathbf{A}_\mu^{(3)}(\rho,x) = \mathbf{A}_\mu^{(3)}(\rho(\cdot - x)), \mu = 1,2,3$, $\rho \in \mathcal{S}_{\frac{1}{2}}$ and I_S the identity operator on $A^{(3)}$. For instance we put $I_{el} \otimes U_{(d)} = U_{(d)}$.

Proposition 3.3 *We assume assumptions (1),(2) and (3) in Proposition 2.1. Then H_S is self-adjoint on $D(H_{el} \otimes I_S) \cap D(I_{el} \otimes H_{rad})$, bounded from below and essentially self-adjoint on any core for $H_{el} \otimes I_S + I_{el} \otimes H_{rad}$. Moreover $U_{(3)}$ maps $D(H)$ onto $D(H_S)$ with*

$$H = U_{(3)}^{-1} H_S U_{(3)}. \tag{3. 4}$$

Proof: From Proposition 2.1 and Lemma 3.2 it follows the desired results. □

3.2 Definition of Hamiltonian and Functional Integral Representation

Let $\mathbf{b}(b,t) = \mathbf{b}(t) = \{b_\mu(t), t \geq 0, \mu = 1,2,3\}$ be the three dimensional Brownian motion starting at the origin, $\mathbf{b}(0) = 0$, on a probability measure space (M, db). Let $\mathcal{K}$ be a Hilbert space. For $f = f(s,x) \in C_b^1(\mathbf{R} \times \mathbf{R}^3; \mathcal{K})$, the set of $\mathcal{K}$-valued strongly continuously differentiable functions with $\sup_{|l|=0,1,(s,x)\in\mathbf{R}\times\mathbf{R}^3} \|\nabla^l f(s,x)\|_{\mathcal{K}} < \infty$, where $l = (l_0, l_1, l_2, l_3)$, $|l| = \sum_{i=0}^3 l_i$ and $\nabla^l = \nabla_s^{l_0} \nabla_{x_1}^{l_1} \nabla_{x_2}^{l_2} \nabla_{x_3}^{l_3}$, we define

$$J_n^\mu(f,t) = \sum_{k=1}^{2^n} f\left(\frac{k-1}{2^n}t, \mathbf{b}\left(\frac{k-1}{2^n}t\right)\right)\left\{b_\mu\left(\frac{k}{2^n}t\right) - b_\mu\left(\frac{k-1}{2^n}t\right)\right\}, \mu = 1,2,3, t \geq 0.$$

One sees that $\{J_n^\mu(f,t)\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(M;\mathcal{K})$, $\mathcal{K}$-valued L^2-space over M. Then we define $\mathcal{K}$-valued stochastic integral of f by the strong limit of $J_n^\mu(f,t)$ as $n \to \infty$ in $L^2(M;\mathcal{K})$ and write the limit as follows:

$$\int_0^t f(s, \mathbf{b}(s)) db_\mu(s), \mu = 1,2,3, t \geq 0.$$

Similarly, for $g = g(x) \in C_b^1(\mathbf{R}^3; \mathcal{K})$,

$$\int_0^t g(\mathbf{b}(s)) db_\mu(s), \mu = 1,2,3, t \geq 0,$$

is defined by the strong limit of

$$I_n^\mu(g,t) = \sum_{k=1}^{2^n} g\left(\mathbf{b}\left(\frac{k-1}{2^n}t\right)\right)\left\{b_\mu\left(\frac{k}{2^n}t\right) - b_\mu\left(\frac{k-1}{2^n}t\right)\right\}, \mu = 1,2,3, t \geq 0,$$

as $n \to \infty$ in $L^2(M,\mathcal{K})$. We introduce Hypothesis as follows:

Hypothesis (RI) $\hat{\rho} \in \mathcal{M}_{-1}^{(3)} \cap \mathcal{M}_1^{(3)} \cap \mathcal{M}_3^{(3)}$ and $\overline{\hat{\rho}(k)} = \hat{\rho}(-k)$.

Hypothesis (RII) The external potential $V = V_+ - V_-$, where V_+ is the positive part of V and V_- the nonpositive part, is so that $V_+ \in L_{loc}^1(\mathbf{R}^3)$ and V_- is relatively form bounded with respected to the Laplacian Δ in $L^2(\mathbf{R}^3)$ with relative bound < 1. Moreover $H_{el} = -\frac{1}{2}\Delta \dot{+} V_+ \dot{-} V_-$ has a ground state Ψ_{el}.

Remark 3.4 *(1) Hypothesis (RI) implies that* $\rho \in \mathcal{S}_{\frac{1}{2}}$ *and* $\rho(\cdot - x) \in C_b^2(\mathbf{R}^3; \mathcal{S}_{\frac{1}{2}})$ *with respect to the variable* $x \in \mathbf{R}$. *(2) Hypothesis (RII) implies that* $e^{-tH_{el}}$ *is positivity improving. Thus the ground state* Ψ_{el} *of* H_{el} *is unique and strictly positive [Fa2,FaSi,Si3].*

Define a bilinear form $B_t^\rho(\cdot,\cdot)$ on $\mathcal{H}_S$ by

$$B_t^\rho(F,G) \equiv \int_{\mathcal{M}} \left(F(\omega(t)), e^{ic\sum_{\mu=1}^3 \mathbf{A}_\mu^{(3)}\left(\int_0^t \rho(\cdot-\omega(s))db_\mu(s)\right)} G(\omega(0)) \right)_{A^{(3)}} d\omega, F, G \in \mathcal{H}_S,$$

where $\omega(t) = x + \mathbf{b}(t), x \in \mathbf{R}^3$, $\mathcal{M} = \mathbf{R}^3 \times M$, $d\omega = dx \otimes db$ and $\int_0^t \rho(\cdot - \omega(s))db_\mu(s)$ denotes $\mathcal{S}_{\frac{1}{2}}$-valued stochastic integral of $f(u) = \rho(\cdot - x - u)$.

Proposition 3.5 ([Hi1, Lemmas 4.6,4.7 and 4.8]) *We assume Hypothesis (RI). Then there exists a nonnegative self-adjoint operator* H_p *in* $\mathcal{H}_S$ *so that*

$$B_t^\rho(F,G) = (F, e^{-tH_p}G)_{\mathcal{H}_S}, F, G \in \mathcal{H}_S,$$

with $D(H_p) \supset C_0^\infty(\mathbf{R}^3)\hat{\otimes}A^{(3)\infty}$ *and*

$$H_p \lceil_{C_0^\infty(\mathbf{R}^3)\hat{\otimes}A^{(3)\infty}} = \frac{1}{2}\left(P \otimes I_S - e\mathbf{A}^{(3)}(\rho,\cdot)\right)^2 \lceil_{C_0^\infty(\mathbf{R}^3)\hat{\otimes}A^{(3)\infty}}, \tag{3. 5}$$

where $\hat{\otimes}$ *denotes algebraic tensor product.*

Since $C_0^\infty(\mathbf{R}^3)\hat{\otimes}[A^{(3)\infty} \cap D(H_{rad})] \subset D(H_p) \cap D(I_{el} \otimes H_{rad}) \cap D(V_+ \otimes I_S)$, $D(H_p) \cap D(I_{el} \otimes H_{rad}) \cap D(V_+ \otimes I_S)$ is dense. Hence we can define the form sum $H_p\dot{+}(I_{el} \otimes H_{rad})\dot{+}(V_+ \otimes I_S)$. Hypothesis (RII) implies that $V_- \otimes I_S$ is also form bounded with respected to $H_p\dot{+}(I_{el} \otimes H_{rad})$ with relative bound < 1 [Hi1, Lemma 4.11]. Hence we see that a quadratic form t

$$\begin{aligned} t(h,h) = &\left(\sqrt{H_p\dot{+}(I_{el} \otimes H_{rad})}h, \sqrt{H_p\dot{+}(I_{el} \otimes H_{rad})}h\right)_{\mathcal{H}_S} + \left(\sqrt{V_+ \otimes I_S}h, \sqrt{V_+ \otimes I_S}h\right)_{\mathcal{H}_S} \\ &- \left(\sqrt{V_- \otimes I_S}h, \sqrt{V_- \otimes I_S}h\right)_{\mathcal{H}_S}, \\ h \in Q(t) = &\, Q\left(\sqrt{H_p\dot{+}(I_{el} \otimes H_{rad})}\right) \cap Q(V_+ \otimes I_S), \end{aligned}$$

is a positive closed form on $Q(t)$, where $Q(t)$ denotes the form domain of t. We denote the positive self-adjoint operator associated with t by $\widetilde{H}_S$, i.e.,

$$\widetilde{H}_S = H_p \dot{+} (I_{el} \otimes H_{rad}) \dot{+} (V_+ \otimes I_S) \dot{-} (V_- \otimes I_S). \tag{3. 6}$$

Proposition 3.6 ([Hi1, Theorem 4.10]) *We assume Hypothesis (RI) and (RII). Then*

$$\left(F, e^{-t\widetilde{H}_S} G\right)_{\mathcal{H}_S} = \int_{\mathcal{M}} e^{-\int_0^t V(\omega(s))ds} \left(J_t F(\omega(t)), e^{ie\mathbf{A}^{(4)}(\mathcal{W}(t))} J_0 G(\omega(0))\right)_{A^{(4)}} d\omega, \tag{3. 7}$$

where $F, G \in \mathcal{H}_S$ and

$$\mathcal{W}(t) = \left(\int_0^t j_s\rho(\cdot - \omega(s))db_1(s), \int_0^t j_s\rho(\cdot - \omega(s))db_2(s), \int_0^t j_s\rho(\cdot - \omega(s))db_3(s)\right)$$

and $\int_0^t j_s\rho(\cdot - \omega(s))db_\mu(s)$ denotes $\mathcal{S}_1$-valued stochastic integral of $f(s,u) = j_s\rho(\cdot - x - u)$.

Remark 3.7 *(1) In the case where the coupling constant e is* **sufficiently large**, *we may define the Hamiltonian of the system by $\widetilde{H}_S$. For we have equations (3.4) and (3.5). More explicitly, Lemma 3.2 implies that, under Hypothesis RI and RII,*

$$H\lceil_{C_0^\infty(\mathbf{R}^3)\hat{\otimes}[\mathcal{F}^{(3)\infty}\cap D(H_E)]} \cong H_S\lceil_{C_0^\infty(\mathbf{R}^3)\hat{\otimes}[A^{(3)\infty}\cap D(H_{rad})]} \subset \widetilde{H}_S, \tag{3. 8}$$

In this sense, $\widetilde{H}_S$ is non-perturbative definition of the Hamiltonian of the system. (2) In the case where the coupling constant e is sufficiently small, under the conditions (1),(2) and (3) in Proposition 2.1, it holds that

$$H_S = \widetilde{H}_S. \tag{3. 9}$$

In this case, (3.7) also holds with the condition (1), (2) and (3) in Proposition 2.1. (3) Even in the case where the coupling constant e is sufficiently large, H_S defines a nonnegative symmetric operator. Then Friedrichs extension of H_S exists. However, as far as we know, we do not know whether the Friedrichs extension and $\widetilde{H}_S$ coincides or not.

4 UNIQUENESS OF GROUND STATE

This section is the main section in this paper. We shall show that the ground state of $\widetilde{H}_S$ is unique.

4.1 Positivity Preserving

The Plancherel-Wiener transformation [Sel, Corollary 3.3] in $A^{(d)}$ is defined by

$$\vartheta_{(d)} = \exp\left(-i\frac{\pi}{2}N_S^{(d)}\right).$$

Let us define

$$\mathbf{E}_\mu^{(d)}(f) \equiv \vartheta_{(d)}^{-1}\mathbf{A}_\mu^{(d)}(f)\vartheta_{(d)}, \quad \mu = 1,2,3.$$

For notational convenience, we put $\vartheta_{(3)} = \vartheta$. For $f \in \mathcal{S}_r(\mathbf{R}^3 \times \mathbf{R})$, in accordance with Lemma 3.2, we have

$$\begin{aligned} U_{(4)}^{-1}\mathbf{E}_\mu^{(4)}(f)U_{(4)} &= e^{i\frac{\pi}{2}N^{(4)}}A_\mu^{(4)}(\hat{f})e^{-i\frac{\pi}{2}N^{(4)}} \\ &= i\sum_{r=1}^{2}\int \frac{\left\{\hat{f}(k,k_0)e_\mu^r(k)a_{(4)}^{\dagger r}(k,k_0) - \hat{f}(-k,-k_0)e_\mu^r(k)a_{(4)}^r(k,k_0)\right\}dkdk_0}{\sqrt{|k|^2+|k_0|^2}} \\ &\equiv E_\mu^{(4)}(\hat{f}). \end{aligned} \tag{4. 1}$$

Similarly, for $f \in \mathcal{S}_r(\mathbf{R}^3)$, we have

$$\begin{aligned} U_{(3)}^{-1}\mathbf{E}_\mu^{(3)}(f)U_{(3)} &= e^{i\frac{\pi}{2}N^{(3)}}A_\mu^{(3)}(\hat{f})e^{-i\frac{\pi}{2}N^{(3)}} \\ &= i\sum_{r=1}^{2}\int \frac{\left\{\hat{f}(k)e_\mu^r(k)a_{(3)}^{\dagger r}(k) - \hat{f}(-k)e_\mu^r(k)a_{(3)}^r(k)\right\}dk}{\sqrt{2|k|}} \\ &\equiv E_\mu^{(3)}(\hat{f}). \end{aligned} \tag{4. 2}$$

By a limiting argument, (4.1) and (4.2) extend to $f \in \mathcal{S}_1$ and $f \in \mathcal{S}_{\frac{1}{2}}$, respectively. Note that, for $\rho = (\rho_1,\rho_2,\rho_3)$ and $\lambda = (\lambda_1,\lambda_2,\lambda_3)$,

$$\sum_{\mu,\nu=1}^{3}[A_\mu^{(d)}(\hat{\rho}_\mu), E_\nu^{(d)}(\hat{\lambda}_\nu)] = iq^{(d)}(\rho,\lambda),$$

$$\sum_{\mu,\nu=1}^{3}\left(A_{\mu}^{(d)}(\hat{\rho}_{\mu})\Omega^{(d)}, E_{\nu}^{(d)}(\hat{\lambda}_{\nu})\Omega^{(d)}\right)_{\mathcal{F}^{(d)}} = \frac{i}{2}q^{(d)}(\rho,\lambda).$$

For $f=(f_1,f_2,f_3)\in(\mathcal{S}_1)^{\times 3}$ for $d=4$ and $f=(f_1,f_2,f_3)\in(\mathcal{S}_{\frac{1}{2}})^{\times 3}$ for $d=3$, we set $\mathbf{E}^{(d)}(f)=\sum_{\mu=1}^{3}\mathbf{E}_{\mu}^{(d)}(f_{\mu})$. Using (4.1) and (4.2), we see that, for $\alpha\in\mathbf{C}$,

$$:e^{\alpha\mathbf{A}^{(d)}(G)}:_{A^{(d)}}\Omega_S^{(d)} = e^{-\frac{\alpha^2}{4}q^{(d)}(G)}e^{\alpha\mathbf{A}^{(d)}(G)}\Omega_S^{(d)}, \tag{4.3}$$

$$:e^{\alpha\mathbf{E}^{(d)}(G)}:_{A^{(d)}}\Omega_S^{(d)} = e^{-\frac{\alpha^2}{4}q^{(d)}(G)}e^{\alpha\mathbf{E}^{(d)}(G)}\Omega_S^{(d)}, \tag{4.4}$$

$$:e^{\alpha\mathbf{E}^{(d)}(G)}:_{A^{(d)}}\Omega_S^{(d)} =: e^{\alpha i\mathbf{A}^{(d)}(G)}:_{A^{(d)}}\Omega_S^{(d)}. \tag{4.5}$$

Hence (4.3), (4.4) and (4.5) imply that, for $\alpha\in\mathbf{C}$,

$$\begin{aligned} e^{\alpha\mathbf{E}^{(d)}(G)}\Omega_S^{(d)} &= e^{\frac{\alpha^2}{4}q^{(d)}(G)}:e^{\alpha\mathbf{E}^{(d)}(G)}:_{A^{(d)}}\Omega_S^{(d)} \\ &= e^{\frac{\alpha^2}{4}q^{(d)}(G)}:e^{\alpha i\mathbf{A}^{(d)}(G)}:_{A^{(d)}}\Omega_S^{(d)} \\ &= e^{\frac{\alpha^2}{2}q^{(d)}(G)}e^{\alpha i\mathbf{A}^{(d)}(G)}\Omega_S^{(d)}. \end{aligned} \tag{4.6}$$

Especially, by (4.6), we have

$$e^{i\mathbf{E}^{(d)}(G)}\Omega_S^{(d)} = e^{-\frac{1}{2}q^{(d)}(G)}e^{-\mathbf{A}^{(d)}(G)}\Omega_S^{(d)}. \tag{4.7}$$

The Weyl relation follows from (4.1) and (4.2):

$$e^{i\mathbf{E}^{(d)}(G)}e^{i\mathbf{A}^{(d)}(H)} = e^{iq^{(d)}(H,G)}e^{i\mathbf{A}^{(d)}(H)}e^{i\mathbf{E}^{(d)}(G)}. \tag{4.8}$$

Lemma 4.1 *Let $G,G_1,G_2\in(\mathcal{S}_1)^{\times 3}$ for $d=4$ and $G,G_1,G_2\in(\mathcal{S}_{\frac{1}{2}})^{\times 3}$ for $d=3$. Then we have*

$$\left(e^{i\mathbf{A}^{(d)}(G_1)}\Omega_S^{(d)}, e^{i\mathbf{E}^{(d)}(G)}e^{i\mathbf{A}^{(d)}(G_2)}\Omega_S^{(d)}\right)_{A^{(d)}} = e^{\frac{i}{2}q^{(d)}(G_1+G_2,G)}e^{-\frac{1}{4}q^{(d)}(G_1-G_2)}e^{-\frac{1}{4}q^{(d)}(G)}. \tag{4.9}$$

Proof: By (4.8) we see that

$$\left(e^{i\mathbf{A}^{(d)}(G_1)}\Omega_S^{(d)}, e^{i\mathbf{E}^{(d)}(G)}e^{i\mathbf{A}^{(d)}(G_2)}\Omega_S^{(d)}\right)_{A^{(d)}} = \left(e^{i\mathbf{A}^{(d)}(G_1-G_2)}\Omega_S^{(d)}, e^{i\mathbf{E}^{(d)}(G)}\Omega_S^{(d)}\right)_{A^{(d)}} e^{iq^{(d)}(G_2,G)}.$$

Moreover from (4.7) it follows that

$$\begin{aligned}\left(e^{i\mathbf{A}^{(d)}(G_1-G_2)}\Omega_S^{(d)}, e^{i\mathbf{E}^{(d)}(G)}\Omega_S^{(d)}\right)_{A^{(d)}} &= \left(e^{i\mathbf{A}^{(d)}(G_1-G_2)}\Omega_S^{(d)}, e^{-\mathbf{A}^{(d)}(G)}\Omega_S^{(d)}\right)_{A^{(d)}} e^{-\frac{1}{2}q^{(d)}(G)} \\ &= e^{\frac{i}{2}q^{(d)}(G_1-G_2,G)} e^{-\frac{1}{4}q^{(d)}(G_1-G_2)} e^{-\frac{1}{4}q^{(d)}(G)}.\end{aligned}$$

Hence (4.9) follows. □

Lemma 4.2 *Let $G \in (\mathcal{S}_1)^{\times 3}$ for $d = 4$ and $G \in (\mathcal{S}_{\frac{1}{2}})^{\times 3}$ for $d = 3$. Then $e^{i\mathbf{E}^{(d)}(G)}$ is positivity preserving on $A^{(d)}$.*

Proof: We must show that, for all positive $\theta_j \in A^{(d)}, j = 1, 2,$

$$\left(\theta_1, e^{i\mathbf{E}^{(d)}(G)}\theta_2\right)_{A^{(d)}} \geq 0. \tag{4. 10}$$

For any positive $\Psi \in A^{(d)}$, we can take a positive sequence $\{\Psi_n\}_{n=1}^{\infty} \subset \mathcal{Q}^{(d)}$ so that $\Psi_n \to \Psi$ in $A^{(d)}$ strongly [HoSi,Theorem 3.2]. Then it suffices to show (4.10) for any positive $\theta_1, \theta_2 \in \mathcal{Q}^{(d)}$. Assume that

$$\begin{aligned}\theta_1 &= \theta_1(\mathbf{A}^{(d)}(K_1), ..., \mathbf{A}^{(d)}(K_n)) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbf{R}^n} d\vec{t}\,\widehat{\theta_1}(\vec{t}) e^{i\mathbf{A}^{(d)}(\sum_{j=1}^n t_j K_j)}, \\ \theta_2 &= \theta_2(\mathbf{A}^{(d)}(L_1), ..., \mathbf{A}^{(d)}(L_m)) = \frac{1}{(\sqrt{2\pi})^m}\int_{\mathbf{R}^m} d\vec{s}\,\widehat{\theta_2}(\vec{s}) e^{i\mathbf{A}^{(d)}(\sum_{i=1}^m s_i L_i)},\end{aligned}$$

where $K_j, L_i \in (\mathcal{S}_r(\mathbf{R}^d))^{\times 3}, j = 1, ..., n, i = 1, .., m$, $\vec{t} = (t_1, ..., t_n), \vec{s} = (s_1, ..., s_m)$. We have

$$\begin{aligned}&\left(\theta_1, e^{i\mathbf{E}^{(d)}(G)}\theta_2\right)_{A^{(d)}} \\ &= \frac{1}{(\sqrt{2\pi})^{n+m}}\int_{\mathbf{R}^{n+m}} d\vec{t}d\vec{s}\,\overline{\widehat{\theta_1}(\vec{t})}\widehat{\theta}_2(\vec{s})\left(e^{i\mathbf{A}^{(d)}(\sum_{j=1}^n t_j K_j)}, e^{i\mathbf{E}^{(d)}(G)} e^{i\mathbf{A}^{(d)}(\sum_{i=1}^m s_i L_i)}\right)_{A^{(d)}} \\ &= \frac{1}{(\sqrt{2\pi})^{n+m}}\int_{\mathbf{R}^{n+m}} d\vec{t}d\vec{s}\,\overline{\widehat{\theta_1}(\vec{t})}\widehat{\theta}_2(\vec{s}) e^{\frac{i}{2}\sum_{j=1}^n q^{(d)}(K_j,G)t_j} e^{\frac{i}{2}\sum_{i=1}^m q^{(d)}(L_i,G)s_i}\mathcal{G}(\vec{t},\vec{s}) e^{-\frac{1}{4}q^{(d)}(G)} \\ &= \left[({\theta_1}^T \otimes \theta_2) * \mathcal{G}^{\vee}\right](u_1, ., u_n, v_1, .., v_m) e^{-\frac{1}{4}q^{(d)}(G)},\end{aligned} \tag{4. 11}$$

where ${\theta_1}^T(x) = \theta(-x)$, the symbol $*$ denotes convolution,

$$\mathcal{G}(\vec{t},\vec{s}) = e^{-\frac{1}{4}q^{(d)}(\sum_{j=1}^n K_j t_j - \sum_{i=1}^m L_i s_i)},$$

and

$$u_1 = \frac{1}{2}q^{(d)}(K_1, G), .., u_n = \frac{1}{2}q^{(d)}(K_n, G), v_1 = \frac{1}{2}q^{(d)}(L_1, G), .., v_m = \frac{1}{2}q^{(d)}(L_m, G).$$

The second equality in (4.11) is due to (4.9). Since $\mathcal{G}^{\vee} > 0$, (4.10) follows. □

Next lemma is explained by J.Fröhlich in private communication.

Lemma 4.3 *We assume Hypothesis (RI) and (RII). Then $\vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta$ is positivity preserving.*

Proof: Assume that $F, G \in \mathcal{H}_S$ are positive. Note that $J_s\vartheta = \vartheta_{(4)}J_s$. From Proposition 3.6 it follows that

$$\left(F, \vartheta^{-1}e^{-t\widetilde{H}_S}\vartheta G\right)_{\mathcal{H}_S} = \int_{\mathcal{M}} e^{-\int_0^t V(\omega(s))ds}\left(J_t F(\omega(t)), e^{ie\mathbf{E}^{(4)}(\mathcal{W}(t))}J_0 G(\omega(0))\right)_{\mathcal{A}^{(4)}} d\omega.$$

Since J_t and J_0 are positivity preserving, Lemma 4.2 yields that

$$\left(J_t F(\omega(t)), e^{ie\mathbf{E}^{(4)}(\mathcal{W}(t))}J_0 G(\omega(0))\right)_{\mathcal{A}^{(4)}} \geq 0$$

for almost everywhere $(x, b) \in \mathcal{M}$. Thus lemma follows. □

4.2 Positivity Improving

In this subsection, we shall prove that $J_t^* e^{i\mathbf{E}^{(4)}(\mathcal{W}(t))}J_0$ is positivity improving and the ground state of $\widetilde{H}_S$ is unique. For $\alpha \in \mathbf{C}$, we note that

$$D(e^{\alpha\mathbf{A}^{(d)}(h)}) \supset \mathcal{Q}^{(d)},$$

where $h \in (\mathcal{S}_1)^{\times 3}$ for $d = 4$ and $h \in (\mathcal{S}_{\frac{1}{2}})^{\times 3}$ for $d = 3$. The following lemma is key lemma in this subsection.

Lemma 4.4 *Let $G \in \mathcal{Q}^{(3)}$ and $h \in (\mathcal{S}_1)^{\times 3}$. Then $J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^* h)}e^{\mathbf{A}^{(3)}(\mathbf{j}_0^* h)}G \in D\left(e^{-\mathbf{A}^{(4)}(h)}\right)$ with*

$$e^{-\mathbf{A}^{(4)}(h)}J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^* h)}e^{\mathbf{A}^{(3)}(\mathbf{j}_0^* h)}G = e^{\frac{1}{2}\left(q^{(4)}(h)+q^{(3)}(\mathbf{j}_0^* h)\right)}e^{i\mathbf{E}^{(4)}(h)}J_0 G. \tag{4.12}$$

Proof: We assume that

$$G = G(\mathbf{A}^{(3)}(K_1), ..., \mathbf{A}^{(3)}(K_n)) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbf{R}^n} d\vec{t}\widehat{G}(\vec{t}) e^{i\mathbf{A}^{(3)}(\sum_{j=1}^n t_j K_j)}, \qquad (4.\ 13)$$

where $G \in \mathcal{S}(\mathbf{R}^n)$, $K_j \in (\mathcal{S}_r(\mathbf{R}^3 \times \mathbf{R}))^{\times 3}, j = 1, ..., n$, $\vec{t} = (t_1, ..., t_n)$. Since J_0 is isometry, we see that

$$J_0 G(\mathbf{A}^{(3)}(K_1), ..., \mathbf{A}^{(3)}(K_n))\Omega_S^{(3)} = G(\mathbf{A}^{(4)}(\mathbf{j}_0 K_1), ..., \mathbf{A}^{(4)}(\mathbf{j}_0 K_n))\Omega_S^{(4)}.$$

Note that, for $K_j = (K_{j1}, K_{j2}, K_{j3})$ and $h = (h_1, h_2, h_3)$,

$$\begin{aligned} q^{(4)}(\mathbf{j}_0 K_j, h) &= \sum_{\mu,\nu=1}^{3} \left(d_{\mu\nu}(j_0 \widehat{K_{j\mu}}), \widehat{h_\nu}\right)_{-1} \\ &= \sum_{\mu,\nu=1}^{3} \left(j_0 (d_{\mu\nu}\widehat{K_{j\mu}})^\vee, h_\nu\right)_{\mathcal{S}_1} \\ &= \sum_{\mu,\nu=1}^{3} \left((d_{\mu\nu}\widehat{K_{j\mu}})^\vee, j_0^* h_\nu\right)_{\mathcal{S}_{\frac{1}{2}}} \\ &= q^{(3)}(K_j, \mathbf{j}_0^* h), \quad j = 1, ..., n. \end{aligned}$$

By virtue of (4.7),(4.8) and (4.13) we have

$$\begin{aligned} & e^{i\mathbf{E}^{(4)}(h)} J_0 G \Omega_S^{(3)} \\ &= G\left(\mathbf{A}^{(4)}(\mathbf{j}_0 K_1) + q^{(4)}(\mathbf{j}_0 K_1, h), ..., \mathbf{A}^{(4)}(\mathbf{j}_0 K_n) + q^{(4)}(\mathbf{j}_0 K_n, h)\right) e^{i\mathbf{E}^{(4)}(h)} \Omega_S^{(4)} \\ &= e^{-\frac{1}{2}q^{(4)}(h)} e^{-\mathbf{A}^{(4)}(h)} G\left(\mathbf{A}^{(4)}(\mathbf{j}_0 K_1) + q^{(4)}(\mathbf{j}_0 K_1, h), ..., \mathbf{A}^{(4)}(\mathbf{j}_0 K_n) + q^{(4)}(\mathbf{j}_0 K_n, h)\right) \Omega_S^{(4)} \\ &= e^{-\frac{1}{2}q^{(4)}(h)} e^{-\mathbf{A}^{(4)}(h)} G\left(\mathbf{A}^{(4)}(\mathbf{j}_0 K_1) + q^{(3)}(K_1, \mathbf{j}_0^* h), ..., \mathbf{A}^{(4)}(\mathbf{j}_0 K_n) + q^{(3)}(K_n, \mathbf{j}_0^* h)\right) \Omega_S^{(4)} \\ &= e^{-\frac{1}{2}q^{(4)}(h)} e^{-\mathbf{A}^{(4)}(h)} J_0 G\left(\mathbf{A}^{(3)}(K_1) + q^{(3)}(K_1, \mathbf{j}_0^* h), ..., \mathbf{A}^{(3)}(K_n) + q^{(3)}(K_n, \mathbf{j}_0^* h)\right) \Omega_S^{(3)} \\ &= e^{-\frac{1}{2}q^{(4)}(h)} e^{-\mathbf{A}^{(4)}(h)} J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^* h)} G e^{-i\mathbf{E}^{(3)}(\mathbf{j}_0^* h)} \Omega_S^{(3)} \\ &= e^{-\frac{1}{2}q^{(4)}(h)} e^{-\mathbf{A}^{(4)}(h)} J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^* h)} G e^{-\frac{1}{2}q^{(3)}(\mathbf{j}_0^* h)} e^{\mathbf{A}^{(3)}(\mathbf{j}_0^* h)} \Omega_S^{(3)} \\ &= e^{-\frac{1}{2}\left(q^{(4)}(h) + q^{(3)}(\mathbf{j}_0^* h)\right)} e^{-\mathbf{A}^{(4)}(h)} J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^* h)} e^{\mathbf{A}^{(3)}(\mathbf{j}_0^* h)} G \Omega_S^{(3)}. \end{aligned}$$

Thus (4.12) follows. □

The right hand side of (4.12) is a bounded operator from $A^{(3)}$ to $A^{(4)}$. Hence we see that the bounded operator extension of $e^{-\mathbf{A}^{(4)}(h)}J_0 e^{i\mathbf{E}^{(3)}(\mathrm{j}_0^* h)}e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}\restriction_{\mathcal{Q}^{(3)}}$ has

$$\overline{e^{-\mathbf{A}^{(4)}(h)}J_0 e^{i\mathbf{E}^{(3)}(\mathrm{j}_0^* h)}e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}\restriction_{\mathcal{Q}^{(3)}}} = e^{\frac{1}{2}\left(q^{(4)}(h)+q^{(3)}(\mathrm{j}_0^* h)\right)}e^{i\mathbf{E}^{(4)}(h)}J_0.$$

Let us define bounded multiplication operators by $\left(e^{-\mathbf{A}^{(4)}(h)}\right)_N$ and $\left(e^{\mathbf{A}^{(3)}(j_0^* h)}\right)_N$ as follows:

$$\left(e^{-\mathbf{A}^{(4)}(h)}\right)_N = \begin{cases} e^{-\mathbf{A}^{(4)}(h)}, & e^{-\mathbf{A}^{(4)}(h)} < N, \\ N, & e^{-\mathbf{A}^{(4)}(h)} \geq N, \end{cases}$$
$$\left(e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}\right)_N = \begin{cases} e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}, & e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)} < N, \\ N, & e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)} \geq N. \end{cases}$$

Lemma 4.5 *Assume that $h \in (\mathcal{S}_1)^{\times 3}$. Then the bounded operator*

$$P_N = J_t^* \left(e^{-\mathbf{A}^{(4)}(h)}\right)_N J_0$$

is positivity improving on $A^{(3)}$.

Proof: Let $\theta_j \in A^{(3)}, j = 1, 2$, be positive. It is clear that the operator P_N is positivity preserving, i.e.,

$$(\theta_1, P_N \theta_2)_{A^{(3)}} \geq 0.$$

We assume that

$$(\theta_1, P_N \theta_2)_{A^{(3)}} = \left(\left(e^{-\mathbf{A}^{(4)}(h)}\right)_N J_t \theta_1, J_0 \theta_2\right)_{A^{(4)}} = 0. \tag{4. 14}$$

Since $\left(e^{-\mathbf{A}^{(4)}(h)}\right)_N J_t$ is positivity preserving, (4.14) implies that

$$\left|\mathrm{supp}\left\{\left(e^{-\mathbf{A}^{(4)}(h)}\right)_N J_t \theta_1\right\} \cap \mathrm{supp} J_0 \theta_2\right|_{(4)} = 0, \tag{4. 15}$$

where $|S|_{(d)}$ denotes the measure of a set S in $\mathcal{A}^{(d)}$. Since $\int_{\mathcal{A}^{(4)}} |\mathbf{A}^{(4)}(h)|^2 d\mu_{\mathcal{A}^{(4)}} = \frac{1}{2}q^{(4)}(h) < \infty$, $|\mathbf{A}^{(4)}(h)|$ is bounded for almost surely $\mathbf{A}^{(4)} \in \mathcal{A}^{(4)}$. Thus we see that

$$\left|\left\{\mathbf{A}^{(4)} \in \mathcal{A}^{(4)} \,\middle|\, \left(e^{-\mathbf{A}^{(4)}(h)}\right)_N = 0\right\}\right|_{(4)} = 0. \tag{4. 16}$$

Thus (4.15) and (4.16) imply that

$$|\mathrm{supp} J_t\theta_1 \cap \mathrm{supp} J_0\theta_2|_{(4)} = 0. \tag{4. 17}$$

(4.17) deduces that

$$(J_t\theta_1, J_0\theta_2)_{A^{(4)}} = \left(\theta_1, e^{-tH_{rad}}\theta_2\right)_{A^{(3)}} = 0. \tag{4. 18}$$

Since $e^{-tH_{rad}}$ is positivity improving, (4.18) is contradiction. Hence there exist no vectors θ_j as in (4.14). Thus lemma follows. □

Lemma 4.6 *Assume that $h \in (\mathcal{S}_1)^{\times 3}$. Then the bounded operator*

$$Q_N = e^{-\frac{1}{2}\left(q^{(4)}(h)+q^{(3)}(\mathrm{j}_0^* h)\right)} P_N e^{i\mathbf{E}^{(3)}(\mathrm{j}_0^* h)} \left(e^{-\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}\right)_N$$

is positivity improving on $A^{(3)}$.

Proof: By Lemma 4.5, P_N is positivity improving and the both $e^{i\mathbf{E}^{(3)}(\mathrm{j}_0^* h)}$ and $\left(e^{-\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}\right)_N$ are positivity preserving. Hence Q_N is positivity improving. □

Theorem 4.7 *Assume that $h \in (\mathcal{S}_1)^{\times 3}$. Then the bounded operator*

$$J_t^* e^{i\mathbf{E}^{(4)}(h)} J_0$$

is positivity improving on $A^{(3)}$.

Since J_t^*, J_0 and $e^{i\mathbf{E}^{(3)}(\mathrm{j}_0^* h)}$ are positivity preserving and

$$\left(e^{-\mathbf{A}^{(4)}(h)}\right)_N \leq e^{-\mathbf{A}^{(4)}(h)},$$
$$\left(e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)}\right)_N \leq e^{\mathbf{A}^{(3)}(\mathrm{j}_0^* h)},$$

we see that, for positive $\theta \in \mathcal{Q}^{(3)}$,

$$\begin{aligned} Q_N\theta &= e^{-\frac{1}{2}\left(q^{(4)}(h)+q^{(3)}(\mathbf{j}_0^*h)\right)} J_t^* \left(e^{-\mathbf{A}^{(4)}(h)}\right)_N J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^*h)} \left(e^{\mathbf{A}^{(3)}(\mathbf{j}_0^*h)}\right)_N \theta \\ &\leq e^{-\frac{1}{2}\left(q^{(4)}(h)+q^{(3)}(\mathbf{j}_0^*h)\right)} J_t^* e^{-\mathbf{A}^{(4)}(h)} J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^*h)} e^{\mathbf{A}^{(3)}(\mathbf{j}_0^*h)} \theta \\ &= e^{-\frac{1}{2}\left(q^{(4)}(h)+q^{(3)}(\mathbf{j}_0^*h)\right)} J_t^* \overline{\left(e^{-\mathbf{A}^{(4)}(h)} J_0 e^{i\mathbf{E}^{(3)}(\mathbf{j}_0^*h)} e^{\mathbf{A}^{(3)}(\mathbf{j}_0^*h)} \lceil_{\mathcal{Q}^{(3)}}\right)} \theta \\ &= J_t^* e^{i\mathbf{E}^{(4)}(h)} J_0 \theta. \end{aligned} \tag{4. 19}$$

Since arbitrary positive Ψ in $A^{(3)}$ can be strongly approximated by a positive sequence $\{\theta_n\}_{n=1}^{\infty} \subset \mathcal{Q}^{(3)}$ in $A^{(3)}$, (4.19) yields that, for any positive $\Psi \in A^{(3)}$,

$$Q_N\Psi \leq J_t^* e^{i\mathbf{E}^{(4)}(h)} J_0 \Psi.$$

Thus, for all positive $\theta_j \in A^{(3)}, j = 1, 2,$

$$\left(\theta_1, J_t^* e^{i\mathbf{E}^{(4)}(h)} J_0 \theta_2\right)_{A^{(3)}} \geq (\theta_1, Q_N\theta_2)_{A^{(3)}} > 0.$$

Hence theorem follows. □

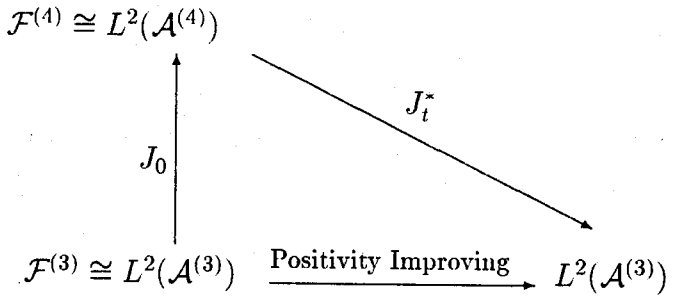

Figure 4.1(Lemma 3.1)

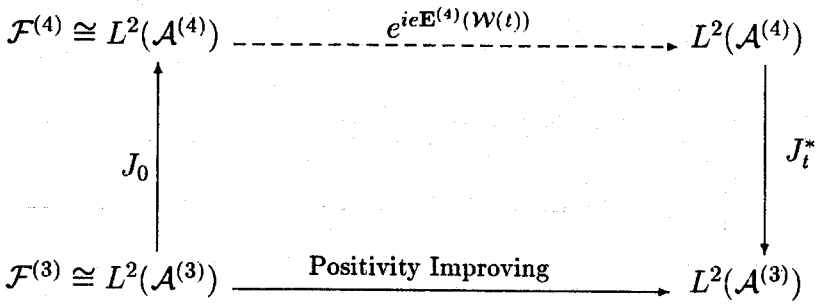

Figure 4.2 (Theorem 4.7)

Theorem 4.8 *We assume Hypothesis (RI),(RII) and (S). Then $\vartheta^{-1}e^{-t\tilde{H}_S}\vartheta$ is positivity improving.*

Proof: We must show that, for all positive $F, G \in \mathcal{H}_S$,

$$\left(F, \vartheta^{-1}e^{-t\tilde{H}_S}\vartheta G\right)_{\mathcal{H}_S} > 0. \tag{4. 20}$$

Since the semigroup

$$e^{-t(H_{el}\otimes I_S + I_{el}\otimes H_{rad})} \tag{4. 21}$$

is positivity preserving and $H_{el}\otimes I_S + I_{el}\otimes H_{rad}$ has the strictly positive unique ground state

$$\Psi_{el}\otimes\Omega_S^{(3)}.$$

Hence (4.21) is ergodic and then positivity improving. By Proposition 3.6, we have

$$\begin{aligned}\left(F, e^{-t(H_{el}\otimes I_S + I_{el}\otimes H_{rad})}G\right)_{\mathcal{H}_S} &= \int_{\mathcal{M}} e^{-\int_0^t V(\omega(s))ds}\left(F(\omega(t)), e^{-tH_{rad}}G(\omega(0))\right)_{A^{(3)}} d\omega \\ &> 0. \end{aligned} \tag{4. 22}$$

Since

$$e^{-\int_0^t V(\omega(s))ds}\left(F(\omega(t)), e^{-tH_{rad}}G(\omega(0))\right)_{A^{(3)}} \geq 0, \quad (x,b)\in\mathcal{M},$$

(4.22) implies that there exists a subset $\mathcal{M}' \subset \mathcal{M}$ with positive measure so that

$$\left(F(\omega(t)), e^{-tH_{rad}} G(\omega(0))\right)_{A^{(3)}} > 0, \quad (x,b) \in \mathcal{M}'.$$

Hence, for $(x,b) \in \mathcal{M}'$, it follows that $F(\omega(t)) \not\equiv 0$ and $G(\omega(0)) \not\equiv 0$ a.e. in $\mathcal{A}^{(3)}$. Since, by Theorem 4.7, $J_t^* e^{ie\mathbf{E}^{(4)}(\mathcal{W}(t))} J_0$ is positivity improving, we have

$$\left(F(\omega(t)), J_t^* e^{ie\mathbf{E}^{(4)}(\mathcal{W}(t))} J_0 G(\omega(0))\right)_{A^{(3)}} > 0, \quad (x,b) \in \mathcal{M}'.$$

Thus it follows that

$$\begin{aligned} \left(F, \vartheta^{-1} e^{-t\widetilde{H}_F(\hat{\rho})} \vartheta G\right)_{\mathcal{H}_S} &\geq \int_{\mathcal{M}'} e^{-\int_0^t V(\omega(s))ds} \left(F(\omega(t)), J_t^* e^{ie\mathbf{E}^{(4)}(\mathcal{W}(t))} J_0 G(\omega(0))\right)_{A^{(3)}} d\omega \\ &> 0. \end{aligned}$$

Hence (4.20) follows. □

Corollary 4.9 (Uniqueness) *We assume Hypothesis (RI), (RII), (S) and that the ground state of $\widetilde{H}_S$ exists. Then the ground state of $\widetilde{H}_S$ is unique and unitary transform of a strictly positive function.*

Proof: By Theorem 4.8 and Proposition 1.3, the ground state of $\vartheta^{-1}\widetilde{H}_S\vartheta$ is unique and strictly positive, which implies the desired result. □

Corollary 4.10 (Overlap) *We assume Hypothesis (RI), (RII), (S) and that the ground state of $\widetilde{H}_S$, Ω_{ground}, exist. Then*

$$(\Psi_{el} \otimes \Omega_S^{(3)}, \Omega_{ground})_{\mathcal{H}_S} \neq 0. \tag{4. 23}$$

Proof: By Corollary 4.9, the ground state Ω_{ground} has the form $\Omega_{ground} = \vartheta\tilde{\Omega}_0$, where $\tilde{\Omega}_0$ is strictly positive in $\mathcal{H}_S$. Hence

$$(\Psi_{el} \otimes \Omega_S^{(3)}, \Omega_{ground})_{\mathcal{H}_S} = (\Psi_{el} \otimes \vartheta^{-1}\Omega_S^{(3)}, \tilde{\Omega}_0)_{\mathcal{H}_S} = (\Psi_{el} \otimes \Omega_S^{(3)}, \tilde{\Omega}_0)_{\mathcal{H}_S} > 0.$$

Thus (4.23) follows. □

Corollary 4.11 (Ground State Energy) *We assume Hypothesis (RI), (RII), (S) and that the ground state of* $\widetilde{H}_S$ *exist. Then*

$$\inf \sigma(\widetilde{H}_S) = \lim_{\beta\to\infty} -\frac{1}{\beta} \log \int_{\mathcal{M}} d\omega \Psi_{el}(\omega(\beta))\Psi_{el}(\omega(0)) e^{-\int_0^\beta V(\omega(s))ds} e^{-\frac{e^2}{4} q^{(4)}(\mathcal{W}(\beta))}. \quad (4.\ 24)$$

Proof: From Corollary 4.10 it follows that

$$\inf \sigma(\widetilde{H}_S) = \lim_{\beta\to\infty} -\frac{1}{\beta} \log \left(\Psi_{el} \otimes \Omega_S^{(3)}, e^{-\beta \widetilde{H}_S} \Psi_{el} \otimes \Omega_S^{(3)} \right)_{\mathcal{H}_S}.$$

Hence, by Proposition 3.6, we see that (4.24) holds. □

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