DETERMINACY, TRANSVERSALITY AND LAGRANGE STABILITY

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

1. Introduction. In the present article, as a continuation of [14], it is given Arnol'd-Mather type characterisation of Lagrange stability for a class of singular Lagrange varieties, open Whitney umbrellas, via the transversality in isotropic jet spaces. Also the determinacy of isotropic map-germs by jets under Lagrange equivalence is considered. We give an example of Lagrange stable isotropic map-germ of corank one in itself and of corank two after the Lagrange projection. Lastly we mention open questions.

Let X be an n-dimensional manifold, and (M, ω) a $2n$-dimensional symplectic manifold with a symplectic form ω, $(n \geq 1)$. A C^∞ mapping $f : X \to M$ is called *isotropic* if $f^*\omega = 0$. Then f is a *Lagrange immersion* off the singular locus $\Sigma(f) = \{x \in X \mid f \text{ is not immersive at } x\}$.

The main object we are studing is a special class of singularities of isotropic mappings, namely, the open Whitney umbrellas, which are first recognised by Arnol'd, Givental' and Zakalyukin [6] [7].

We are mainly interested in the case $M = T^*Y$, the cotangent bundle over an n-manifold Y, endowed with the symplectic form $\omega = d\theta_Y$, the exterior differential of the Liouville 1-form θ_Y on T^*Y. Consider the canonical Lagrange projection $\pi : T^*Y \to Y$. Then singularities of Lagrange projections of Lagrange submanifolds are called *Lagrange singlarities.* The study of Lagrange singularities is reduced by Hörmander and Arnol'd to the theory of deformations of functions by means of generating families of Morse type. Based on this reduction, Lagrange singularities are studied extensively. See [1][26][4].

An open Whitney umbrella is obtained as a component of a singular Lagrangian variety induced by a non-Morse generating family. For this direction, see [15][27]. In this

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paper we study singularities of Lagrangian projections of open Whitney umbrellas, from the view point of Thom-Mather's theory of differentiable mapping.

After Thom's work, Mather, in the series of papers [18], gives the theory on C^∞ stable mappings. Restricting ourselves to local C^∞ theory, we recall the following results due to Mather on C^∞ map-germs $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$: (A) Infinitesimal characterisation of stable map-germs. (B) Determinacy: If f is stable then f is $(p+1)$-determined. (C) Classification by $\mathbf{R}$-algebras of stable map-germs. Construction of a stable germ of given algebra type. (D) Characterization of stablity by transversality. (E) Determination of "nice range" where stable maps are generic.

Then we are naturally led to the question: *Are there analogies to Thom-Mather's theory, for Lagrange stable projections of open Whitney umbrellas ?*

A germ of submersion $\pi' : (M, y_0) \to (Y, z_0)$ is called a *Lagrange projection* if all fibers of π' are Lagrange submanifolds, in other word, if any pairs of components of π' are Poisson commutative.

Consider a pair (f, π) (resp. (f', π')) of map-germs $f : (X, x_0) \to (M, y_0)$ (resp. $f' : (X', x'_0) \to (M', y'_0)$) and a Lagrange projection $\pi : (M, y_0) \to (Y, z_0)$ (resp. $\pi' : (M', y'_0) \to (Y', z'_0)$). Then (f, π) and (f', π') are called *Lagrange equivalent* if there exist a diffeomorphism-germ $\sigma : (X, x_0) \to (X', x'_0)$, a symplectomorphism-germ $\tau : (M, y_0) \to (M', y'_0)$ and a diffeomorphism-germ $\bar{\tau} : (Y, z_0) \to (Y', z'_0)$ such that $\tau \circ f = f' \circ \sigma$ and that $\bar{\tau} \circ \pi = \pi' \circ \tau$.

In [14], we show the equivalence of the "homotopical" Lagrange stability and the infinitesimal Lagrange stablity. An isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ is *homotopically Lagrange stable* with respect to the standard Lagrange projection $\pi : T^*Y \to Y$, if any 1-parameter isotropic deformations f_t of f are trivialised under Lagrange equivalence, namely, if the pair (f_t, π) and (f, π) are Lagrange equivalent by families $(\sigma_t, \tau_t, \bar{\tau}_t)$. *Infinitesimally Lagrange stability* is defined naturally in [14]. See also §3.

In this paper we define the Lagrange stability as follows: Roughly speaking, an isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ is Lagrange stable if, by any sufficiently small isotropic perturbations, the Lagrange equivalence class of f_{x_0} is not removed. To formulate accurately, denote by $C^\infty_I(X, M)$ the space of C^∞ isotropic mappings from X to M, endowed with the Whitney C^∞ topology. Then an isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ is *Lagrange stable* if, for any isotropic representative $f : U \to T^*Y$ of f, there exists a neighborhood W in $C^\infty_I(X, M)$ such that, for any $f' \in W$, the original pair of germs (f, π) is Lagrange equivalent to $(f'_{x'_0}, \pi)$ for some $x'_0 \in U$ (cf. [4] page 325).

To characterise the Lagrange stability by by means of transversality, we recall the isotropic jet spaces [13]. Denote by $J^r_I(X, M)$ the set of r-jets of isotropic map-germs $f : (X, x_0) \to (M, y_0)$ *of corank at most one*:

$$J^r_I(X, M) = \{ j^r f(x_0) \mid f : (X, x_0) \to (M, y_0) \text{ isotropic, } \operatorname{corank}_{x_0} f \leq 1 \}.$$

Then $J^r_I(X, M)$ is a submanifold of the ordinary jet space $J^r(X, M)$ ([13]). Moreover, for $z = j^r f(x_0) \in J^r_I(X, M)$, r-jets of map-germs which are Lagrange equivalent to $f : (X, x_0) \to (M, y_0)$ form a submanifold of $J^r_I(X, M)$.

If $f : X \to M$ is an isotropic mapping of corank at most one, then the image of the r-jet section $j^r f : X \to J^r(X, M)$ is contained in $J^r_I(X, M)$. Then we regard $j^r f$ as a mapping to $J^r_I(X, M)$.

For a manifold-germ (X, x_0), we denote by E_{X,x_0} the $\mathbf{R}$-algebra consisting of C^∞ function-germs $(X, x_0) \to \mathbf{R}$, and by m_{X,x_0} the unique maximal ideal of E_{X,x_0}. If the base point x_0 is clear in the context, we abbreviate E_{X,x_0} and m_{X,x_0} to E_X and m_X respectively.

Now set

$$r_0 = \inf\{r \in \mathbf{N} \mid f^*E_{T^*Y} \cap m_X^{r+2} \subset f^*m_{T^*Y}^{n+3}\}.$$

Then, by Artin-Ree's type theorem, r_0 is a finite positive integer, determined by n and k, the type of the open Whitney umbrella. Actually r_0 depends only on the right-left equivalent class of f.

The purpose of this paper is to show the following result, which is an analogue to the points (A) and (D):

THEOREM 1.1. *Let* $\dim X = \dim Y = n$ *and* $f : (X, x_0) \to (T^*Y, f(x_0))$ *an open Whitney umbrella. Then the following conditions are equivalent to each other for* $r \geq r_0$*:*

(s) *f is Lagrange stable.*

(hs) *f is homotopically Lagrange stable.*

(is) *f is infinitesimally Lagrange stable.*

(a) $f^*E_{T^*Y}$ *is generated by* $1, p_1 \circ f, \ldots, p_n \circ f$ *as* E_Y*-module via* $(\pi \circ f)^*$*.*

(a′) $f^*E_{T^*Y}/(\pi \circ f)^*m_Y f^*E_{T^*Y}$ *is generated by* $1, p_1 \circ f, \ldots, p_n \circ f$ *over* $\mathbf{R}$*.*

(a_r'') $f^*E_{T^*Y}/\{(\pi \circ f)^*m_Y f^*E_{T^*Y} + f^*E_{T^*Y} \cap m_X^{r+2}\}$ *is generated by* $1, p_1 \circ f, \ldots, p_n \circ f$ *over* $\mathbf{R}$*.*

(t_r) *The jet extension* $j^r f : (X, x_0) \to J_I^r(X, T^*Y)$ *is transversal to the Lagrange equivalence class of* $j^r f(x_0)$*.*

For the notation, see [14] and §2, §3.

In the case f is a Lagrange immersion, the condition (a′) is equivalent to that a generationg family of f is R_+-versal [4]. In this case we see $r_0 = n + 1$.

COROLLARY 1.2. (Arnol'd, Tsukada) *A Lagrange immersion-germ is Lagrange stable if and only if its generating family is* R_+*-versal.*

This is clearly formulated in [4], while the explicit proof is ommited, as far as the author knows: T. Tsukada has given an explicit proof in his unpublished work [23]. In the proof by Tsukada, the perturbations of Lagrange immersions and those of their generating families are studied explicitly, to show the equivalence of Lagrange stability and stability of the generating families as unfoldings of functions. In our proof, Lagrange stability are directly described, in the natural way, by the transversality in the space of isotropic jets.

To establish the description, we need the determinacy result for isotropic map-germs. On the ordinary theory of determinacy of map-germs, refer to the excellent survey [24]. Here we treat its isotropic counterpart.

An isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ is called *Lagrange r-determined* if, for any Lagrange projection $\pi' : (T^*Y, y_0) \to Y$ with $j^r\pi'(y_0) = j^r\pi(y_0)$, two pairs (f, π') and (f, π) are Lagrange equivalent. An isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ is called *strictly Lagrange r-determined* if, for any isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ with $j^r f'(x_0) = j^r f(x_0)$, (f', π) and (f, π) are Lagrange equivalent.

We easily see that, if f is strictly Lagrange r-determined, then f is Lagrange r-determined. In the case f is a Lagrange immersion, these two notions coincide.

For the point (B), we show the following result, which seems to be a special case of a theorem due to Givental' ([7], Cor.1, "sufficient jet theorem"):

THEOREM 1.3. *Let* $\dim X = \dim Y = n$. *If an open Whitney umbrella* $f : (X, x_0) \to (T^*Y, y_0)$ *is infinitesimally Lagrange stable, then* f *is Lagrange* $(n+1)$*-determined and* f *is strictly Lagrange* r_0*-determined.*

Since no explicit proof is given in [7], we give a proof to assure ourselves.
For analogies to the points (C) and (E), see §6.

In the next section we recall the objects, open Whitney umbrellas. Theorem 1.3 is proved in §3, recalling the notion of infinitesimal Lagrange stability. In §4, we describe the transversality in the isotropic jet space, as the infinitesimal Lagrange stability up to finite order. Theorem 1.1 is proved in 5.

For an isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$, the carank of $\pi \circ f : (X, x_0) \to (Y, \pi(y_0))$ at x_0 is called L*-corank* of f. The classification of Lagrange stable open Whitney umbrella with L-corank ≤ 1 is given explicitely [27], [12]. In §6, we give an example $f : (\mathbf{R}^5, 0) \to (T^*\mathbf{R}^5, 0)$ of Lagrange stable open Whitney umbrella with L-corank 2. Such example seems to have never been given in any literature so far.

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2. Open Whitney Umbrellas. We recall the definition in [14].

The local model of *an open Whitney umbrella* $f = f_{n,k} : (\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n 0)$ *of type* $k, (0 \leq k \leq \left[\frac{n}{2}\right])$ is concretely given by $q_1 \circ f = x_1, \dots, q_{n-1} \circ f = x_{n-1}$,

$$q_n \circ f = \frac{x_n^{k+1}}{(k+1)!} + x_1 \frac{x_n^{k-1}}{(k-1)!} + \cdots + x_{k-1} x_n \ (=: u),$$

$$p_n \circ f = x_k \frac{x_n^k}{k!} + \cdots + x_{2k-1} x_n \ (=: v),$$

and

$$p_i = \int_0^{x_n} \frac{\partial(v, u)}{\partial(x_i, x_n)} dx_n, \ 1 \leq i \leq n-1,$$

where $\dfrac{\partial(v, u)}{\partial(x_i, x_n)}$ is the Jacobian.

Remark that $f_{n,k}$ is isotropic, that is, $f_{n,k}^* \omega = 0$, where $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ is the standard symplectic form on $T^*\mathbf{R}^n$. Moreover, $f_{n,k}$ is a Lagange immesion if and only if $k = 0$, and, if $k \neq 0$, then the singular locus of $f_{n,k}$ is given by $\{\frac{\partial u}{\partial x_n} = \frac{\partial v}{\partial x_n} = 0\}$ and, therefore, of codimension two.

In general a C^∞ map-germ $f : (X, x_0) \to (M, y_0)$ is called an *open Whitney umbrella of type* k if f is symplectically equivalent to $f_{n,k}$, namely, if there exist a diffeomorphism-germ $\sigma : (X, x_0) \to (\mathbf{R}^n, 0)$ and a symplectomorphism-germ $\tau : (M, y_0) \to (T^*\mathbf{R}^n, 0)$ such that $\tau \circ f = f_{n,k} \circ \sigma$.

Thus Lagrange immersions are naturally generalised to open Whitney umbrellas: In [14], we introduce the notion of symplectic stability and characterise open Whitney umbrellas as symplectically stable isotropic map-germs of corank at most one.

In [14] Prop. 4.1, it is proved that, if $f : (X, x_0) \to (T^*Y, y_0)$ is an open Whitney umbrella, then the *ramification module*

$$R_f = \{e \in E_X \mid de \in \langle d(p_1 \circ f), \ldots, d(p_n \circ f), d(q_1 \circ f), \ldots, d(q_n \circ f)\rangle_{E_X}\},$$

is equal to the image $f^* E_{T^*Y}$ of the pull-back $f^* : E_{T^*Y} \to E_X$. Moreover we see the following (cf. [11]), which is needed later:

LEMMA 2.1. *Let $f = f_{n,k} : (\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$ be the local model of the open Whitney umbrella of type k, and denote by $m(R_f)$ the unique maximal ideal of R_f. Then $x_1, \ldots, x_{n-1}, u, v, p_i \circ f, (1 \le i \le 2k-1)$ form a basis of the "Zariski tangent space" $m(R_f)/m(R_f)^2 \cong f^* m_{T^*Y}/f^* m^2_{T^*Y}$ over $\mathbf{R}$.*

We call an isotropic map-germ $f : (X, x_0) \to (T^*Y, y_0)$ *symplectically r-determined* if any isotropic map-germ f' with $j^r f'(x_0) = j^r f(x_0)$ is symplectically equivalent to f.

Clearly a Lagrange immersion is symplectically 1-determined. Similarly we have

LEMMA 2.2. *Let $f : (X, x_0) \to (M, y_0)$ be an open Whitney umbrella of type k, $0 \le k \le \left[\frac{n}{2}\right]$. Then f is symplectically $(k+1)$-determined. In particular, an open Whitney umbrella is symplectically n-determined.*

Proof. By [10], the condition that f is an open Whitney umbrella of type k is described by the transversality of k-jet extension of (some components of) f. Therefore the condition depends only on its $(k+1)$-jet at the base point. This implies the result. ∎

3. Determinacy. The following is a fundamental fact we need (cf. [7]):

LEMMA 3.1. *Let $r \ge 0$, and $\pi' : (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ a Lagrange projection with $j^r\pi'(0) = j^r\pi(0)$, for the standard projection $\pi : T^*\mathbf{R}^n \to \mathbf{R}^n$. Then there exists a symplectic diffeomorphism $\tau : (T^*\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$ such that $\pi' = \pi \circ \tau$ and $j^r\tau(0) = j^r\mathrm{id}(0)$.*

Proof. The result for the case $r = 0$ is just Darboux theorem for Lagrange projections ([4], Theorem 18.4). The proof for arbitrary r follows from that for $r = 0$. ∎

Therefore we see

LEMMA 3.2. *Let $r \ge 1$ and $f : (X, x_0) \to (T^*Y, y_0)$ an isotropic map-germ. Then the followings are equivalent to each other:*

(1) *For any symplectic diffeomorphism $\tau : (T^*Y, y_0) \to (T^*Y, y_0)$ with $j^r\tau(y_0) = j^r\mathrm{id}(y_0)$, $\tau \circ f$ is Lagrange equivalent to f with respect to π.*

(2) *For any Lagrange projection $\pi' : (T^*Y, y_0) \to Y$ with $j^r\pi'(y_0) = j^r\pi(y_0)$, (f, π') is Lagrange equivalent to (f, π).*

We recall that the infinitesimal Lagrange stability of f is written as

$$VI_f = tf(V_X) + wf(VL_{T^*Y}).$$

We denote by VI_f the set of *infinitesimal isotropic deformations* of f. Remark that the symplectic structure on T^*Y induces the isomorphism, therefore a diffeomorphism $T(T^*Y) \cong T^*(T^*Y)$. Besides, $T^*(T^*Y)$ has the natural symplectic structure $\omega = d\theta_{T^*Y}$, where θ_{T^*Y} is the Liouville 1-form on $T^*(T^*Y)$. Therefore we have naturally a symplectic structure $\tilde{\omega} = d\tilde{\theta}_{T^*Y}$ on $T(T^*Y)$, via the above isomorphism. Then an infinitesimal deformation $v : (X, x_0) \to T(T^*Y)$ of f is called *isotropic*, if the pull-back 2-form $v^*\tilde{\omega} = 0$. V_X means the set of germs of vector fields $\xi : (X, x_0) \to TX$ along the identity. Moreover we denote by VL_{T^*Y} the set of infinitesimal *Lagrange* diffeomorphisms, namely, the set of germs of Hamiltonian vector fields $\eta : (T^*Y, y_0) \to T(T^*Y)$ with affine Hamiltonian of type $a_0(q) + a_1(q)p_1 + \cdots + a_n(q)p_n$. Then we set $tf(\xi) = f_*\xi$ and $wf(\eta) = \eta \circ f$, for $\xi \in V_X, \eta \in VL_{T^*Y}$.

If $v \in VI_f$, then $d(v^*\tilde{\theta}_{T^*Y}) = v^*\tilde{\omega} = 0$. Then there exists a function-germ $e \in E_X = \{(X, x_0) \to \mathbf{R}\}$ such that $de = v^*\tilde{\theta}_{T^*Y}$. We call e a *generating function* of v. Then

$$R_f = \{e \in E_X \mid e \text{ is a generating function for some } v \in VL_f\}$$

is a sub $\mathbf{R}$-algebra of E_X containing $f^*E_{T^*Y}$.

Notice that VI_f has an E_{T^*Y}-module structure and VL_{T^*Y} has an E_Y-module structure [14]. In particular, for $h \in E_{T^*Y}$ and $v \in VI_f$, the E_{T^*Y}-multiplication is defined by

$$h * v = h \circ f \cdot v - e \cdot X_h \circ f,$$

where $\cdot$ is the pointwise multiplication, e is the generating function of v with $e(x_0) = 0$, and X_h is the germ of Hamiltonian vector field with Hamiltonian h so that $i_{X_h}\omega = -dh$.

Set $M = T^*Y$. Since f is an open Whitney umbrella, we see $R_f = f^*E_M$. Remark that R_f is an E_M-module via f^* and an E_Y-module via $(\pi \circ f)^*$. Then $m_M R_f = m(R_f) = f^*(m_M)$.

LEMMA 3.3. *Let $f : (X, x_0) \to (T^*Y, y_0)$ be an infinitesimally Lagrange stable open Whitney umbrella. Then we have*

(1) $m_M^{n+1} R_f \subset m_Y R_f$.

(2) *If $\pi' : (T^*Y, y_0) \to Y$ is a Lagrange projection with $j^n\pi'(y_0) = j^n\pi(y_0)$, then f is infinitesimally Lagrange stable also with respect to π'.*

(3) $m_{T^*Y}^{n+2} VI_f \subset tf(m_X V_X) + wf(m_Y VL_{T^*Y})$.

Proof. (1) Suppose f is infinitesimally Lagrange stable, that is, $VI_f = tf(V_X) + wf(VL_{T^*Y})$. Set $Q_f = f^*E_{T^*Y}/m_Y f^*E_{T^*Y} = R_f/m_Y R_f$. Then Q_f is generated by $1, p_1 \circ f, \dots, p_n \circ f$ over $\mathbf{R}$ by the equivalence of (is) and (a) ([14] Theorem 1.2). Therefore $\dim_{\mathbf{R}} Q_f \le n + 1$. Then considering the sequence of E_M-modules:

$$Q_f \supset m_M Q_f \supset m_M^2 Q_f \supset \cdots \supset m_M^{n+1} Q_f,$$

we see that $m_M^{n+1} Q_f = 0$ and that $m_M^{n+1} R_f \subset m_Y R_f$, using Nakayama's lemma.

(2) Take a symplectic diffeomorphism τ as in Lemma 3.1 such that $\pi' = \pi \circ \tau$. Set $g = \tau \circ f$. Then $R_g = R_f$, $q_i \circ g - q_i \circ f \in m_M^{n+1} R_f$ and $p_j \circ g - p_j \circ f \in m_M^{n+1} R_f$. Then by (1), $m_Y R_g = m_Y R_f$, with respect to π. Thus the condition (a) is satisfied also for g. Thus g is Lagrange stable with respect to π, and therefore f is Lagrange stable with respect to π'.

(3) Take $v \in m_M^{n+2} VI_f$. Then v has a generating function $e \in m_M^{n+3} R_f$. By (1), we see $m_M^{n+3} R_f \subset m_Y m(R_f)^2$. Therefore we have

$$e = \left(\sum_{i=1}^{s} a_i(q) b_i(p, q)\right) \circ f,$$

for some $a_i \in m_Y$ and affine functions $b_i \in m_{T^*Y}$ with respect to π-fibers satisfying $b_i \circ f \in m(R_f)^2$, $1 \le i \le n$. Set $h = \sum_{i=1}^{s} a_i b_i$ and consider the Hamiltonian vector filed X_h with the Hamiltonian h on T^*Y. Then $X_h \in m_Y VL_{T^*Y}$, and $v - wf(X_h)$ has the generating function 0. Therefore, by [14] Lemma 4.3, there exists $\xi \in V_X$ such that $v - wf(X_h) = tf(\xi)$. We show $\xi(0) = 0$.

If f is an immersion, then it is clear. Assume f is an open Whitney umbrella of type $k \geq 1$ and assume $\xi(0) \neq 0$. Then, with respect to the symplectic coordinates of normal forms for open Whitney umbrellas (§2), the coeffiecient of $\frac{\partial}{\partial q_n} \circ f$ of both side of $v - wf(X_h) = tf(\xi)$ should be of order one. On the other hand, we see that, by Lemma 2.1, $\frac{\partial b_i}{\partial p_n}(0) \neq 0$ with respect to the coördinates of normal forms. Therefore the coeffiecient of $\frac{\partial}{\partial q_n} \circ f$ should be of order ≥ 2. This leads a contradiction, and we see $\xi(0) = 0$. Thus $v = tf(\xi) + wf(X_h)$ with $\xi \in m_X V_X, X_h \in m_Y VL_{T^*Y}$, and this proves (3). ∎

Let s be a positive integer. Consider the space $\mathrm{Sp}_s(n)$ of germs of symplectic diffeomorphisms $(T^*\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$ with identity s-jets. We need the following result on "connectivity" of $\mathrm{Sp}_s(n)$.

PROPOSITION 3.4. *Let $s \geq 1$. Then, for any pair $\tau_0, \tau_1 \in \mathrm{Sp}_s(n)$, there exists a smooth family $\tau_t, 0 \leq t \leq 1$, connecting τ_0 and τ_1.*

Proof. Consider the graphs Γ_0, Γ_1 of τ_0, τ_1 respectively in $T^*\mathbf{R}^n \times T^*\mathbf{R}^n$, which are Lagrange submanifolds with respect to the symplectic form $\pi_1^*\omega - \pi_2^*\omega$. Take a Lagrange projection $\Pi : T^*\mathbf{R}^n \times T^*\mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ such that Γ_0, and therefore Γ_1, is mapped diffeomorphically. Then we can take generating functions e_0, e_1 of Γ_0, Γ_1 with respect to Π such that $j^{s+1}e_0(0,0) = j^{s+1}e_1(0,0)$. Then set $e_t = (1-t)e_0 + te_1$. Then the family of Lagrange submanifolds Γ_t generated by e_t corresponds to a family $\tau_t \in \mathrm{Sp}_s(n)$ connecting τ_0 and τ_1. ∎

We denote by I_f^s the set of isotropic map-germs $f' : (X, x_0) \to (M, y_0)$ with $j^s f'(x_0) = j^s f(x_0)$. To show the sirict Lagrange determinacy we need the following:

PROPOSITION 3.5. *Let $s \geq k+1$. Let $f : (X, x_0) \to (M, y_0)$ be an open Whitney umbrella of type k, $0 \leq k \leq \left[\frac{n}{2}\right]$, $\dim X = n = \frac{1}{2}\dim M$. Then, for any pair $f_0, f_1 \in I_f^s$, there exists a 1-parameter smooth family $f_t \in I_f^s, 0 \leq t \leq 1$, connecting f_0 and f_1.*

Proof. We can assume that $f = f_{n,k} : (\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$. Then there exist a family of diffeomorphisms $\sigma_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $\bar{\tau}_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $j^s\sigma_t(0) = j^s\mathrm{id}(0), j^s\bar{\tau}_t(0) = j^s\mathrm{id}(0)$ and that $\bar{\tau}_1 \circ \pi \circ f_0 \circ \sigma_1^{-1} = \pi \circ f_1$. We then set $\tau_t = \bar{\tau}_t^{-1*} : (T^*\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$. Then τ_t is a family of symplectomorphisms with $j^s\tau_t(0) = j^s\mathrm{id}(0)$ and $\pi \circ \tau_t = \bar{\tau}_t$ (cf. Lemma 3.1). Consider the family $f'_t = \tau_t \circ f \circ \sigma_t^{-1} \in I_f^s$. Then $\pi \circ f'_1 = \pi \circ f_1$. Take the generating function e' and e of f'_1 and f_1 respectively so that $f_1'^*\theta = de'$ and $f_1^*\theta = de$ with $e'(0) = e(0) = 0$. Then $j^{s+1}e'(0) = j^{s+1}e(0)$. Now set $e_t = (1-t)e' + te$. Then there exists a family $f''_t \in I_f^s$ such that $f_t''^*\theta = e_t$ and $\pi \circ f''_t = \pi \circ f'_1 (= \pi \circ f_1)$. Then $f''_0 = f'_1$ and $f''_1 = f_1$. This proves the Lemma. ∎

Proof of Theorem 1.3: Assume

$$f : (X, x_0) = (\mathbf{R}^n, 0) \to (T^*Y, y_0) = (T^*\mathbf{R}^n, 0)$$

is infinitesimally Lagrange stable and take $\tau \in \mathrm{Sp}_{n+1}(n)$. Then it suffices to show that, for the standard Lagrange projection $\pi : (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, $(\tau \circ f, \pi)$ and (f, π) are Lagrange equivalent. Set $g = \tau \circ f$. Then, by Proposition 3.4, there exists a smooth family $\tau_t \in \mathrm{Sp}_{n+1}(n)$ such that $\tau_0 = \mathrm{id}$ and $\tau_1 = \tau$. Consider the family $f_t = \tau_t \circ f$ of isotropic map-germs. By Lemma 3.3 (2), we see each f_t is infinitesimally Lagrange stable. Then, by Lemma 3.3 (3),

$$m_{T^*Y}^{n+2} VI_{f_t} \subset tf_t(m_X V_X) + wf_t(m_Y VL_{T^*Y}).$$

Moreover this equality holds for vector fields smoothly depending on t. See [14], Lemma 5.4. Therefore, for each $t_0 \in [0, 1]$, the family f_t is trivialised under Lagrange equivalence (σ_t, τ_t') fixing base points; namely we have $f_{t_0} = \tau_t' \circ f_t \circ \sigma_t$, τ_t is a Lagrange diffeomorphism, $\tau_t'(0) = 0$, and $\sigma_t(0) = 0$. Thus $f = f_0$ and $g = \tau \circ f = f_1$ are Lagrange equivalent with respect to π.

Now we recall that r_0 is determined as the least positive integer satisfying $R_f \cap m_X^{r_0+2} \subset f^* m_{T^*Y}^{n+3}$. For any $f' \in I_f^{r_0}$, by Lemma 3.5, we connect f' and f by a smooth family $f_t \in I_f^{r_0}$. Remark that $r_0 \geq n+1$. Then, for the family f_t of isotropic map-germs with $j^{r_0} f_t(0) = j^{r_0} f(0)$, we see, similarly as above,

$$VI_{f_t} \cap m_X^{r_0+1} V_f \subset tf_t(m_X V_X) + wf_t(m_Y VL_{T^*Y}).$$

Thus f_t is trivialised under Lagrange equivalence fixing base points. ∎

4. Isotropic Jets. Let $f : (X, x_0) \to (M, y_0)$ be an isotropic map-germ of corank at most one. We set

$$VI_f^s = \{v \in VI_f \mid j^s v(x_0) = 0\} = VI_f \cap m_X^{s+1} V_f, \ (s = 0, 1, 2, \ldots).$$

Let $z \in J_I^r(n, 2n)$. Define $\pi_r : VI_f^0 \to T_z J^r(n, 2n)$ as follows: For each $v \in VI_f^0$, take an isotropic deformation f_t of f with $v = \frac{df_t}{dt}|_{t=0}$, and set $\pi_r(v) = \frac{d(j^r f_t(0))}{dt}|_{t=0}$. Then the image of the linear map π_r coincides with $T_z J_I^r(n, 2n)$.

Let $z \in J_I^r(n, 2n)$ and $z = j^r f(0)$ for a $f : (\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$. Hereafter we set $X = (\mathbf{R}^n, 0), Y = (\mathbf{R}^n, 0)$ and $M = (T^*Y, 0)$. Then under the identification $T_z J^r(n, 2n) \cong m_X V_f / m_X^{r+1} V_f$ we have

$$T_z J_I^r(n, 2n) \cong VI_f^0 / VI_f^r.$$

If we denote by $\mathcal{S}^r z$ (resp, $\mathcal{L}^r z$) the orbit of z under the symplectic equivalence (resp. Lagrange equivalence), we have

$$T_z \mathcal{S}^r z \cong VI_f^0 / \{(tf(m_X V_X) + wf(m_M VH_M)) \cap V_f^r\},$$

$$T_z \mathcal{L}^r z \cong VI_f^0 / \{(tf(m_X V_X) + wf(m_Y VL_M)) \cap V_f^r\}.$$

Set $z = j^r f(x_0)$. For $(w, v) \in T_{x_0} X \oplus VI_f$, take a curve x_t in X with the velocity vector w at $t = 0$ and take an isotropic deformation f_t of f with $v = \frac{df_t}{dt}|_{t=0}$ (cf. [14], Lemma 3.4), and define a linear map

$$\Pi_r : T_{x_0} X \oplus VI_f \to T_z J^r(X, M),$$

by

$$\Pi_r(w, v) = \frac{j^r df_t(x_t)}{dt}|_{t=0}.$$

Then $\Pi_r(T_{x_0}X \oplus VI_f) = T_z J_I^r(X,M)$ and $\mathrm{Ker}\Pi_r = \{0\} \oplus VI_f^r$. Moreover we have, for the Lagrange equivalence class

$$[z] = \{j^r f'(x_0') \mid x_0' \in X, f' \text{ is Lagrange equivalent to } f\}$$

in $J_I^r(X,M)$,

$$T_z[z] = \Pi_r(T_{x_0}X \oplus (tf(m_X V_X) + wf(VL_M))).$$

For the jet extension $j^r f : (X, x_0) \to J_I^r(n, M)$, we have

$$(j^r f)_*(\frac{\partial}{\partial x_i}) = \Pi_r(\frac{\partial}{\partial x_i}, f_*(\frac{\partial}{\partial x_i})).$$

Now the condition that $j^r f$ is trasnverse to $[z] = [j^r f(x_0)]$ at x_0 is equivalent to that

$$(j^r f)_*(T_{x_0}X) + T_z[z] = T_z J_I^r(X,M),$$

and to that

$$(\Pi_r)^{-1}((j^r f)_*(T_{x_0}X)) + T_{x_0}X \oplus (tf(m_X V_X) + wf(VL_M)) + \{0\} \oplus VI_f^r$$

coincides with $T_{x_0}X \oplus VI_f$. This condition is equivalent to that

$$VI_f = \langle f_*(\frac{\partial}{\partial x_1}), \dots, f_*(\frac{\partial}{\partial x_n})\rangle_{\mathbf{R}} + tf(m_X V_X) + wf(VL_M) + VI_f^r,$$

namely that

$$VI_f = tf(V_X) + wf(VL_M) + VI_f^r.$$

We recall that

$$C_I^\infty(X,M)^1 := \{f \in C^\infty(X,M) \mid f \text{ is isotropic and of corank} \le 1\}$$

is a Bair space ([13]). Furthermore we have

THEOREM 4.1. ([13]) *Let Q be a submanifold $J_I^r(X,M)$. Then the set*

$$T = \{f \in C_I^\infty(X,M)^1 \mid j^r f : X \to J_I^r(X,M) \text{ is transverse to } Q\}$$

is residual and therefore dense in $C_I^\infty(X,M)^1$.

5. Transversality and Lagrange Stability.

Proof of Theorem 1.1: The equivalence of (hs), (is), (a) and (a′) is shown already in [14]. We show the remaining implications.

(s) $\Rightarrow$ (t$_r$): Take a representative $f : U \to T^*Y$ of f such that $f \in C_I^\infty(X,M)^1$. By Theorem 4.1, f is approximated by $f' \in C_I^\infty(X,M)^1$ such that $j^r f' : U \to J_I^r(X,M)$ is transverse to the Lagrange orbit $[j^r f(x_0)]$. Since f is Lagrange stable, there exists $x_0' \in U$ such that $(f'_{x_0'}, \pi)$ and (f_{x_0}, π) are Lagrange equivalent. Then $j^r f'$ is transverse to $[j^r f(x_0)]$ at x_0', and therefore $j^r f$ is transverse to $[j^r f(x_0)]$ at x_0.

(t$_r$) $\Rightarrow$ (a$_r''$): As we see in §4, the condition (t$_r$) is equivalent to that

$$VI_f = tf(V_X) + wf(VL_M) + VI_f^r.$$

Taking generating functions of both sides, we have

$$R_f = (\pi \circ f)^* E_Y + \sum_{i=1}^n (\pi \circ f)^* E_Y p_i \circ f + R_f \cap m_X^{r+2}.$$

Remarking $R_f = f^* E_{T^*Y}$, we have (a$_r''$).

(a_r'') $\Rightarrow$ (a') Since $R_f \cap m_X^{r+2} \subset m_M^{n+3}R_f$, ($a_r''$) implies that $R_f/(m_Y R_f + m_M^{n+3}R_f)$ is generated by $1, p_1 \circ f, \dots, p_n \circ f$ over $\mathbf{R}$. Then we have $m_M^{n+1}R_f \subset m_Y R_f + m_M^{n+3}R_f$, therefore, by Nakayama's lemma, $m_M^{n+1}R_f \subset m_Y R_f$. Then $m_Y R_f + m_M^{n+3}R_f = m_Y R_f$, so we have that $R_f/m_Y R_f$ is generated by $1, p_1 \circ f, \dots, p_n \circ f$ over $\mathbf{R}$, namely, the condition (a').

Thus we see the implication (t_r) $\Rightarrow$ (is).

(t_r) & (is) $\Rightarrow$ (s): If $j^r f$ is transverse to $[j^r f(x_0)]$ at x_0, then there exists a neighborhood $W \subset C_I^\infty(X, M)^1$ of an isotropic representative $f : U \to T^*Y$ such that, for any $f' \in W$, $j^r f'$ is transverse to $[j^r f(x_0)]$ at a point $x_0' \in U$. Since $j^r f'(x_0') \in [j^r f(x_0)]$, there exists an isotropic map-germ $f'' : (X, x_0) \to T^*Y$ which is Lagrange equivalent to $f'_{x_0'}$ with respect to π and $j^r f''(x_0) = j^r f(x_0)$. On the other hand, since f is infinitesimally Lagrange stable, by Theorem 1.3, f is strictly Lagrange r-determined. Therefore (f'', π) and (f, π) are Lagrange equivalent. Thus $(f'_{x_0'}, \pi)$ and (f, π) are Lagrange equivalent, and f is Lagrange stable.

6. Supplementary Remarks. On the point (C): Let $f, f' : (\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$ be Lagrange stable open Whitney umbrella. Then we naturally ask that whether the isomorphism $(Q_f, e) \cong (Q_{f'}, e')$ implies that f and f' are Lagrange equivalent or not. Here $Q_f = f^* E_{T^*\mathbf{R}^n}/(\pi \circ f)^* m_{\mathbf{R}^n} f^* E_{T^*\mathbf{R}^n}$ and e is the generating function of f with $e(0) = 0$. In the case of complex analytic Lagrange immersions, the implication is true by Yau's theorem [25]. See also [19]. In real case, there exists a counter example [8]. See also [21]. So the problem is how to change the formulation in the real case.

Also in Legendre case, similar question can be posed, analoguously to Mather-Yau type theorem. For Legendre immersions, it is solved affermatively from Mather's theorem on K-versal unfoldings of functions and Arnol'd's thoerem on Legendre singularities, even in real case. This fact is imformed by S. Izumiya to the author.

On the point (E): Lagrange immersions $X^n \to T^*Y$, $\dim X = \dim Y = n$, with Lagrange stable jets are dense in the space of Lagrange immersions, if $n \leq 5$ [4]. This is based on Arnol'd classification of simple Lasgrange singularities [1]. So we can say that the nice range for Lagrange immersions is $\{n \in \mathbf{N} \mid n \leq 5\}$.

It is known the generic Lagrange classification, consisting of finite lists, of open Whitney umbrellas for $n \leq 3$, and in this case all germs are of L-corank ≤ 1. See [12].

Then it is natural to ask that $n = 4, 5$ belong to the nice range or not, for Lagrange projections of open Whitney umbrellas. Ilya A. Bagaevski poses to the author the necessity of the theory of simple singularities of open Whitney umbrellas.

Besides, in the case $n = 4$, there appear generaic isotropic map-germs of corank one and of L-corank two. See [13]. Generic classification is unknown for $n \geq 4$. Then we ask, according to V.M. Zakalyukin, to begin with, whether there exists a Lagrange stable projection of an open Whitney umbrella of L-corank ≥ 2, or not.

The following example is found in 23 June 1998, during the workshop "Caustics" in Warszawa. The example is based on Scherbak's parametrisation of the variety of irregular orbits of the reflection group H_4 [22]. This answers to the question posed by V.M. Zakalyukin on the occasion of author's talk.

EXAMPLE 6.1. Let x, y, z, w, λ be the coordinates of $\mathbf{R}^5$ and $f : (\mathbf{R}^5, 0) \to (T^*\mathbf{R}^5, 0)$

a map-germ defined by

$$p_1 \circ f = x,\ p_2 \circ f = \tfrac{1}{2}y^2,\ p_3 \circ f = -\tfrac{1}{2}xy^2,\ p_4 \circ f = -\tfrac{1}{3}y^3,\ p_5 \circ f = -xy^3,$$
$$q_1 \circ f = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2z + \lambda y^3,\ q_2 \circ f = xz + yw + y^2 + \tfrac{3}{2}\lambda xy,$$
$$q_3 \circ f = z,\ q_4 \circ f = w,\ q_5 \circ f = \lambda.$$

Then f is an open Whitney umbrella of type 1, Lagrange stable with respect to the standard projection $\pi : T^*\mathbf{R}^5 \to \mathbf{R}^5$ defined by $\pi(p,q) = q$, corank$(f) = 1$, and L-corank$(f) = 2$. The generating function of f is given by

$$e = \frac{1}{3}x^3 + \frac{1}{4}y^4 + \frac{1}{2}xy^2z + \frac{1}{6}y^3w + \frac{3}{4}xy^3\lambda,$$

and the caustic of f, namely, the set of singular values of $\pi \circ f$ is given by

$$xw + 2xy + \frac{3}{2}x^2\lambda - yz^2 - \frac{9}{2}y^2z\lambda - \frac{9}{2}y^3\lambda^2 = 0,$$

in $(\mathbf{R}^5, 0)$.

Proof. (1) Set $X = (\mathbf{R}^5,0), Y = (\mathbf{R}^5,0)$. Then

$$\begin{aligned} R_f &= \{h \in E_X \mid dh \in E_X d(f^*E_{T^*Y})\} \\ &= \{h \in E_X \mid \tfrac{\partial h}{\partial y} \in \langle y, w + \tfrac{3}{2}\lambda x\rangle_{E_X}\}, \end{aligned}$$

and R_f is generated by

$$x, z, w, \lambda, y^2, y^3, y(w + \frac{3}{2}\lambda x), y^2(w + \frac{3}{2}\lambda x)$$

as differentiable algebra. Therefore we see $R_f = f^*E_{T^*Y}$. Moreover we see codim$\Sigma(f_{\mathbf{C}}) = 2$. Then, by [14] Prop. 5.1, page 230, and by considering the multiplicity of f, we conclude that f is an open Whithey umbrella of type 1.

(2) By a straightforward caluculation, we see that

$$\begin{aligned} Q_f &= f^*E_{T^*Y}/m_Y f^*E_{T^*Y} \\ &= R_f/\langle x^2, yw + y^2 + \tfrac{3}{2}\lambda xy, z, w, \lambda\rangle_{R_f}, \end{aligned}$$

and Q_f is generated by $1, p_1 \circ f, \ldots, p_5 \circ f$. Therefore by Theorem 1.1, f is Lagrange stable. ∎

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