L^∞ -ESTIMATE OF FIRST-ORDER SPACE DERIVATIVES OF STOKES FLOW IN A HALF SPACE

Yasuyuki Shimizu

Series #434. November 1998

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#409 Y. Giga and K. Ito, Loss of convexity of simple closed curves moved by surface diffusion, 16 pages. 1998.

#410 Y. Giga, K. Inui and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, 34 pages. 1998.

#411 S. Izumiya, D. Pei and T. Sano, The lightcone gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, 16 pages. 1998.

#412 M. Tsujii, Absolutely continuous invarient measures for piecewise real-analytic expanding maps on the plane, 18 pages. 1998.

#413 J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, 40 pages. 1998.

#414 T. Suwa, Dual class of a subvariety, 19 pages. 1998.

#415 M. Tsujii, Piecewise expanding maps on the plane with singular ergodic properties, 9 pages. 1998.

#416 T. Yoshida, Categorical aspects of generating functions(I): Exponential formulas and Krull-Schmidt categories, 44 pages. 1998.

#417 K. Yamaguchi, G_2-Geometry of overdetermined systems of second order, 22 pages. 1998.

#418 M. Ishikawa and S. Matsui, Existence of a forward self-similar stagnation flow of the Navier-Stokes equations, 8 pages. 1998.

#419 S. Izumiya, H. Katsumi and T. Yamasaki, The rectifying developable and the spherical Darboux image of a space curve, 16 pages. 1998.

#420 R. Kobayashi and Y. Giga, Equations with singular diffusivity, 45 pages. 1998.

#421 D. Pei and T. Sano, The focal developable and the binormal indicatrix of a nonlightlike curve in Minkowski 3-space, 14 pages. 1998.

#422 R. Kobayashi, J. A. Warren and W. C. Carter, Modeling grain boundaries using a phase field technique, 12 pages. 1998.

#423 T. Tsukada, Reticular Legendrian Singularities, 28 pages. 1998.

#424 A. N. Kirillov and M. Shimozono, A generalization of the Kostka-Foulkes polynomials, 37 pages. 1998.

#425 M. Nakamura, Strichartz estimates for wave equations in the homogeneous Besov space, 17 pages. 1998.

#426 A. Arai, On the essential spectra of quantum field Hamiltonians, 18 pages. 1998.

#427 T. Sano, Bifurcations of affine invariants for one parameter family of generic convex plane curves, 11 pages. 1998.

#428 F. Hiroshima, Ground states of a model in quantum electrodynamics, 48 pages. 1998.

#429 F. Hiroshima, Uniqueness of the ground state of a model in quantum electrodynamics: A functional integral approach, 32 pages. 1998.

#430 J. F. Van Diejen and A. N. Kirillov, Formulas for q-spherical functions using inverse scattering theory of reflectionless Jacobi operators, 33 pages. 1998.

#431 G. Ishikawa, Determinacy, transversality and Lagrange stability, 13 pages. 1998.

#432 T. Yoshida, Categorical aspects of generating functions(II): Operations on categories and functors, 65 pages. 1998.

#433 K. Ito, Loss of convexity of compact hypersurfaces moved by surface diffusion, 20 pages. 1998.

L^∞-ESTIMATE OF FIRST-ORDER SPACE DERIVATIVES OF STOKES FLOW IN A HALF SPACE

YASUYUKI SHIMIZU

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

§1 Introduction

We consider the Stokes equation

$$
\begin{aligned}
(1.1) \qquad u_t - \Delta u + \nabla p = 0, \mathrm{div} u &= 0 \text{ in } \Omega \times (0,\infty), \\
u &= u_0 \text{ at } t = 0, \\
u &= 0 \text{ on } \partial\Omega \times (0,\infty)
\end{aligned}
$$

in a domain Ω in $\mathbb{R}^n (n \geq 2)$ with smooth boundary. Here $u = (u^1, \dots, u^n)$ is unknown velocity field and p is unknown pressure field. Initial data u_0 is assumed to satisfy a *compatibility condition* : $\mathrm{div} u_0 = 0$ in Ω and the normal component of u_0 equals zero on $\partial\Omega$. This system is a typical parabolic equation and it has several properties resembling the heat equation.

Regularity-decay estimates like $L^p - L^q$ estimates are extensively studied especially for $1 < p, q < \infty$ as explained in Giga-Matsui-Shimizu [8]. We reproduce its explanation for reader's convenience. If $\Omega = \mathbb{R}^n$, u is reduced to a solution of the heat equation with initial data u_0 because there is no boundary condition. In this case regularity-decay estimate

$$
(1.2) \qquad \|\nabla u(t)\|_p \leq C t^{-1/2} \|u_0\|_p \text{ for } t > 0
$$

The author is grateful to the referees for numerous expository suggestions on this paper. He is also grateful to Professor Shin'ya Matsui for several valuable comments.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX holds for all $1 \leq p \leq \infty$ with C independent of t and u_0, where

$$\|f(t)\|_p := \left(\int_\Omega |f(t,x)|^p dx \right)^{1/p}$$

and ∇ denotes the gradient in space variables. If $p = 2$, the estimate (1.2) is still valid for any domain. Indeed, since the Stokes operator A is self–adjoint and nonnegative, the operator A generates an analytic semigroup e^{-tA}. This yields

$$\|A^{1/2} e^{-tA} u_0\|_2 \leq C t^{-1/2} \|u_0\|_2.$$

Since $u = e^{-tA} u_0$ and $\|A^{1/2} u\|_2 = \|\nabla u\|_2$, (1.2) follows for $p = 2$.(See Borchers and Miyakawa [3] for applications.) For $1 < p < \infty$, (0.2) is valid for bounded domains (Giga [7]) and for a half space (Ukai [14]). The estimate (1.2) is also valid for exterior domain with $n \geq 3$, with extra restriction $1 < p < n$.(See Borchers and Miyakawa [2], Giga and Sohr [9], Iwashita [11].)

However, few results are available for $p = 1$ or $p = \infty$ in the case where the boundary $\partial\Omega$ of Ω is nonempty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in L^1 and L^∞ type spaces, because it involves singular integral operators such as Riesz operators. In the case $p = 1$, Giga-Matsui-Shimizu [8] showed (1.2) when Ω is a half-space by estimating Ukai's representation of solutions in Hardy space $\mathcal{H}^1$ of Fefferman and Stein ([6,13]).

In this paper, we prove (1.2) for $p = \infty$ where Ω is a half space $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n); x_n > 0\}$.

Theorem 1.1. *Let u be the solution of the Stokes equation* (1.1) *in $\Omega = \mathbb{R}^n_+$ with initial data $u_0 \in L^\infty(\mathbb{R}^n)$, which satisfies* $\mathrm{div} u_0 = 0$. *Then there is a constant C independent of u_0 such that*

$$\|\nabla u(t)\|_\infty \leq C t^{-1/2} \|u_0\|_\infty \tag{1.3}$$

for all $t > 0$.

Since the Helmholtz projection is not bounded in L^1 and L^∞, it seems impossible to derive (1.3) from L^1-estimate $\|\nabla u(t)\|_1 \leq C t^{-1/2} \|u_0\|_1$ of

[8] by duality arguement. Also, the method described in [8] does not directly apply to get (1.3) since many terms in Ukai's formula [14] have singularities on the hyperplane $\{x_n = 0\}$ when we estimate its derivatives in L^∞ spaces.

A key idea of this paper is to rearrange Ukai's formula for ∇u by modifying way of extension so that the terms involving the square root of minus tangential Laplacian Λ have no singularities on $\{x_n = 0\}$. This enables us to estimate all terms in L^∞ following idea of [8]. Of course, we should estimate terms involving singularities. However, since there are no square root of the minus tangential Laplacian, we are able to estimate these terms by estimating its integral kernels directly in L^1.

The proof of this theorem is divided in three sections. In section 2, we rearrange the solution formula obtained by ukai [14]. We eliminate some of these singularities in Ukai's formula by modifying a way of extending solutions u to $\{x_n < 0\}$, so that the terms involving the square root of minus tangential Laplacian $\Lambda = (-\Delta')^{-1/2}$ have no singularities on $x_n = 0$. In section 3, we estimate the terms which do not include Λ. These terms involve singularities on $\{x_n = 0\}$, but they can be estimated in L^∞ by estimating the corresponding integral kernels directly in the space L^1. Finally, in section 4 we estimate the terms involving Λ. Since these terms do not have singularities on $\{x_n = 0\}$, one can estimate the corresponding integral kernels in the Hardy space as in [8]. These estimates then provide L^1-estimates for integral kernels which are needed, as in section 3, in estimating the terms in L^∞.

§2 Solution formula

In this section, we deduce a new solution formula of (0.1). First, we fix some notation. For an n-dimensional vector a, we denote the tangential component $(a_1, \dots, a_{n-1})$ by $a' \in \mathbb{R}^{n-1}$, so that $a = (a', a_n)$. We set $\partial_j = \partial/\partial x_j$ and let $\nabla' = (\partial_1, \cdots, \partial_{n-1})$. Hereafter, C denotes a positive constant which may differ from one occasion to another.

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

and let $\hat{f}$ be the Fourier tranform of f with respect to the tangential variables x':

$$\hat{f}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} f(x', x_n)dx'.$$

The Riesz operators R_j $(j = 1, \ldots, n)$, and the operator Λ are defined by

$$\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi),$$
$$\mathcal{F}(\Lambda f)(\xi) = |\xi'|\mathcal{F}f(\xi)$$

amd we set $R' = (R_1, \ldots, R_{n-1})$.

We also define the operator $E(t)$ and $F(t)$ by

$$[E(t)f](x) = \int_{\mathbb{R}^n_+} \{G_t(x-y) - G_t(x'-y', x_n+y_n)\} f(y)dy,$$
$$[F(t)f](x) = \int_{\mathbb{R}^n_+} \{G_t(x-y) + G_t(x'-y', x_n+y_n)\} f(y)dy,$$

where G_t is the Gauss kernel $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Furthermore, we define the operator $E_+(t)$ by

$$[E_+(t)f](x) = \begin{cases} [E(t)f](x) & \text{for } x_n > 0, \\ [E(t)f](x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Note that $z = E(t)f$ (resp. $F(t)f$) solves the heat equation in $\mathbb{R}^n_+$ with zero-Dirichlet (resp. zero-Neumann) boundary condition;

$$z_t - \Delta z = 0 \text{ in } \mathbb{R}^n_+ \times (0, T),$$
$$z|_{t=0} = f,$$
$$z|_{x_n=0} \equiv 0. \text{ (resp. } \partial_n z|_{x_n=0} = 0.)$$

Now we are ready to show the new formula.

Theorem 2.1. *Assume that u_0 is in $L^p(\mathbb{R}^n_+)$, $1 \le p \le \infty$ and satisfies* $\mathrm{div} u_0 = 0$. *Then the function*

(2.1a)
$$\begin{aligned}u^n(t) =& -\Lambda(-\Delta)^{-1}\nabla' \cdot E(t)u_0' + \partial_n(-\Delta)^{-1}\nabla' \cdot E_+(t)u_0' \\ &+ (-\Delta)^{-1}\Delta' E_+(t)u_0^n - \partial_n(-\Delta)^{-1}\Lambda E(t)u_0^n,\end{aligned}$$

(2.1b)
$$\begin{aligned}u'(t) =& E(t)u_0' + \Lambda^{-1}\nabla' E(t)u_0^n \\ &+ \nabla'(-\Delta)^{-1}\{\nabla' \cdot E_+(t)u_0'\} - \partial_n(-\Delta)^{-1}\Lambda^{-1}\nabla'\{\nabla' \cdot E(t)u_0'\} \\ &- \nabla'(-\Delta)^{-1}\{\Lambda E(t)u_0^n\} + \partial_n(-\Delta)^{-1}\nabla' E_+(t)u_0^n\end{aligned}$$

satisfies (1.1) *in* $\Omega = \mathbb{R}^n_+$.

Proof. From (1.1) we get $\Delta p = 0$ so that

$$(\partial_n^2 - |\xi'|^2)\hat{p} = 0.$$

Solving this under the additional condition that $\hat{p} \to 0$ as $x_n \to \infty$, we have $\hat{p}(x_n) = C(\xi')e^{-|\xi'|x_n}$. So $\hat{p}$ satisfies

$$(\partial_n + |\xi'|)\hat{p} = 0. \tag{2.2}$$

We next introduce the function v^n by

$$\hat{v}^n = (\partial_n + |\xi'|)\hat{u}^n. \tag{2.3}$$

Then we have

$$v_t^n - \Delta v^n = 0 \text{ in } (0,\infty) \times \mathbb{R}^n, \tag{2.4a}$$

$$\begin{aligned} v^n|_{t=0} &= (\partial_n + \Lambda)u^n \\ &= -\nabla' \cdot u_0' + \Lambda u_0^n, \end{aligned} \tag{2.4b}$$

$$v^n|_{x_n=0} = 0 \tag{2.4c.}$$

Solving this gives

$$\begin{aligned} v^n =& -\nabla' \cdot E(t)u_0' + \Lambda E(t)u_0^n \\ \equiv& V_1 + V_2. \end{aligned} \tag{2.5}$$

Inserting this in the left-hand side of (2.3) and solving the resulting equation gives

$$
\begin{aligned}
\hat{u}^n &= \int_0^{x_n} e^{-|\xi'|(x_n-y_n)}\hat{v}^n(\xi', y_n)dy_n \\
&= \int_0^{x_n} e^{-|\xi'|(x_n-y_n)}\hat{V}_1(\xi', y_n)dy_n \\
&\quad + \int_0^{x_n} e^{-|\xi'|(x_n-y_n)}\hat{V}_2(\xi', y_n)dy_n \\
&\equiv \hat{U}_1 + \hat{U}_2.
\end{aligned} \tag{2.6}
$$

Here, we consider the following two kinds of extensions of the functions $\hat{U}_k$ $(k = 1, 2)$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$:

$$
\widehat{U_{k+}}(x_n) = \begin{cases} \hat{U}_k(x_n) & (x_n > 0), \\ \hat{U}_k(-x_n) & (x_n < 0), \end{cases}
$$

$$
\widehat{U_{k-}}(x_n) = \begin{cases} \hat{U}_k(x_n) & (x_n > 0), \\ -\hat{U}_k(-x_n) & (x_n < 0), \end{cases}
$$

and set

$$
\begin{aligned}
\widehat{u_{+-}}^n &= \widehat{U_{1+}} + \widehat{U_{2-}}, \\
\widehat{u_{-+}}^n &= \widehat{U_{1-}} + \widehat{U_{2+}}.
\end{aligned}
$$

Since $[\mathcal{F}f](\xi) = \int_{-\infty}^{+\infty} e^{-i\xi_n x_n} \hat{f}(\xi', x_n) dx_n$, we have

$$\begin{aligned}
\mathcal{F}U_{k+}(\xi) = & \int_{-\infty}^{+\infty} e^{-i\xi_n x_n} \widehat{U_{k+}}(\xi', x_n) dx_n \\
= & \int_0^{+\infty} e^{i\xi_n x_n} \int_0^{x_n} e^{-|\xi'|(x_n - y_n)} \hat{V}_k(\xi', y_n) dy_n dx_n \\
& + \int_0^{+\infty} e^{-i\xi_n x_n} \int_0^{x_n} e^{-|\xi'|(x_n - y_n)} \hat{V}_k(\xi', y_n) dy_n dx_n \\
= & \int_0^{+\infty} e^{|\xi'| y_n} \hat{V}_k(\xi', y_n) \int_{y_n}^{+\infty} e^{(-|\xi'| + i\xi_n) x_n} dx_n dy_n \\
& + \int_0^{+\infty} e^{|\xi'| y_n} \hat{V}_k(\xi', y_n) \int_{y_n}^{+\infty} e^{(-|\xi'| - i\xi_n) x_n} dx_n dy_n \\
= & \frac{|\xi'| + i\xi_n}{|\xi|^2} \int_0^{+\infty} e^{i\xi_n y_n} \hat{V}_k(\xi', y_n) dy_n \\
& + \frac{|\xi'| - i\xi_n}{|\xi|^2} \int_0^{+\infty} e^{-i\xi_n y_n} \hat{V}_k(\xi', y_n) dy_n \\
= & \frac{|\xi'|}{|\xi|} \mathcal{F}V_{k+}(\xi) - \frac{i\xi_n}{|\xi|} \mathcal{F}V_{k-}(\xi),
\end{aligned}$$

and, similary,

$$\mathcal{F}U_{k-}(\xi) = \frac{|\xi'|}{|\xi|} \mathcal{F}V_{k-}(\xi) - \frac{i\xi_n}{|\xi|} \mathcal{F}V_{k+}(\xi),$$

where

$$V_{k+}(x_n) = \begin{cases} V_k(x_n) & \text{for } x_n > 0, \\ V_k(-x_n) & \text{for } x_n < 0, \end{cases}$$
$$V_{k-}(x_n) = \begin{cases} V_k(x_n) & \text{for } x_n > 0, \\ -V_k(-x_n) & \text{for } x_n < 0. \end{cases}$$

We note that $E(t)f$ is odd in x_n. Direct calculation gives

(2.7a)

$$\begin{aligned} u^n_{+-} =& -\Lambda(-\Delta)^{-1}\nabla'\cdot E_+(t)u'_0 + \partial_n(-\Delta)^{-1}\nabla'\cdot E(t)u'_0 \\ &+ (-\Delta)^{-1}\Delta' E(t)u^n_0 - \partial_n(-\Delta)^{-1}\Lambda E_+(t)u^n_0, \end{aligned}$$

(2.7b)

$$\begin{aligned} u^n_{-+} =& -\Lambda(-\Delta)^{-1}\nabla'\cdot E(t)u'_0 + \partial_n(-\Delta)^{-1}\nabla'\cdot E_+(t)u'_0 \\ &+ (-\Delta)^{-1}\Delta' E_+(t)u^n_0 - \partial_n(-\Delta)^{-1}\Lambda E(t)u^n_0. \end{aligned}$$

The formula (2.7b) proves (2.1a). We shall use u_{+-} later on. To get the desired formula (2.1b) for u', we set $\hat{v}' = \hat{u}' + \frac{i\xi'}{|\xi'|}\hat{u}^n$. Then

$$\begin{aligned} \hat{v}'_t =& \hat{u}'_t + \frac{i\xi'}{|\xi'|}\hat{u}^n_t \\ =& (-|\xi'|^2 + \partial_n^2)\hat{v}' + \frac{i\xi'}{|\xi'|}(|\xi'| + \partial_n)\hat{p} \\ =& (-|\xi'|^2 + \partial_n^2)\hat{v}', \end{aligned}$$

hence $v'_t - \Delta v' = 0$ in $\mathbb{R}^n$. We also have the initial condition $v'|_{t=0} = u'_0 + \nabla'\Lambda^{-1}u^n_0$ and the boundary condition $v'|_{x_n=0} = 0$. Solving these gives $v' = E(t)u'_0 + \nabla'\Lambda^{-1}u^n_0$, and

$$u' = E(t)u'_0 + \nabla'\Lambda^{-1}u^n_0 - \nabla'\Lambda^{-1}u^n. \tag{2.8}$$

Putting $u^n = u^n_{+-}$ in (2.7a) and computing operators, we obtain the desired formula. □

Remark. Ukai[14] used the zero-extension operator e, that is, $ef = f$ for $x_n > 0$ and $ef = 0$ for $x_n < 0$ when extending the function $\hat{u_0}$ in (2.6) from $\mathbb{R}^n_+$ over $\mathbb{R}^n$. Using this extension, we have the following formula:

(2.9a)

$$\begin{aligned} u^n(t) =& -\Lambda(-\Delta)^{-1}\nabla'\cdot eE(t)u'_0 + \partial_n(-\Delta)^{-1}\nabla'\cdot eE(t)u'_0 \\ &+ (-\Delta)^{-1}\Delta' eE(t)u^n_0 - \partial_n(-\Delta)^{-1}\Lambda eE(t)u^n_0, \end{aligned}$$

(2.9b)

$$\begin{aligned} u'(t) =& E(t)u'_0 + \Lambda^{-1}\nabla' E(t)u^n_0 \\ &+ \nabla'(-\Delta)^{-1}\{\nabla'\cdot eE(t)u'_0\} - \partial_n(-\Delta)^{-1}\Lambda^{-1}\nabla'\{\nabla'\cdot eE(t)u'_0\} \\ &- \nabla'(-\Delta)^{-1}\{\Lambda eE(t)u^n_0\} + \partial_n(-\Delta)^{-1}\nabla' eE(t)u^n_0 \end{aligned}$$

In this formula, the terms involving Λ contain singularities on the boundary which will cause difficulties in estimating the solutions in L^∞. See section 4 for more details.

We now deduce our desired formula for the function ∇u. Noting that $\mathrm{div} u_0 = 0$ and $R_j = \partial_j(-\Delta)^{-1}$, we have the following theorem:

Theorem 2.2. *Let u is the function in Theorem 2.1. Then*

(2.10a)
$$\begin{aligned}\partial_j u^n = &- R_j R' \cdot \Lambda E(t) u_0' + R_j R_n \nabla' \cdot E_+(t) u_0' \\ &+ R_j R' \cdot \nabla' E_+(t) u_0^n - R_j R_n \Lambda E(t) u_0^n,\end{aligned}$$

(2.10b)
$$\begin{aligned}\partial_j u' =&\partial_j E(t) u_0' + w_j \\ &+ R_j R' \{\nabla' \cdot E_+(t) u_0'\} - R_j R_n \Lambda^{-1} \nabla' \{\nabla' \cdot E(t) u_0'\} \\ &- R_j R' \{\Lambda E(t) u_0^n\} + R_j R_n \nabla' E_+(t) u_0^n,\end{aligned}$$

where

(2.10c)
$$w_j = \begin{cases} \Lambda^{-1} \partial_j \nabla' E(t) u_0^n & \text{for } j < n, \\ \Lambda^{-1} \nabla' \{\nabla' \cdot E(t) u_0'\} & \text{for } j = n. \end{cases}$$

Note that the terms containing Λ do not contain E_+ (which has singularities at $x_n = 0$).

§3 Estimates of solutions

In this section, we estimate the term without Λ. Note that $\partial_j E(t) u_0' = (\partial_j G_t) * \bar{u_0}'$, where $\bar{f}$ is the odd extension of f with respect to x_n, that is,

$$\bar{f}(x) = \begin{cases} f(x', x_n) & \text{for } x_n > 0, \\ -f(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

So the estimate of this term is easily obtained by Young's theorem:

(3.1)
$$\|\partial_j E(t) u_0'\|_{L^\infty(\mathbb{R}^n_+)} \le \|\partial_j G_t\|_{L^1(\mathbb{R}^n)} \|\bar{u_0}'\|_{L^\infty(\mathbb{R}^n_+)} \le C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Next we estimate the term $R_j R_k \partial_i E_+(t) u_0$. By the definition of $E_+(t)$, we have $E_+(t) u_0(x) = \chi_+(x)[E(t) u_0](x) - \chi_-(x)[E(t) f](x)$, where $\chi_\pm$ is defined as

$$\chi_+(x_n) = \begin{cases} 1 & \text{for } x_n > 0, \\ 0 & \text{for } x_n < 0, \end{cases} \tag{3.2}$$
$$\chi_-(x_n) = 1 - \chi_+(x_n).$$

So we have

(3.3)
$$R_j R_k \partial_i E_+(t) u_0(x) = \int_{\mathbb{R}^n} \bar{u_0}(y) R_{x_j} R_{x_k} \partial_{x_i} (\chi_+(x_n) G_t(x,y)) dy$$
$$- \int_{\mathbb{R}^n} \bar{u_0}(y) R_{x_j} R_{x_k} \partial_{x_i} (\chi_-(x_n) G_t(x,y)) dy,$$

where $G_t(x,y) = G_t(x-y)$.

To estimate (3.3) in L^∞, we have only to estimate the integral kernel $R_{x_j} R_{x_k} \partial_{x_i} (\chi_\pm(x_n) G_t(x,y))$ in the space L^1 of function of y.

Lemma 3.1. *Assume that $1 \le i \le n-1$, $1 \le j,k \le n$, and the function f is in $L^\infty(\mathbb{R})$. Then*

$$\|R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_t(x,\cdot))\|_{L^1(\mathbb{R}^n)} \le C t^{-1/2} \|f\|_{L^\infty(\mathbb{R})} \tag{3.4}$$

with C independent of x.

Proof. It suffices to show (3.4) for $t=1$. Indeed, setting $x = t^{1/2} X$ and $y = t^{1/2} Y$, we then obtain

(3.5)
$$R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_t(x,y)) = t^{-(n+1)/2} R_{X_j} R_{X_k} \partial_{X_i} (f(t^{1/2} X_n) G_1(X,Y))$$

and $\|f(t^{1/2}\cdot)\|_{L^\infty} = \|f\|_{L^\infty}$. Moreover we may assume $j \ne n$, because if $j = k = n$, then we can reduce the problem to the $j \ne n$, using $\sum_{\alpha=1}^n R_\alpha^2 = -I$.

Note that $R_j R_k = \partial_j \partial_k (-\Delta)^{-1}$ and the integral kernel of $\partial_k(-\Delta)^{-1}$ is $c_n x_k |x|^{-n}$. So we have

(3.6)
$$R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_1(x,y)) = c_n \partial_{x_i} \partial_{x_j} \int_{\mathbb{R}^n} \frac{x_k - z_k}{|x-z|^n} f(z_n) G_1(z,y) dz.$$

Let ψ_1 be a smooth function in $\mathbb{R}^{n-1}$ such that $0 \le \psi_1 \le 1$, supp $\psi \subset \{|x| \le 1\}$, and $\psi_1|_{|x|<1/2} \equiv 1$. Set $\psi_2 = 1 - \psi_1$. Then

$$\begin{aligned} &c_n^{-1} R_{x_j} R_{x_k} \partial_{x_i}(f(x_n)G_1(x,y)) \\ =&\partial_{x_i}\partial_{x_j} \int_{\mathbb{R}^n} \frac{(x_k - z_k)\psi_1(x-z)}{|x-z|^n} f(z_n) G_1(z,y) dz \\ +&\partial_{x_i}\partial_{x_j} \int_{\mathbb{R}^n} \frac{(x_k - z_k)\psi_2(x-z)}{|x-z|^n} f(z_n) G_1(z,y) dz \\ =&I_1(x,y) + I_2(x,y). \end{aligned} \tag{3.7}$$

The estimate of I_1: Recalling that $i, j \neq n$, we have

$$\begin{aligned} I_1(x,y) = \int_{\mathbb{R}^n} &\left\{ \frac{w_k \psi_1(w)}{|w|^n} \right\} f(x_n - w_n) \\ &\left\{ \frac{(x_i - y_i - w_i)(x_j - y_j - w_j)}{4} - \frac{\delta_{ij}}{2} \right\} G_1(x-w,y) dw, \end{aligned}$$

where δ_{ij} is Kronecker's delta.

Recalling $|x-y-w| \le |x-y| + 1$ and $|x-y-w|^2 \ge |x-y|^2/2 - 1$ hold for $|w| \le 1$, we get

$$\begin{aligned} |I_1(x,y)| \le C \int_{|w|\le 1} &|w|^{-(n-1)} |f(x_n - w_n)| \left\{ (|x-y|+1)^2 + 1 \right\} \\ &e^{-(|x-y|^2-2)/8} dw \\ \le C(&|x-y|^2+1) e^{-|x-y|^2/8} \|f\|_{L^\infty(\mathbb{R})} \end{aligned} \tag{3.9}$$

and obtain $\|I_1(x,\cdot)\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{L^\infty(\mathbb{R})}$.

The estimate of I_2: We have

$$\begin{aligned} I_2(x,y) &= \sum_{\alpha=0}^{2} \int_{\mathbb{R}^n} K_\alpha(x-z) f(z_n) G_1(z,y) dz \\ &= \sum_{\alpha=0}^{2} J_\alpha(x,y), \end{aligned} \tag{3.10}$$

where

(3.11a)

$$K_0(x) =\frac{x_k(\partial_i\partial_j\phi_2)(x)}{|x|^n},$$

(3.11b)

$$K_1(x) = \left(\frac{\delta_{jk}}{|x|^n} - n\frac{x_jx_k}{|x|^{n+2}}\right)(\partial_i\phi_2)(x) + \left(\frac{\delta_{ik}}{|x|^n} - n\frac{x_ix_k}{|x|^{n+2}}\right)(\partial_j\phi_2)(x)$$

(3.11c)

$$K_2(x) =n\left\{(n+2)\frac{x_jx_kx_k}{|x|^{n+4}} - \frac{\delta_{jk}x_i + \delta_{ki}x_j + \delta_{ij}x_k}{|x|^{n+2}}\right\}\phi_2(x).$$

Noting that $\operatorname{supp}\nabla\phi_2 \subset 1/2 < |x| < 1$, we can estimate $J_\alpha(\alpha = 0, 1)$ as in the case of I_1, to obtain

(3.12)

$$\begin{aligned}|J_\alpha(x,y)| \leq& C\int_{1/2\leq|x-z|\leq1} |x-z|^{-n-\alpha+1}|f(z_n)|G_1(z,y)dz\\ =&C\int_{1/2\leq|w|\leq1} |w|^{-n-\alpha+1}|f(x_n-w_n)|G_1(x-w,y)dw\\ \leq& C(|x-y|^2+1)e^{-|x-y|^2/8}\|f\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

To estimate the term J_2, we use the inequality $|x-y|^l \leq C_l(|x-z|^l + |z-y|^l)$ for $l \geq 0$. Assuming that $0 \leq l \leq n+1$, we have

(3.13)

$$\begin{aligned}|J_2(x,y)| \leq& C\int_{|x-z|\geq1/2} |x-z|^{-n-1}|f(z_n)|G_1(z,y)dz\\ \leq& C\int_{|x-z|\geq1/2} |x-y|^{-l}(|x-z|^{l-n-1} + |x-z|^{-n-1}|z-y|^l)\\ &\qquad |f(z_n)|G_1(z,y)dz\\ \leq& C|x-y|^{-l}\|f\|_{L^\infty(\mathbb{R})}\\ &\int_{|x-z|\geq1/2} (2^{n-l+1} + 2^{n+1}|z-y|^l)G_1(z,y)dz\\ \leq& C|x-y|^{-l}\|f\|_{L^\infty(\mathbb{R})},\end{aligned}$$

hence

$$|J_2(x,y)| \le \frac{C}{|x-y|^{n+1}+1}\|f\|_{L^\infty(\mathbb{R})}. \tag{3.14}$$

Combining the estimates for $J_\alpha (\alpha = 0,1,2)$, we obtain $\|I_2(x,\cdot)\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{L^\infty(\mathbb{R})}$. Collecting the estimates of I_1 and I_2 completes the proof of Lemma 3.1. □

Using Lemma 3.1, we obtain the estimate of (3.1):

$$\begin{aligned}
&\|R_jR_k\partial_iE_+(t)u_0\|_{L^\infty(\mathbb{R}^n)}\\
\le&\|\bar{u_0}\|_{L^\infty(\mathbb{R}^n)}\|R_{x_j}R_{x_k}\partial_{x_i}(\chi_+(x_n)G_t(x,\cdot))\|_{L^1(\mathbb{R}^n)}\\
+&\|\bar{u_0}\|_{L^\infty(\mathbb{R}^n)}\|R_{x_j}R_{x_k}\partial_{x_i}(\chi_-(x_n)G_t(x,\cdot))\|_{L^1(\mathbb{R}^n)}\\
\le&\|u_0\|_{L^\infty(\mathbb{R}^n_+)}Ct^{-1/2}\|\chi_+\|_{L^\infty(\mathbb{R})}\\
+&\|u_0\|_{L^\infty(\mathbb{R}^n_+)}Ct^{-1/2}\|\chi_-\|_{L^\infty(\mathbb{R})}\\
\le&Ct^{-1/2}\|u_0\|_{L^\infty(\mathbb{R}^n_+)}.
\end{aligned} \tag{3.15}$$

§4 Application of Hardy space theory

Finally, we estimate the term $R_jR_k\partial_h\partial_i\Lambda^{-1}E(t)u_0$ $(1\le h,i\le n-1,$ $1\le j,k\le n)$. Since we have

$$R_jR_k\partial_h\partial_i\Lambda^{-1}E(t)u_0(x) = \int_{\mathbb{R}^n}\bar{u_0}(y)R_{x_j}R_{x_k}\partial_{x_h}\partial_{x_i}\Lambda_x^{-1}G_t(x,y)dy, \tag{4.1}$$

we need to show

$$\|R_{x_j}R_{x_k}\partial_{x_h}\partial_{x_i}\Lambda_x^{-1}G_t(x,\cdot)\|_{L^1(\mathbb{R}^n)} \le Ct^{-1/2}. \tag{4.2}$$

In this section we prove (4.2) with the aid of the theory of Hardy space $\mathcal{H}^1$. First, we recall the definition of $\mathcal{H}^1$.

Definition 4.1. A function $f\in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1=\mathcal{H}^1(\mathbb{R}^n)$ if

$$f^+(x) = \sup_{s>0}|G_s * f(x)| \in L^1(\mathbb{R}^n), \tag{4.3}$$

where the symbol $*$ denotes the convolution with respect to the space variable x. The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{H}^1} := \|f^+\|_{L^1(\mathbb{R}^n)} \tag{4.4}$$

It is known ([6,13]) that anL^1-function f is in $\mathcal{H}^1$ if and only if all its Riesz transforms $R_j f$ are in $L^1(\mathbb{R}^n)$ and that

$$\|f\|_{\mathcal{H}^1} \cong \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)} \texttt{(equivalent norm)}. \tag{4.5}$$

We denote by $|||\cdot|||_{\mathcal{H}^1}$ the operator-norm of R_j in $\mathcal{H}^1$.

Since $G_t(x,y) = G_t(y,x)$, we have

$$R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} \Lambda_x^{-1} G_t(x,y) = R_{y_j} R_{y_k} \partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x,y) \tag{4.6}$$

and

$$\begin{aligned} &\|R_{y_j} R_{y_k} \partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x,\cdot)\|_{\mathcal{H}^1} \\ &\le |||R_j|||_{\mathcal{H}^1} |||R_j|||_{\mathcal{H}^1} \|\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x,\cdot)\|_{\mathcal{H}^1}. \end{aligned} \tag{4.7}$$

So we have only to show that $\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x-y)$ is in $\mathcal{H}^1$ as a function of y.

Remark. Using the formula obtained by Ukai[14], we have

$$\begin{aligned} &R_j R_k \partial_h \partial_i \Lambda^{-1} eE(t) u_0(x) \\ &= \int_{\mathbb{R}^n} \bar{u_0}(y) R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} \Lambda_x^{-1} \{\theta(x_n) G_t(x,y)\} dy, \end{aligned} \tag{4.8}$$

where θ is the Heaviside function, that is, $\theta = 1$ for $x_n > 0$ and $\theta = 0$ for $x_n < 0$. Since $\theta(x_n)G_t(x-y)$ is not symmetric with the respect to x and y, we cannot change the operator R_{x_j} to R_{y_j} and so we have to estimate (4.8) directly.

The following is our key lemma for estimating $\partial_j \partial_k \Lambda^{-1} G_t$ in the space $\mathcal{H}^1$.

Lemma 4.1. [8] *Assume that real parameters l and m satisfy $0 \le l \le n$ and $m \ge 0$. Then there exists a constant $C = C_{l,m}$ which does not depend on $x \in \mathbb{R}^n$ and $t \ge 0$ such that*

$$|\partial_j \partial_k \Lambda^{-1} G_t(x)| \le C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \quad \text{for } 1 \le j,k \le n-1 \text{ with } n \ge 3, \tag{4.9}$$

$$|\Lambda G_t(x)| \le C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \quad \text{with } n \ge 2. \tag{4.10}$$

Proof. The proof is given in [8], but we will reproduce the proof for reader's convenience. Note that the operator Λ^{-1} is equal to $(-\Delta')^{-1/2}$ $= \left(-\sum_{k=1}^{n-1} \partial_k^2\right)^{-1/2}$, so the integral kernel of Λ^{-1} is $c_n |x'|^{-n+2}$ for $n \ge 3$, where c_n is some positive constant. Therefore we have

$$\partial_j \partial_k \Lambda^{-1} G_t(x) = c_n \partial_j \partial_k \int_{\mathbb{R}^{n-1}} |x' - y'|^{-n+2} G_t(y', x_n) dy'. \tag{4.11}$$

Set $x = t^{1/2} z$ to get

$$\partial_{x_j} \partial_{x_k} \Lambda^{-1} G_t(x) = t^{-(n+1)/2} \partial_{z_j} \partial_{z_k} \Lambda^{-1} G_1(z).$$

So it is sufficient to show (4.9) for $t = 1$, i.e.

$$|\partial_j \partial_k \Lambda^{-1} G_1(z)| \le C |z'|^{-l} |z_n|^{-m}. \tag{4.12}$$

In fact, if (4.12) is valid, then we have

$$\begin{aligned} |\partial_{x_j} \partial_{x_k} \Lambda^{-1} G_t(x)| &= t^{-(n+1)/2} |\partial_{z_j} \partial_{z_k} \Lambda^{-1} G_1(z)| \\ &\le C t^{-(n+1)/2} |z'|^{-l} |z_n|^{-m} \\ &= C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \end{aligned}$$

for any $t > 0$.

Let ψ_1 be a smooth function in $\mathbb{R}^{n-1}$ such that $0 \le \psi_1 \le 1$, supp $\psi \subset \{|z'| \le 1\}$, and $\psi_1|_{|z'|<1/2} \equiv 1$. Set $\psi_2 = 1 - \psi_1$. Then

(4.13)

$$\begin{aligned} \partial_j \partial_k \Lambda^{-1} G_1(z) &= \frac{C}{(4\pi)^{n/2}} e^{-z_n^2/4} \Big\{ \partial_j \partial_k \int_{\mathbb{R}^{n-1}} \frac{\psi_1(z'-y')}{|z'-y'|^{n-2}} e^{-|y'|^2/4} dy' \\ &\qquad + \partial_j \partial_k \int_{\mathbb{R}^{n-1}} \frac{\psi_2(z'-y')}{|z'-y'|^{n-2}} e^{-|y'|^2/4} dy' \Big\} \\ &= C e^{-z_n^2/4} \{ I_1(z') + I_2(z') \}. \end{aligned}$$

We first estimate I_1. We have

$$I_1(z') = \partial_j \partial_k \int_{|y'|\le 1} \frac{\psi_1(y')}{|y'|^{n-2}} e^{-|z'-y'|^2/4} dy' \tag{4.14}$$
$$= \int_{|y'|\le 1} \frac{\psi_1(y')}{|y'|^{n-2}} K_{j,k}(z'-y')dy,$$

where

$$K_{j,k}(z') = \left(\frac{z_j z_k}{4} - \frac{\delta_{j,k}}{2} \right) e^{-|z'|^2/4}$$

and $\delta_{j,k}$ is Kronecker's delta. since $|z'-y'| \le |z'|+1$ and $|z'-y'|^2 \ge |z'|^2/2 - 1$ if $|y'| \le 1$, we get

$$|K_{j,k}(z'-y')| \le \left\{ \frac{(|z'|+1)^2}{4} + \frac{1}{2} \right\} e^{-(|z'|^2-2)/8}$$
$$= \frac{e^{1/4}}{4} \left\{ (|z'|+1)^2 + 2 \right\} e^{-|z'|^2/8}$$
$$\le C|z'|^{-l}.$$

Hence we have

$$|I_1(z')| \le C \int_{|y'|\le 1} \frac{1}{|y'|^{n-2}} |z'|^{-l} dy'$$
$$\le C|z'|^{-l}$$

We next estimate I_2. We have

(4.15)

$$I_2(z') = \int_{\mathbb{R}^{n-1}} \frac{(\partial_j \partial_k \psi_2)(z'-y')}{|z'-y'|^{n-2}} e^{-|y'|^2/4} dy'$$
$$- (n-2) \Big\{ \int_{\mathbb{R}^{n-1}} (\partial_j \psi_2)(z'-y') \frac{z_k - y_k}{|z'-y'|^n} e^{-|y'|^2/4} dy'$$
$$+ \int_{\mathbb{R}^{n-1}} (\partial_k \psi_2)(x'-y') \frac{z_j - y_j}{|z'-y'|^n} e^{-|y'|^2/4} dy' \Big\}$$
$$+ \int_{\mathbb{R}^{n-1}} \psi_2(z'-y') L_{j,k}(z'-y') e^{-|y'|^2/4} dy'$$
$$= J_1(z') - (n-1) J_2(z') + J_3(z'),$$

where

$$L_{j,k}(z') = (n-2)\left\{n\frac{x_j x_k}{|z'|^{n+2}} - \frac{\delta_{j,k}}{|z'|^n}\right\}.$$

Since the support of the derivatives of ψ_2 are included in $1/2 \le |z| \le 1$, we can estimate J_1 and J_2 as in the case of I_1:

(4.16)

$$\begin{aligned}|J_1(z')| &= \left|\int_{1/2\le|y'|\le 1} \frac{(\partial_j\partial_k\psi_2)(y')}{|y'|^{n-2}} e^{-|z'-y'|^2/4} dy'\right| \\ &\le \|\nabla^2\psi_2\|_{L^\infty} \int_{1/2\le|y'|\le 1} \frac{1}{|y'|^{n-2}} e^{-(|z'|^2-2)/8} dy' \\ &\le C|z'|^{-l},\end{aligned}$$

(4.17)

$$\begin{aligned}|J_2(z')| &\le \|\nabla\psi_2\|_{L^\infty} \int_{1/2\le|y'|\le 1} \frac{1}{|y'|^{n-1}} e^{-(|z'|^2-2)/8} dy' \\ &\le C|z'|^{-l}.\end{aligned}$$

To estimate the term J_3, we use the inequality $|z'|^l \le C_l(|z'-y'|^l + |y'|^l)$. Since $|L_{j,k}(z')| \le \frac{C}{|z'-y'|^{n+1}}$, we get

(4.18)

$$\begin{aligned}|J_3(z')| &\le C|z'|^{-l} \int_{|z'-y'|\ge 1/2} \left(\frac{|z'-y'|^l}{|z'-y'|^n} + \frac{|y'|^l}{|z'-y'|^n}\right) e^{-|y'|^2/4t} dy' \\ &\le C|z'|^{-l} \int_{|z'-y'|\ge 1/2} (2^{l-n} + 2^n|y'|^l) e^{-|y'|^2/4} dy' \\ &= C|z'|^{-l}.\end{aligned}$$

Combining (4.16)-(4.18) gives $|I_2(z')| \le C|z'|^{-l}$ and

$$\begin{aligned}|\partial_j\partial_k\Lambda^{-1}G_1(z)| &\le Ce^{-x_n^2/4}|z'|^{-l} \\ &\le C_{l,m}|z'|^{-l}|z_n|^{-m}.\end{aligned} \tag{4.19}$$

This proves (4.12) for $n \ge 3$.

The estimate (4.10) for $n \geq 3$ is easily obtained by applying (4.9), since

$$\Lambda = (-\Delta')\Lambda^{-1} = -(\partial_1^2 + \cdots \partial_{n-1}^2)\Lambda^{-1}.$$

Finally, we show (4.10) for $n = 2$. Note that Λ is equal to $|\partial_1| = \partial_1 S_1$. So we have

$$\begin{aligned} \Lambda G_t(x) &= \partial_1 S_1 G_t(x) \\ &= \partial_1 \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y_1|>\epsilon} \frac{1}{y_1} G_t(x_1 - y_1, x_2) dy_1. \end{aligned} \tag{4.20}$$

(See Torchinsky [13], p.266.) Integrating by parts, we get

$$\begin{aligned} \int_{|y_1|>\epsilon} \frac{1}{y_1} G_t(x_1 - y_1, x_2) dy_1 &= \Big[\log |y_1| G_t(x_1 - y_1, x_2)\Big]_\epsilon^\infty \\ &\quad + \Big[\log |y_1| G_t(x_1 - y_1, x_2)\Big]_{-\infty}^{-\epsilon} \\ &\quad - \int_{|y_1|>\epsilon} \log |y_1| \partial_{y_1} G_t(x_1 - y_1, x_2) dy_1 \\ &= \log \epsilon \big(G_t(x_1 + \epsilon, x_2) - G_t(x_1 - \epsilon, x_2)\big) \\ &\quad + \int_{|y_1|>\epsilon} \log |y_1| \frac{x_1 - y_1}{2t} G_t(x_1 - y_1, x_2) dy_1. \end{aligned}$$

Letting $\epsilon \downarrow 0$, we get

$$\Lambda G_t(x) = \frac{1}{\pi} \partial_1 \int_{-\infty}^{\infty} \log |y_1| \frac{x_1 - y_1}{2t} G_t(x_1 - y_1, x_2) dy_1. \tag{4.21}$$

Set $x = t^{1/2} z$ and $y = t^{1/2} w$. Then we have

$$\begin{aligned} (\Lambda G_t)(x) &= \frac{1}{\pi} t^{-1/2} \partial_{z_1} \int_{-\infty}^{\infty} (\log |w_1| + \log t^{1/2}) \\ &\qquad\qquad \frac{z_1 - w_1}{2t^{1/2}} t^{-1} G_1(z_1 - w_1, w_2) t^{1/2} dw_1 \\ &= t^{-3/2} (\Lambda G_1)(z). \end{aligned}$$

So it is sufficient to show (4.10) for $t = 1$. Letting $t = 1$ in (4.21), we have

(4.22)
$$
\begin{aligned}
\Lambda G_1(z) = {} & \frac{1}{\pi}\frac{1}{4\pi}e^{-z_2^2/4}\partial_1\Big\{\int_{|y_1|<1} \log|y_1|\frac{z_1-y_1}{2}e^{-(z_1-y_1)^2/4}dy_1 \\
& + \int_{|y_1|>1} \log|y_1|\frac{z_1-y_1}{2}e^{-(z_1-y_1)^2/4}dy_1\Big\} \\
= {} & \frac{1}{4\pi^2}e^{-z_2^2/4}(I_1(z_1)+I_2(z_1)).
\end{aligned}
$$

We first estimate I_1. We have

$$
I_1(z_1) = \int_{-1}^{1} \log|y_1|\frac{1}{2}\Big(1-\frac{|z_1-y_1|^2}{2}\Big)e^{-(z_1-y_1)^2/4}dy_1.
$$

In the same way as in the case $n \geq 3$, we obtain

(4.23)
$$
\begin{aligned}
|I_1(z_1)| &\leq \frac{1}{2}\int_{-1}^{1} |\log|y_1||\Big(1+\frac{(|z_1|+1)^2}{4}\Big)e^{-\frac{|z_1|^2}{8}+\frac{1}{4}}dy_1 \\
&\leq C(1+|z_1|^2)e^{-|z_1|^2/8}.
\end{aligned}
$$

To estimate I_2, we integrate it by parts and get

$$
\begin{aligned}
I_2(z_1) = {} & \partial_1\Big\{\Big[\log|y_1|e^{-(z_1-y_1)^2/4}\Big]_1^{+\infty} \\
& + \Big[\log|y_1|e^{-(z_1-y_1)^2/4}\Big]_{-\infty}^{-1} \\
& - \int_{|y_1|>1} \frac{1}{y_1}e^{-(z_1-y_1)^2/4}dy_1\Big\} \\
= {} & \int_{|y_1|>1} \frac{1}{y_1}\frac{z_1-y_1}{2}e^{-(z_1-y_1)^2/4}dy_1 \\
= {} & e^{-(z_1+1)^2/4} - e^{-(z_1-1)^2/4} + \int_{|y_1|>1} \frac{1}{y_1^2}e^{-(z_1-y_1)^2/4}dy_1.
\end{aligned}
$$

We set $w_1 = z_1 - y_1$ and obtain

$$
I_2(z_1) = e^{-(z_1+1)^2/4} - e^{-(z_1-1)^2/4} + \int_{|z_1-w_1|>1} \frac{1}{(z_1-w_1)^2}e^{-w_1^2/4}dw_1.
$$

Using $|z_1|^l \le C(|z_1 - w_1|^l + |w_1|^l)$, we obtain

(4.24)
$$\begin{aligned}|I_2(z_1)| &\le |e^{-(z_1+1)^2/4}| + |e^{-(z_1-1)^2/4}| \\ &\quad + C\int_{|z_1-w_1|>1} \frac{1}{|z_1|^l}\left(|z_1 - w_1|^{l-2} + \frac{|w_1|^l}{|z_1 - w_1|^2}\right) e^{-|w_1|^2/4} dw_1 \\ &\le C|z|^{-l}\end{aligned}$$

since $l \le 2$ and so $|z_1 - w_1|^{l-2} \le 1$. Combining (4.23) and (4.24) gives (4.10) for $n = 2$. for $n = 2$. $\square$

Now we are ready to show the key lemma for the main theorem.

Lemma 4.2. *Assume that* $1 \le j, k \le n-1$. *Then*

$$\|\partial_{y_h}\partial_{y_i}\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1} \le Ct^{-1/2}. \tag{4.25}$$

Proof. Note that $G_s * (\partial_{y_h}\partial_{y_i}\Lambda_y^{-1}G_t(x,y))(y) = \partial_{y_h}\partial_{y_i}\Lambda_y^{-1}G_{s+t}(x,y)$. Using Lemma 4.1, we have

(4.26)
$$\begin{aligned}|\partial_{y_h}\partial_{y_i}\Lambda_y^{-1}G_{s+t}(x,y)| \le& C(s+t)^{(l+m-n-1)/2}|x'-y'|^{-l}|x_n - y_n|^{-m} \\ \le& Ct^{(l+m-n-1)/2}|x'-y'|^{-l}|x_n - y_n|^{-m},\end{aligned}$$

where l and m satisfy the assumption in Lemma 4.1. Therefore we obtain

(4.27)
$$\begin{aligned}&\|\partial_{y_h}\partial_{y_i}\Lambda_y^{-1}G_t(x,y)\|_{\mathcal{H}^1_y} \\ &\le \sum_{k=1}^{4} C_{l,m}t^{(l+m-n-1)/2}\int_{\Omega_k}|x'-y'|^{-l}|x_n-y_n|^{-m}dy,\end{aligned}$$

where

$$\begin{aligned}\Omega_1 &= \{|x'-y'| \le t^{1/2}, |x_n - y_n| \le t^{1/2}\}, \\ \Omega_2 &= \{|x'-y'| > t^{1/2}, |x_n - y_n| \le t^{1/2}\}, \\ \Omega_3 &= \{|x'-y'| \le t^{1/2}, |x_n - y_n| > t^{1/2}\}, \\ \Omega_4 &= \{|x'-y'| > t^{1/2}, |x_n - y_n| > t^{1/2}\}.\end{aligned}$$

We estimate the integrals on the right-hand side of (4.27), taking $l = m = 0$ for $k = 1$, $l = n$, $m = 0$ for $k = 2$, $l = 0$, $m = 2$ for $k = 3$ and $l = n - 1/2$, $m = 3/2$ for $k = 4$, to find that the integrals of (4.27) are all bounded above by a constant multiple of $t^{-1/2}$. This proves (4.25). $\square$

Recalling $\|f\|_{L^1} \leq \|f\|_{\mathcal{H}^1}$ and using Lemma 4.2, we have

(4.28)
$$\begin{aligned}
&\|R_j R_k \partial_h \partial_i \Lambda^{-1} E(t) u_0\|_{L^\infty(\mathbb{R}^n_+)} \\
\leq &\|\bar{u_0}\|_{L^\infty(\mathbb{R}^n)} \|R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} G_t(x,\cdot)\|_{L^1(\mathbb{R}^n)} \\
\leq &\|u_0\|_{L^\infty(\mathbb{R}^n_+)} \|R_{y_j} R_{y_k} \partial_{y_h} \partial_{y_i} G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
\leq &\|u_0\|_{L^\infty(\mathbb{R}^n_+)} |||R_j|||_{\mathcal{H}^1} |||R_k|||_{\mathcal{H}^1} \|\partial_{y_h} \partial_{y_i} G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
\leq &C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}^n_+)}.
\end{aligned}$$

Combining the estimate in (3.15) and (4.28), we obtain the desired estimate:
$$\|\nabla u\|_{L^\infty(\mathbb{R}^n_+)} \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}^n_+)}.$$

REFERENCES

1. W. Borchers and T. Miyakaya, *L^2 decay for the Navier-Stokes flow in halfspaces*, Math. Ann. **282** (1988), 139–155.
2. ———, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains*, Acta Math. **165** (1990), 189–227.
3. ———, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains, II*, Hiroshima Math. J. **21** (1991), 621–640.
4. A.Carpio, *Large time behavior in incompressible Navier-Stokes equations*, SIAM J. Math. Anal. **27** (1996), 449-475.
5. Z.-M.Chen, *Solution of the stationary and nonstationary Navier-Stokes equations in exterior domains*, Pacific J. Math. **159** (1993), 227–240.
6. C.Fefferman and E.Stein, *$\mathcal{H}^p$ spaces of several variavles*, Acta Math. **129** (1972), 137–197.
7. Y.Giga, *Analyticity of the semigroup generated by the Stokes operator in L^r spaces*, Math. Z. **178** (1981), 297–329.
8. Y.Giga, S.Matsui and Y.Shimizu, *On Estimates in Hardy Spaces for the Stokes Flow in a Half Space*, To appear in Math. Z., Hokkaido University Preprint Series in Mathematics **381** (1997).
9. Y.Giga and H.Sohr, *On the Stokes operator in exterior domains*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **36** (1989), 103–130.
10. Y.Giga and H.Sohr, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. of Functional Analysis **102** (1991), 72–94.

11. H.Iwashita, $L_q - L_r$ *estimate for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in* L_q *spaces*, Math. Ann. **285** (1989), 265–288.
12. T.Miyakawa, *Hardy spaces of solenoidal vector fields, with application to the Navier-Stokes equations*, Kyushu J. Math. **50** (1997), 1–64.
13. A.Torchinsky, *Real-variable methods in harmonic anarysis*, Academic Press, 1986.
14. S.Ukai, *A solution formula for the Stokes equation in* $\mathbb{R}^n_+$, Comm. Pure Appl. Math. **XL** (1987), 611–621.