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**Takeshi Uemura**

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# Morita-Mumford classes on finite cyclic subgroups of the mapping class group of closed surfaces

Takeshi Uemura

## Abstract

Let  $G$  be a finite cyclic subgroup of the mapping class group of order  $m$ . We prove the Morita-Mumford classes restricted to  $G$  admit a certain kind of periodicity whose period is given by the Euler function  $\phi(m)$ . Using this periodicity theorem, we compute the Morita-Mumford classes on arbitrary finite cyclic subgroups of the automorphism group of Klein's quartic curve.

## Introduction

Let  $\Sigma_g$  be a closed oriented surface of genus  $g \geq 2$ , and  $M_g$  the mapping class group of  $\Sigma_g$ , which is the group of isotopy classes of orientation preserving diffeomorphisms of  $\Sigma_g$ . The cohomological study of  $M_g$  has been developed rapidly and has yielded many interesting results. The Morita-Mumford classes, defined by Morita [Mo1] and Mumford [Mu] independently, are a series of cohomology classes of  $M_g$ , whose zeroth term is equal to the Euler number  $2 - 2g$  of  $\Sigma_g$ . Many mathematicians, including Harer [H2] [H3], Miller [Mi], and Morita [Mo1] [Mo2] [Mo3] [Mo4], have pointed out the importance of these classes for the study of the stable cohomology ring of  $M_g$ . Moreover, recently it is revealed by Akita that the Morita-Mumford classes play an important role in the study of the  $\eta$ -invariant of mapping tori of periodic mapping classes (see [Ak]). We are convinced that the Morita-Mumford classes contribute largely to the unstable cohomological study of  $M_g$  in the future.

The Morita-Mumford classes of surface bundles are defined as follows. Let  $\pi : E \rightarrow B$  be an oriented fiber bundle whose fiber is  $\Sigma_g$ . (We call such a bundle a "surface bundle") The relative tangent bundle  $T_{E/B}$  is the oriented real 2-dimensional vector bundle over  $E$  consisting of all the tangent vectors along the fibers. Take its Euler class  $e := e(T_{E/B}) \in H^2(E; \mathbf{Z})$ , then  $e^{n+1} \in H^{2(n+1)}(E; \mathbf{Z})$ . Let  $\pi_! : H^n(E; \mathbf{Z}) \rightarrow H^{n-2}(B; \mathbf{Z})$  be the Gysin homomorphism, which is also called the "integral along the fibers", derived from the Serre spectral sequence of the surface bundle. Then the  $n$ -th Morita-Mumford class  $e_n$  is defined as follows:

$$e_n = e_n(E) := \pi_!(e^{n+1}) \in H^{2n}(B; \mathbf{Z}).$$

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*Key words:* Morita-Mumford class, mapping class group, Klein curve.

*1991 Mathematics Subject Classification:* Primary, 57R20. Secondary, 14H45.

It is equal to the pull-back of  $e_n \in H^{2n}(M_g; \mathbf{Z})$  by the holonomy homomorphism of  $\pi_1(B)$  into  $M_g$ . Especially if  $n = 0$ , then  $e_0$  is equal to the Euler number  $2 - 2g$  of  $\Sigma_g$ .

The main purpose of this paper is to compute the Morita-Mumford classes on arbitrary finite cyclic subgroups of the automorphism group of the Klein curve. The Klein curve is defined by the equation

$$X^3Y + Y^3Z + Z^3X = 0$$

in the complex projective plane  $CP^2$ , and it has been studied by many people, including Baker [Ba], Matsuura [Ma], Morifuji [Mf2], Prapavessi [P] and others. As is known, its genus is 3, and its automorphism group is isomorphic to the projective special linear group  $PSL(2, 7)$ .

We will use a general formula for the Morita-Mumford classes (Theorem 2.1) to prove the main result in section 3. Let  $C$  be a compact Riemann surface of genus  $g$  and  $G$  a finite cyclic group of order  $m$ . Suppose  $G$  acts on  $C$  in a faithful and holomorphic way. Consider the homotopy quotient  $\pi : C_G \rightarrow B_G$  of this action, which is a surface bundle with fiber  $C$ . Let  $\zeta = \exp(2\pi\sqrt{-1}/m)$ , and  $u_0 \in H^2(G; \mathbf{Z})$  the Euler class associated with the complex 1-dimensional  $G$ -module  $R$  given by multiplication by  $\zeta$ . It is equal to the Euler class of the complex line bundle  $R_G$  over the classifying space  $B_G$ . Then the Morita-Mumford classes admit a certain kind of periodicity, whose period is  $\phi(m)$ , the number of integers between 1 and  $m$  relatively prime to  $m$ . Then

**Theorem 2.1**  $e_{n+\phi(m)}(C_G) = e_n(C_G)u_0^{\phi(m)} \in H^{2(n+\phi(m))}(G; \mathbf{Z})$  for  $n \geq 0$ .

Theorem 2.1 is discussed in section 2. In [Ak], Akita notices it for the case where  $m$  is a prime. In view of the affirmative solution of the Nielsen realization problem by Kerckhoff [Ke], any finite subgroup of  $M_g$  is realized as a holomorphic automorphism group of a suitable Riemann surface. Hence the periodicity theorem (Theorem 2.1) also holds for any cyclic subgroup of  $M_g$ . The main result of this paper is the following.

**Theorem 3.1** *Let  $C$  be the Klein curve and  $G$  a finite cyclic group. Suppose  $G$  acts on  $C$  in a faithful and holomorphic way. Let  $\zeta = \exp(2\pi\sqrt{-1}/7)$ , and  $\omega = \exp(2\pi\sqrt{-1}/3)$ . Then the Morita-Mumford classes of this action are given as follows:*

(1) If  $G \cong \mathbf{Z}/7$ , then

$$e_n(C_G) = \begin{cases} 3u_0^n, & \text{if } n \text{ is a multiple of } 3, \\ 0, & \text{otherwise,} \end{cases}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/7$ , where  $u_0 \in H^2(G; \mathbf{Z})$  denotes the Euler class associated with the complex 1-dimensional  $G$ -module given by multiplication by  $\zeta$ .

(2) If  $G \cong \mathbf{Z}/3$ , then

$$e_n(C_G) = \begin{cases} 2v_0^n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/3$ , where  $v_0 \in H^2(G; \mathbf{Z})$  denotes the Euler class associated with the complex 1-dimensional  $G$ -module given by multiplication by  $\omega$ .

(3) If  $G \cong \mathbf{Z}/2$  or  $\mathbf{Z}/4$ , then  $e_n(C_G) = 0$  for  $n \geq 0$  in  $H^{2n}(G; \mathbf{Z})$ .

Theorem 3.1 implies that there exist two kinds of finite cyclic subgroups of  $M_3$ . One satisfies  $e_1 = 0$  and  $e_2 \neq 0$ , the other  $e_1 = e_2 = 0$  and  $e_3 \neq 0$ . In section 4, we construct an action of finite cyclic group on a closed oriented surface satisfying  $e_1 = e_2 = \cdots = e_{n-1} = 0$  and  $e_n \neq 0$  when  $n(\geq 4)$  is an even number or a multiple of 3. Finally in section 5, we consider the case where  $C$  is a hyperelliptic curve, and give two actions of finite cyclic groups. Especially if the genus of  $C$  is one, one of them satisfies  $e_{\text{odd}} \neq 0$  and  $e_{\text{even}} = 0$ .

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## 1 Preliminaries

In this section, we recall a fixed-point formula of the Morita-Mumford classes on finite groups ([KU]). In [KU], we studied the Morita-Mumford classes on finite subgroups of  $M_g$  in the following situation. Let  $G$  be a finite group and  $C$  a compact Riemann surface of genus  $g \geq 0$ . Suppose  $G$  acts on  $C$  in a faithful and holomorphic way. Consider the universal principal  $G$ -bundle  $E_G \rightarrow B_G$ . Then it induces the homotopy quotient (which is also called “the Borel construction”)  $\pi : C_G \rightarrow B_G$  of this action. The space  $C_G$  is the quotient of  $E_G \times C$  by the diagonal action of  $G$ . The map  $\pi$  induced by the first projection provides an oriented fiber bundle with fiber  $C$

$$C \rightarrow C_G \xrightarrow{\pi} B_G.$$

Its Morita-Mumford class  $e_n(C_G) \in H^{2n}(B_G; \mathbf{Z}) = H^{2n}(G; \mathbf{Z})$  is equal to the restriction of  $e_n$  to the subgroup  $G$

Denote the isotropy group at a point  $p \in C$  by  $G_p$ . The singular set

$$S := \{p \in C | G_p \neq \{1\}\}$$

is a  $G$ -stable finite subset of  $C$ , since the action is faithful and holomorphic. Let  $\xi_p = (E_{G_p} \times T_p C)/G_p$  be the oriented real 2-dimensional vector bundle over  $B_{G_p}$  associated with the action of  $G_p$  on the tangent space  $T_p C$  and  $e(\xi_p) \in H^2(B_{G_p}; \mathbf{Z}) = H^2(G_p; \mathbf{Z})$  its Euler class. Since the transfer map  $\text{cor}_{G_p}^G : H^*(G_p; \mathbf{Z}) \rightarrow H^*(G; \mathbf{Z})$  is invariant under conjugation, the cohomology class  $\text{cor}_{G_p}^G(e(\xi_p)^n) \in H^{2n}(G; \mathbf{Z})$  is constant on each  $G$ -orbit (see for example [Br].) Then we obtain an explicit formula for the Morita-Mumford classes  $e_n(C_G)$  in terms of fixed-point data.

**Theorem 1.1 (Kawazumi-Uemura)** *In the situation stated above we have*

$$e_n(C_G) = \sum_{p \in S/G} \text{cor}_{G_p}^G(e(\xi_p)^n) \in H^{2n}(B_G; \mathbf{Z}) = H^{2n}(G; \mathbf{Z})$$

for any  $n \geq 1$ .

It should be noted that this fixed-point formula is deduced from a general formula of Morita-Mumford classes for fiberwise branched coverings of surface bundles by Miller [Mi] and Morita [Mol]. The right-hand side depends only on the isotropy groups and their actions on the tangent spaces at the fixed-points.

## 2 A periodicity theorem for the Morita-Mumford classes

Let  $C$  be a compact Riemann surface of genus  $g$ . Suppose a finite cyclic group  $G$  of order  $m$  acts on  $C$  in a faithful and holomorphic way. Let  $\zeta = \exp(2\pi\sqrt{-1}/m)$ , and

choose a generator  $\gamma$  of  $G$ . Then we consider the complex 1-dimensional  $G$ -module  $R$  where the action of  $\gamma$  is given by the multiplication by  $\zeta$ , and define  $u_0 \in H^2(G; \mathbf{Z})$  by the Euler class associated with  $R$ . Throughout this paper, we will call  $u_0$  simply "the Euler class given by multiplication by  $\zeta$ ". Then the Morita-Mumford classes admit a certain kind of periodicity, whose period is  $\phi(m)$ , the number of integers between 1 and  $m$  relatively prime to  $m$ . In other words,  $\phi(m)$  is the Euler function of  $m$ . Then we obtain the following result.

**Theorem 2.1**  $e_{n+\phi(m)}(C_G) = e_n(C_G)u_0^{\phi(m)} \in H^{2(n+\phi(m))}(G; \mathbf{Z})$  for  $n \geq 0$ .

**Proof.** Let  $S = \coprod_{i=1}^l G \cdot p_i$  be the  $G$ -stable decomposition of the singular set and  $m_i$  the order of  $G \cdot p_i$ , so that  $\frac{m}{m_i} = |G_{p_i}|$ . Let  $\zeta_i = \exp(2\pi\sqrt{-1}/\frac{m}{m_i})$ . Then the action  $\gamma^{m_i}$  on the tangent space  $T_{p_i}C$  is equal to the multiplication by  $\zeta_i^{k_i}$  for some integer  $k_i$  relatively prime to  $m$ . From Theorem 1.1, when  $n \geq 1$ , the Morita-Mumford classes of this action is given as follows:

$$e_n(C_G) = \left( \sum_{i=1}^l m_i k_i^n \right) u_0^n.$$

As is well-known,  $k_i^{\phi(\frac{m}{m_i})} \equiv 1 \pmod{\frac{m}{m_i}}$ . Since  $\phi(\frac{m}{m_i})$  divides  $\phi(m)$ , this congruence implies  $m_i k_i^{\phi(m)} \equiv m_i \pmod{m}$ . Therefore we obtain

$$e_{n+\phi(m)}(C_G) = \left( \sum_{i=1}^l m_i k_i^{n+\phi(m)} \right) u_0^{n+\phi(m)} = \left( \sum_{i=1}^l m_i k_i^n \right) u_0^n u_0^{\phi(m)} = e_n(C_G)u_0^{\phi(m)}$$

in  $H^{2(n+\phi(m))}(G; \mathbf{Z}) \cong \mathbf{Z}/m$ . In the case where  $n = 0$  we have  $\sum_{i=1}^l m_i \equiv 2-2g = e_0(C_G) \pmod{m}$  from the classical Riemann-Hurwitz formula. Hence we obtain

$$e_{\phi(m)}(C_G) = (2-2g)u_0^{\phi(m)} = e_0(C_G)u_0^{\phi(m)}$$

similarly. This concludes the proof.

**Corollary 2.1**  $e_{s\phi(m)}(C_G) = (2-2g)u_0^{s\phi(m)} = e_0(C_G)u_0^{s\phi(m)}$  for any integer  $s \geq 1$ .

When  $m = 2, 3, 4$  and  $6$ , then  $\phi(m) \leq 2$ . Using Theorem 2.1 and Corollary 2.1, we deduce the following corollaries.

**Corollary 2.2** If  $G \cong \mathbf{Z}/2$ , then  $e_n(C_G) = 0$  for  $n \geq 0$ .

**Corollary 2.3** If  $G \cong \mathbf{Z}/3, \mathbf{Z}/4$  or  $\mathbf{Z}/6$ , then

$$e_n(C_G) = \begin{cases} (2-2g)u_0^n, & \text{if } n \text{ is even,} \\ e_1 u_0^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$



### 3 An application to the Klein curve

Let  $C$  be the complex algebraic curve defined by the equation

$$X^3Y + Y^3Z + Z^3X = 0 \quad (1)$$

in the complex projective plane  $\mathbf{CP}^2$ . The curve  $C$  is of genus 3, and called the Klein curve. It is known that the automorphism group  $\text{Aut}(C)$  is isomorphic to the projective special linear group  $PSL(2, 7)$  which has order 168. Moreover  $\text{Aut}(C)$  has the presentation

$$PSL(2, 7) = \langle s, t \mid s^2 = t^3 = (st)^7 = [s, t]^4 = 1 \rangle,$$

where  $[s, t] = sts^{-1}t^{-1}$  denotes the commutator of  $s$  and  $t$ . We may regard it as a subgroup of  $M_3$ .

The purpose of this section is to compute the Morita-Mumford classes on arbitrary cyclic subgroups of  $PSL(2, 7)$  as an application of Theorem 1.1 and Theorem 2.1. The conjugacy classes of  $PSL(2, 7)$  are as follows (see [I]):

| Conjugacy class    | 1 | 2  | 3  | 4  | $7_1$ | $7_2$ |
|--------------------|---|----|----|----|-------|-------|
| Number of elements | 1 | 21 | 56 | 42 | 24    | 24    |

**Table 1.** Conjugacy classes of  $PSL(2, 7)$

In Table 1, each conjugacy class is denoted by the order of its elements, and  $7_1$  and  $7_2$  mean the different classes. This Table 1 indicates that any two cyclic subgroups of  $PSL(2, 7)$  are conjugate to each other if they have the same order, and each of them is isomorphic to  $\mathbf{Z}/2$ ,  $\mathbf{Z}/3$ ,  $\mathbf{Z}/4$  or  $\mathbf{Z}/7$ .

The main result in this paper is the following.

**Theorem 3.1** *Let  $C$  be the Klein curve and  $G$  a finite cyclic group. Suppose  $G$  acts on  $C$  in a faithful and holomorphic way. Let  $\zeta = \exp(2\pi\sqrt{-1}/7)$ , and  $\omega = \exp(2\pi\sqrt{-1}/3)$ . Then the Morita-Mumford classes of this action are given as follows:*

(1) If  $G \cong \mathbf{Z}/7$ , then

$$e_n(C_G) = \begin{cases} 3u_0^n, & \text{if } n \text{ is a multiple of } 3, \\ 0, & \text{otherwise,} \end{cases}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/7$ , where  $u_0$  denotes the Euler class given by multiplication by  $\zeta$ .

(2) If  $G \cong \mathbf{Z}/3$ , then

$$e_n(C_G) = \begin{cases} 2v_0^n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/3$ , where  $v_0$  denotes the Euler class given by multiplication by  $\omega$ .

(3) If  $G \cong \mathbf{Z}/2$ , then  $e_n(C_G) = 0$  for  $n \geq 0$  in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/2$ .

(4) If  $G \cong \mathbf{Z}/4$ , then  $e_n(C_G) = 0$  for  $n \geq 0$  in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/4$ .

**Proof.** We recall that the genus of the Klein curve is 3. We see from [KU] that  $e_1 = 0$ , since  $PSL(2, 7)$  is a perfect group. Hence (2),(3) and (4) follow from Corollary 2.2 and 2.3 immediately.

In order to prove (1), we define an automorphism  $\gamma$  of  $C$  as follows: (see for example [AR], [Kl])

$$\gamma(X, Y, Z) := (\zeta X, \zeta^4 Y, \zeta^2 Z),$$

where  $\zeta = \exp(2\pi\sqrt{-1}/7)$ . It induces an element  $\gamma$  of order 7 of the automorphism group  $PSL(2, 7)$ . We put  $G = \langle \gamma \rangle < PSL(2, 7)$ . Since any cyclic subgroups of  $PSL(2, 7)$  of order 7 is conjugate to  $G$ , it suffices to compute  $e_n(C_G)$ .

On the open subset  $\{Z \neq 0\}$ , substituting  $x := X/Z$  and  $y := Y/Z$  into (1), we obtain the following function of two variables:

$$f := x^3 y + y^3 + x.$$

Then  $\gamma(x) = \zeta^{-1}x$  and  $\gamma(y) = \zeta^2 y$ . We can easily see that  $[0 : 0 : 1]$  is the unique fixed point of  $\gamma$  on  $\{Z \neq 0\}$ . By the implicit function theorem, the variable  $y$  can serve as a coordinate at  $(x, y) = (0, 0)$  since  $f_x(0, 0) \neq 0 = f_y(0, 0)$ . Let  $u_0 \in H^2(G; \mathbf{Z})$  be the Euler class given by multiplication by  $\zeta$ . Then we can see that the contribution at  $[0 : 0 : 1]$  is  $(2u_0)^n$ .

In a similar way, on  $\{X \neq 0\}$ ,  $[1 : 0 : 0]$  is the unique fixed point and its contribution is  $u_0^n$ , and on  $\{Y \neq 0\}$ ,  $[0 : 1 : 0]$  is the unique fixed point and its contribution is  $(-3u_0)^n$ . Therefore we obtain

$$\begin{aligned} e_n(C_G) &= (2u_0)^n + u_0^n + (-3u_0)^n \\ &= \{2^n + 1 + 2^{2n}\}u_0^n \end{aligned}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/7$ . This concludes the proof.

**Remark 3.1** As is known, we have another action  $\gamma_0$  of order 7 such that

$$\gamma_0(X, Y, Z) := (\zeta X, \zeta^2 Y, \zeta^4 Z)$$

(see for example [Ba].) If we compute the Morita-Mumford classes using this action, we obtain the following:

$$e_n(C_G) = \begin{cases} -3u_0^n, & \text{if } n \text{ is a multiple of } 3, \\ 0, & \text{otherwise,} \end{cases}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/7$ .

**Remark 3.2** The cyclic actions on the Klein curve  $C$  are explicitly given by [Kl], [P], and [Ba]. We can also compute the Morita-Mumford classes on  $\mathbf{Z}/3$  by using the action  $\tau$  of order 3 given by

$\tau(X, Y, Z) := (Y, Z, X)$  (cyclic permutation.)

In fact, the fixed points of  $\tau$  are  $[1 : \omega : \omega^2]$  and  $[1 : \omega^2 : \omega]$ , so using  $e_1 = 0$  (recall that  $PSL(2, 7)$  is perfect), we obtain the same result as in Theorem 3.1.

## 4 Some actions of cyclic groups on surfaces

Theorem 3.1 implies the existence of a finite cyclic subgroup of  $M_3$  satisfying  $e_1 = 0$ ,  $e_2 \neq 0$ , and  $e_1 = e_2 = 0$ ,  $e_3 \neq 0$ . So we consider the following problem.

**Problem** Construct a finite cyclic subgroup of  $M_g$  satisfying  $e_1 = e_2 = \dots = e_{n-1} = 0$  and  $e_n \neq 0$  for each  $n \geq 4$ .

In this section, we will give two affirmative partial answers to this problem.

**Theorem 4.1** For an arbitrary integer  $m \geq 0$ , there exists an action of a finite cyclic group  $G$  on a closed oriented surface  $C$  satisfying  $e_1(C_G) = e_2(C_G) = \dots = e_{2m-1}(C_G) = 0$  and  $e_{2m}(C_G) \neq 0$ .

**Theorem 4.2** For an arbitrary integer  $m \geq 0$ , there exists an action of a finite cyclic group  $G$  on a closed oriented surface  $C$  satisfying  $e_1(C_G) = e_2(C_G) = \dots = e_{3m-1}(C_G) = 0$  and  $e_{3m}(C_G) \neq 0$ .

**Proof of Theorem 4.1.** By Dirichlet's Theorem, there exists a prime  $p$  satisfying  $p = 2ml + 1$  for some integer  $l \geq 1$ . Let  $k$  be a primitive root of  $p$ , so that  $k^{p-1} \equiv 1 \pmod{p}$  and  $k_0 := k^l$ . We consider the following situation. At first, let  $S_i^2$  be the 2-sphere of radius  $a > 0$  inside  $\mathbf{R}^3$  defined by the following equation:

$$S_i^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + \{z + 3(i-1)a\}^2 = a^2\}$$

for  $1 \leq i \leq m$ . Secondly, take  $2p$  points

$$\begin{aligned} p_{i+}^j &= \left(\frac{\sqrt{3}}{2}a \cos\left(\frac{2j\pi}{p}\right), \frac{\sqrt{3}}{2}a \sin\left(\frac{2j\pi}{p}\right), (-3i + \frac{7}{2})a\right) \\ p_{i-}^j &= \left(\frac{\sqrt{3}}{2}a \cos\left(\frac{2j\pi}{p}\right), \frac{\sqrt{3}}{2}a \sin\left(\frac{2j\pi}{p}\right), (-3i + \frac{5}{2})a\right) \end{aligned}$$

on each  $S_i^2$  ( $0 \leq j \leq p-1$ ). Take sufficiently small open discs  $U_{i\pm}^j$  centered at  $p_{i\pm}^j$  respectively, and connect  $U_{i-}^j$  and  $U_{(i+1)+}^{k_0j}$  with a tube for each  $i, j$ . Then we obtain a closed oriented surface  $C$  of genus  $(p-1)(m-1)$ . We define an action of the cyclic group  $G = \mathbf{Z}/p$  on  $C$  as follows. Rotate  $S_i^2$  by  $2k_0^{i-1}\pi/p$  about the  $z$ -axis. From the construction, these actions extend to the action of  $G = \mathbf{Z}/p$  on the whole surface  $C$ . Let  $u_0 \in H^2(G; \mathbf{Z})$  be the Euler class given by multiplication by  $\zeta = \exp(2\pi\sqrt{-1}/p)$ . Then the isotropy group of each singular point is  $G$ , namely, this action is semi-free. The fixed

points on  $S_i^2$  are  $(0, 0, (-3i + 3 \pm 1)a)$ . Considering the contribution of each fixed point, the  $n$ -th Morita-Mumford class of this action is

$$\begin{aligned} e_n(C_G) &= u_0^n + (-u_0)^n + (k_0 u_0)^n + (-k_0 u_0)^n + \cdots + (k_0^{m-1} u_0)^n + (-k_0^{m-1} u_0)^n \\ &= \{1 + (-1)^n + k_0^n + (-k_0)^n + \cdots + k_0^{(m-1)n} + (-k_0)^{(m-1)n}\} u_0^n \end{aligned}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/p$ . It is obvious that  $e_n(C_G) = 0$  when  $n$  is an odd number. When  $n = 2t$  ( $1 \leq t \leq m-1$ ), then

$$\begin{aligned} e_{2t}(C_G) &= 2(1 + k_0^{2t} + k_0^{4t} + \cdots + k_0^{2(m-1)t}) u_0^{2t} \\ &= 2 \cdot \frac{k_0^{2mt} - 1}{k_0^{2t} - 1} u_0^{2t} \\ &= 0, \end{aligned}$$

since  $k_0^{2mt} = k^{2mlt} = 1$ , and  $k_0^{2t} = k^{2lt} \neq 0$ . When  $n = 2m$ , then

$$\begin{aligned} e_{2m}(C_G) &= 2(1 + k_0^{2m} + k_0^{4m} + \cdots + k_0^{2m(m-1)}) u_0^{2m} \\ &= 2(1 + k^{2ml} + k^{4ml} + \cdots + k^{2ml(m-1)}) u_0^{2m} \\ &= 2 \cdot 1 \cdot m u_0^{2m} \\ &\neq 0 \end{aligned}$$

in  $\mathbf{Z}/p$  since  $m < p-1 = 2ml$ . This concludes the proof.

Consider the case where  $m = 1$  in the proof of Theorem 4.1. Then the genus of  $C$  is zero. Therefore  $C$  is isomorphic to the complex projective line  $\mathbf{P}^1$ . Any action of a finite cyclic group on  $\mathbf{P}^1$  is conjugate to the rotation as above. We can regard  $C$  as the unit sphere  $S^2$  in  $\mathbf{R}^3$  by a suitable diffeomorphism. So we can define the action of  $G = \mathbf{Z}/p$  on  $C$ , which is the rotation of  $C$  by  $2a\pi/p$  about the  $z$ -axis for some integer  $1 \leq a \leq [\frac{p}{2}]$ . Here  $[\frac{p}{2}]$  denotes the largest integer less than or equal to  $\frac{p}{2}$ . Therefore we obtain the following.

**Proposition 4.1** *Let  $C$  be the Riemann sphere  $\mathbf{P}^1$ . Suppose  $G = \mathbf{Z}/p$  acts on  $C$  as above. Let  $u_0 \in H^2(G; \mathbf{Z})$  be the Euler class given by multiplication by  $\zeta = \exp(2\pi\sqrt{-1}/m)$ . Then*

$$e_n(C_G) = \begin{cases} 2au_0^n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof of Theorem 4.2.** By Dirichlet's Theorem, there exists a prime  $p$  satisfying  $p = 3ml + 1$  for some integer  $l \geq 1$ . Let  $k$  be a primitive root of  $p$ , and  $k_0 := k^l$ , and  $a (\geq 2)$  the smallest integer satisfying  $p \mid 1 + a + a^2$ . Define the complex algebraic curve  $C_0$  by

$$X^{a+1}Y + Y^{a+1}Z + Z^{a+1}X = 0$$

in  $\mathbf{CP}^2$ . It is not difficult to see that  $C_0$  is a non-singular curve, and its genus is  $a(a+1)/2$  by *Plücker's formula*. Prepare  $m$  copies  $C_i (1 \leq i \leq m)$  of the curve  $C_0$ . Similarly in the proof of Theorem 3.1, we define an automorphism  $\gamma_i$  on each  $C_i$  as follows:

$$\gamma_i(X, Y, Z) := (\zeta^{k_0^{i-1}} X, \zeta^{k_0^{i-1} a^2} Y, \zeta^{k_0^{i-1} a} Z),$$

where  $\zeta = \exp(2\pi\sqrt{-1}/p)$ . Note that  $\gamma_i = \gamma_1^{k_0^{i-1}}$ .

Each  $\gamma_i$  induces an action of the cyclic group  $G = \mathbf{Z}/p$  on  $C_i$ . We can easily see that the singular set  $S_i \subset C_i$  of  $G$  is

$$S_i = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}.$$

Choose two points  $p_i, q_i \in C_i - S_i$  such that  $G \cdot p_i \cap G \cdot q_i = \emptyset$ . Define  $p_i^j := \gamma_i^j(p_i)$  and  $q_i^j := \gamma_i^j(q_i)$  for  $0 \leq j \leq p-1$ . Note that the action of  $G$  on  $C_i - S_i$  is free. Take sufficiently small open discs  $U_{i,j}$  and  $V_{i,j}$  in  $C_i$  centered at  $p_i^j$  and  $q_i^j$ , respectively. Connect  $V_{i,j}$  and  $U_{i+1,j}$  with a tube for each  $i, j$  ( $1 \leq i \leq m-1$ ). Then we obtain a closed oriented surface  $C$  of genus  $a(a+1)(p-1)(m-1)/2$ . From this construction, the automorphisms  $\gamma_i$ 's extend to the action of  $G = \mathbf{Z}/p$  on the whole surface  $C$ .

Let  $u_0 \in H^2(G; \mathbf{Z})$  be the Euler class given by multiplication by  $\zeta = \exp(2\pi\sqrt{-1}/p)$ . Clearly this action is semi-free, and we can compute the contribution of each fixed point similarly in the proof of Theorem 3.1. Therefore the  $n$ -th Morita-Mumford class of the action on  $C_i$  is

$$e_n((C_i)_G) = [\{k_0^{i-1}(a-1)\}^n + \{k_0^{i-1}(1-a^2)\}^n + \{k_0^{i-1}(a^2-a)\}^n] u_0^n,$$

and that of the action on the whole surface  $C$  is

$$\begin{aligned} e_n(C_G) &= \sum_{i=1}^m e_n((C_i)_G) \\ &= (1 + k_0^n + \dots + k_0^{(m-1)n}) \{(a-1)^n + (1-a^2)^n + (a^2-a)^n\} u_0^n \end{aligned}$$

in  $H^{2n}(G; \mathbf{Z}) \cong \mathbf{Z}/p$ . It is easy to check that  $e_n(C_G) = 0$  when  $n$  is not a multiple of 3. When  $n = 3t$  ( $1 \leq t \leq m-1$ ), then

$$(1 + k_0^{3t} + \dots + k_0^{3t(m-1)}) = \frac{k_0^{3mt} - 1}{k_0^{3t} - 1} = \frac{k^{3lmt} - 1}{k^{3lt} - 1} = 0,$$

since  $k_0^{3mt} = k^{3mlt} = 1$ , and  $k_0^{3t} = k^{3lt} \neq 0$ . Therefore  $e_{3t}(C_G) = 0 \in \mathbf{Z}/p$ . When  $n = 3m$ , then

$$(1 + k_0^{3m} + \dots + k_0^{3m(m-1)}) = 1 + k^{3ml} + \dots + k^{3m(m-1)l} = 1 \cdot m \neq 0.$$

Therefore it is easy to see that  $e_{3m}(C_G) = 3m(a-1)^{3m} u_0^{3m} \neq 0$  in  $\mathbf{Z}/p$ . This concludes the proof.

## 5 Hyperelliptic curves

In this section, we consider the case where  $C$  is a hyperelliptic curve, and give two actions of finite cyclic groups.

**Example 5.1** Consider two complex plane curves

$$w^2 = z(1 - z^{2g}), \quad w_1^2 = z_1(z_1^{2g} - 1)$$

for  $g \geq 1$ . Glueing them each other by the map  $z_1 = z^{-1}$  and  $w_1 = z^{-g-1}w$ , we obtain a hyperelliptic curve  $C$  of genus  $g$ . Let  $\zeta = \exp(2\pi\sqrt{-1}/4g)$ , consider the action

$$\gamma : (z, w) \mapsto (\zeta^{2k}z, \zeta^k w) \quad (k = 1, 2, \dots, 4g - 1).$$

Then it gives an automorphism of  $C$  of order  $4g$ . Its singular set  $S$  is

$$S = \{(0, 0), \infty, (\zeta^{2j}, 0) ; j = 0, 1, \dots, 2g - 1\},$$

where  $\infty$  denotes the point at infinity:  $(z_1, w_1) = (0, 0)$ . This action is not semi-free since the isotropy groups of  $(0, 0)$  and  $\infty$  are  $\langle \gamma \rangle$ , but that of  $(\zeta^{2j}, 0)$  is  $\langle \gamma^{2g} \rangle$ . Here  $\langle \gamma \rangle$  (resp.  $\langle \gamma^{2g} \rangle$ ) denotes the automorphism group of  $C$  generated by  $\gamma$  (resp.  $\gamma^{2g}$ ). Let  $u_0 \in H^2(\langle \gamma \rangle; \mathbf{Z})$  (resp.  $v_0 \in H^2(\langle \gamma^{2g} \rangle; \mathbf{Z})$ ) be the Euler class given by multiplication by  $\zeta$  (resp.  $\zeta^{2g}$ ). Then  $u_0^n$  (resp.  $v_0^n$ ) generates the group  $H^{2n}(\langle \gamma \rangle; \mathbf{Z}) \cong \mathbf{Z}/4g$  (resp.  $H^{2n}(\langle \gamma^{2g} \rangle; \mathbf{Z}) \cong \mathbf{Z}/2$ ) for each  $n$ .

Then Theorem 1.1 implies

$$e_n(C_{\langle \gamma \rangle}) = u_0^n + \{-(2g + 1)\}^n u_0^n + \text{cor}_{\langle \gamma^{2g} \rangle}^{\langle \gamma \rangle} v_0^n \in H^{2n}(\langle \gamma \rangle; \mathbf{Z}).$$

From well-known properties of the transfer map, we can easily see that  $\text{cor}_{\langle \gamma^{2g} \rangle}^{\langle \gamma \rangle} v_0^n = [\langle \gamma \rangle : \langle \gamma^{2g} \rangle] u_0^n = 2g u_0^n$  (see for example [Br].) Therefore we obtain

$$e_n(C_{\langle \gamma \rangle}) = \begin{cases} (2 + 2g)u_0^n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

in  $H^{2n}(\langle \gamma \rangle; \mathbf{Z}) \cong \mathbf{Z}/4g$ . Especially if  $g = 1$ , then  $2 + 2g \equiv 0 \pmod{4}$ . So  $e_n(C_{\langle \gamma \rangle}) = 0$  for any  $n \geq 0$ .

**Example 5.2** Consider two complex plane curves

$$w^2 = z(1 - z^{2g+1}), \quad w_1^2 = z_1^{2g+1} - 1$$

for  $g \geq 1$ . Glueing them each other by the map  $z_1 = z^{-1}$  and  $w_1 = z^{-g-1}w$ , we obtain a hyperelliptic curve  $C$  of genus  $g$ . Let  $\zeta = \exp(2\pi\sqrt{-1}/(4g + 2))$ , consider the action

$$\gamma : (z, w) \mapsto (\zeta^{2k}z, \zeta^k w) \quad (k = 1, 2, \dots, 4g + 1).$$

Then it gives an automorphism of  $C$  of order  $4g + 2$ . Its singular set  $S$  is

$$S = \{(0, 0), \infty_-, \infty_+, (\zeta^{2j}, 0) ; j = 0, 1, \dots, 2g\},$$

where  $\infty_-$  and  $\infty_+$  denote the points at infinity:  $(z_1, w_1) = (0, \pm\sqrt{-1})$ . This action is not semi-free since the isotropy group of  $(0, 0)$  is  $\langle \gamma \rangle$ , but those of  $\infty_-$  and  $\infty_+$  are  $\langle \gamma^2 \rangle$ , and that of  $(\zeta^{2j}, 0)$  is  $\langle \gamma^{2g+1} \rangle$ . Take the Euler classes  $u_0 \in H^2(\langle \gamma \rangle; \mathbf{Z})$ ,  $m_0 \in H^2(\langle \gamma^2 \rangle; \mathbf{Z})$ , and  $v_0 \in H^2(\langle \gamma^{2g+1} \rangle; \mathbf{Z})$  similarly in Example 5.1.

Then Theorem 1.1 implies

$$e_n(C_{\langle \gamma \rangle}) = u_0^n + \text{cor}_{\langle \gamma^2 \rangle}^{\langle \gamma \rangle}(-m_0)^n + \text{cor}_{\langle \gamma^{2g+1} \rangle}^{\langle \gamma \rangle}v_0^n \in H^{2n}(\langle \gamma \rangle; \mathbf{Z}),$$

Note that the actions at the points at infinity are  $z_1 \mapsto \zeta^{-2k}z_1$  and  $w_1 \mapsto \zeta^{-(2g+1)k}w_1 = (-1)^k w_1$ , so the contribution at each point is  $(-m_0)^n$ . Similarly in Example 5.1, we can easily see that  $\text{cor}_{\langle \gamma^2 \rangle}^{\langle \gamma \rangle}(-m_0)^n = [\langle \gamma \rangle : \langle \gamma^2 \rangle]u_0^n = 2(-1)^n u_0^n$ , and  $\text{cor}_{\langle \gamma^{2g+1} \rangle}^{\langle \gamma \rangle}v_0^n = [\langle \gamma \rangle : \langle \gamma^{2g+1} \rangle]u_0^n = (2g + 1)u_0^n$ . Therefore we obtain

$$e_n(C_{\langle \gamma \rangle}) = \begin{cases} 2(2 + g)u_0^n, & \text{if } n \text{ is even,} \\ 2gu_0^n, & \text{if } n \text{ is odd,} \end{cases}$$

in  $H^{2n}(\langle \gamma \rangle; \mathbf{Z}) \cong \mathbf{Z}/(4g + 2)$ . Especially if  $g = 1$ , then  $2(2 + 1) \equiv 0 \pmod{6}$ . So we obtain

$$e_n(C_{\langle \gamma \rangle}) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 2u_0^n, & \text{if } n \text{ is odd,} \end{cases}$$

in  $H^{2n}(\langle \gamma \rangle; \mathbf{Z}) \cong \mathbf{Z}/6$ . This example shows that  $e_{\text{odd}} \neq 0$  and  $e_{\text{even}} = 0$ , and differs from the others described above.

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