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On The Bergman Space**

**T. Nakazi and T. Osawa**

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# Finite Rank Intermediate Hankel Operators On The Bergman Space

by

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Abstract. Let  $L^2 = L^2(D, r dr d\theta/\pi)$  be the Lebesgue space on the open unit disc and  $L_a^2 = L^2 \cap \mathcal{H}ol(D)$  be the Bergman space. Let  $P$  be the orthogonal projection of  $L^2$  onto  $L_a^2$  and let  $Q$  be the orthogonal projection onto  $\bar{L}_{a,0}^2 = \{g \in L^2 ; \bar{g} \in L_a^2, g(0) = 0\}$ . Then  $I - P \geq Q$ . The big Hankel operator and the small Hankel operator on  $L_a^2$  are defined as the following : For  $\phi$  in  $L^\infty$ ,  $H_\phi^{big}(f) = (I - P)(\phi f)$  and  $H_\phi^{small}(f) = Q(\phi f)$  ( $f \in L_a^2$ ). In this paper, the finite rank intermediate Hankel operators between  $H_\phi^{big}$  and  $H_\phi^{small}$  are studied. We are working on the more general space, that is, the weighted Bergman space.

## §1. Introduction

Let  $D$  be the open unit disc in  $\mathbf{C}$  and  $d\mu$  the finite positive Borel measure on  $D$ . Let  $L^2 = L^2(\mu) = L^2(D, d\mu)$  and  $\mathcal{H}ol(D)$  be the set of all holomorphic functions on  $D$ . The weighted Bergman space  $L_a^2 = L_a^2(\mu)$  is the intersection of  $L^2$  and  $\mathcal{H}ol(D)$ . In general,  $L_a^2$  is not closed. In [6, Theorem 8], when  $(\text{supp } \mu) \cap D$  is a uniqueness set for  $\mathcal{H}ol(D)$ , the first author and M. Yamada gave a necessary and sufficient condition for that  $L_a^2$  is closed. Throughout this paper, we assume that  $L_a^2$  is closed. When  $d\mu = r dr d\theta / \pi$ ,  $L_a^2$  is the usual Bergman space.

For  $\mu$  such that  $L_a^2(\mu)$  is closed, when  $\mathcal{M}$  is the closed subspace of  $L^2(\mu)$  and  $z\mathcal{M} \subseteq \mathcal{M}$ ,  $\mathcal{M}$  is called an invariant subspace. Suppose  $\mathcal{M} \supseteq zL_a^2$ .  $P^{\mathcal{M}}$  denotes the orthogonal projection from  $L^2$  onto  $\mathcal{M}$ . For  $\phi$  in  $L^\infty = L^\infty(\mu) = L^\infty(D, d\mu)$ , the intermediate Hankel operator  $H_\phi^{\mathcal{M}}$  is defined by

$$H_\phi^{\mathcal{M}} f = (I - P^{\mathcal{M}})(\phi f) \quad (f \in L_a^2).$$

When  $\mathcal{M} = L_a^2$ ,  $H_\phi^{\mathcal{M}}$  is called a big Hankel operator  $H_\phi^{big}$  and when  $\mathcal{M} = (\bar{z}L_a^2)^\perp$ ,  $H_\phi^{\mathcal{M}}$  is called a small Hankel operator  $H_\phi^{small}$ . Note that  $H_\phi^{\mathcal{M}}$  is called a little Hankel operator when  $\mathcal{M} = (\bar{L}_a^2)^\perp$ .

For arbitrary symbol  $\phi$  in  $L^\infty$ , in the case of  $d\mu = r dr d\theta / \pi$ , both  $H_\phi^{big}$  and  $H_\phi^{small}$  were studied when they are compact operators or Schatten class operators (see [12]). However it seems to have not been studied when they are finite rank operators. When  $\bar{\phi}$  is in  $L_a^2$ , it is known (see [12, p155]) that if  $H_\phi^{big}$  is a finite rank operator then  $H_\phi^{big} = 0$  and if  $\bar{\phi}$  is a polynomial then  $H_\phi^{small}$  is a finite rank operator. In this paper, for arbitrary symbol  $\phi$  in  $L^\infty$  we show that if  $H_\phi^{big}$  is a finite rank operator then  $H_\phi^{big} = 0$ , and we study when  $H_\phi^{small}$  is a finite rank operator. In fact, we study such problems for the intermediate Hankel operators  $H_\phi^{\mathcal{M}}$  on the weighted Bergman space  $L_a^2(\mu)$ . In [2],[7],[9] and [10], intermediate Hankel operators were studied in special weights,  $d\mu = (\alpha + 1)(1 - r^2)^\alpha r dr d\theta / \pi$  for  $-1 < \alpha < \infty$ . In particular, E. Strose [9] studied finite rank intermediate Hankel operators.

Let  $d\mu = d\sigma(r)d\theta$  be a Borel measure on  $D$  where  $d\sigma(r)$  is a positive measure on  $[0,1)$  with  $d\sigma([0,1)) = 1/2\pi$  and  $d\theta$  is the Lebesgue measure on  $\partial D$ .  $L_a^2(\mu)$  is closed if  $d\sigma([t,1)) > 0$  for any  $t > 0$  (see [6]). For this type measures, it is possible to study more precisely the intermediate Hankel operators. In fact,  $L^2$  has the following orthogonal decomposition :

$$L^2 = \sum_{j=-\infty}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}$$

where  $\mathcal{L}^2 = L^2(d\sigma) = L^2([0,1), d\sigma)$ . Set

$$\mathbf{H}^2 = \sum_{j=0}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta},$$

then  $L_a^2 \subset \mathbf{H}^2 \subset (\bar{z}L_a^2)^\perp$  and  $L^2 = \mathbf{H}^2 \oplus e^{-i\theta}\bar{\mathbf{H}}^2$ . If  $\mathcal{M} = \mathbf{H}^2$ , it is easy comparatively to determine finite rank Hankel operators  $H_\phi^{\mathcal{M}}$  and we can do it completely in Section 5.

We can expect that  $H_\phi^{\mathcal{M}}$  is close to  $H_\phi^{big}$  in case  $\mathcal{M} \subseteq \mathbf{H}^2$  (see Section 5) and  $H_\phi^{\mathcal{M}}$  is close to  $H_\phi^{small}$  in case  $\mathcal{M} \supseteq \mathbf{H}^2$  (see Section 6).

In Section 2, we will describe an invariant subspace in  $L_a^2$  whose codimension is of finite. Moreover we will show that there does not exist an invariant subspace which contains  $L_a^2$  properly and in which  $L_a^2$  is of finite codimension. We will also give a lot of examples of invariant subspaces which contain  $L_a^2$  and in which Hankel operators will be studied in this paper. In Section 3, we will describe finite rank intermediate Hankel operators for arbitrary measure  $\mu$  such that  $L_a^2(\mu)$  is closed. Moreover we will show that there does not exist any nonzero finite rank Hankel operators  $H_\phi^{big}$  and there exists a nonzero finite rank Hankel operator  $H_\phi^{small}$ . In fact, we will give two necessary and sufficient conditions for that if  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  then  $H_\phi^{\mathcal{M}} = 0$ . In Sections 3, 4 and 5 we will use the Fourier coefficients  $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$  of  $\mathcal{M}$  and so we will assume  $d\mu = d\sigma(r)d\theta$ . Using the Fourier coefficients of  $\phi$  and  $\mathcal{M}$ , we will give a necessary and sufficient condition for that  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$ . Assuming that  $\phi$  is a harmonic function, we can get a better necessary and sufficient condition. When  $\mathcal{M} \subseteq \mathbf{H}^2$ , using the Fourier coefficients  $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ , we will give a necessary condition and a sufficient condition for that if  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  then  $H_\phi^{\mathcal{M}} = 0$ . Two conditions are very similar but are a little different. Applications will be given to examples in Section 2.

## §2. Invariant subspaces

In this section, we assume that  $d\mu = d\sigma(r)d\theta$  and  $d\sigma([t, 1]) > 0$  for any  $t > 0$ , except Propositions 1 and 2. For our purpose, the invariant subspace  $\mathcal{M}$  must contain  $zL_a^2$  but  $\ker H_\phi^{\mathcal{M}}$  is an invariant subspace in  $L_a^2$ . If  $H_\phi^{\mathcal{M}}$  is of finite rank, then the codimension of  $\ker H_\phi^{\mathcal{M}}$  in  $L_a^2$  is finite. In order to study finite rank intermediate Hankel operators, we need the generalization of a result of S.Axler and P.Bourdon [1] which determines finite codimensional invariant subspaces in  $L_a^2$  when  $d\mu = r dr d\theta / \pi$ . In the following Propositions 1 and 2, the measure  $\mu$  is an arbitrary finite positive Borel measure such that  $L_a^2$  is closed and  $(\text{supp } \mu) \cap D$  is a uniqueness set for  $\mathcal{H}ol(D)$ . Since  $\mathbf{H}^2 \cap L^\infty$  is an extended weak-\*Dirichlet algebra in  $L^\infty$ , Proposition 4 is a corollary of [4, Theorem 1]. We will give several examples of invariant subspaces which contain  $zL_a^2$ .

**Proposition 1.** *Suppose  $\mathcal{M}$  is an invariant subspace in  $L_a^2$  and  $\ell$  is a positive integer. The codimension of  $\mathcal{M}$  in  $L_a^2$  is  $\ell$  if and only if  $\mathcal{M} = qL_a^2$  where  $q = \prod_{j=1}^{\ell} (z - a_j)$  and  $a_j \in D$  ( $1 \leq j \leq \ell$ ).*

*Proof.* The proof is almost parallel to that in [1, Theorem 1]. We will give a sketch of it. Suppose  $\mathcal{M}^\perp = L_a^2 \ominus \mathcal{M}$  and  $\dim \mathcal{M}^\perp = \ell$ . Put

$$S_z f = P(zf) \quad (f \in \mathcal{M}^\perp)$$

where  $P$  is an orthogonal projection. Since  $\ell < \infty$ , there exists an analytic polynomial  $b$  such that  $b(S_z) = S_{b(z)} = 0$  and the degree of  $b$  is less than equal to  $\ell$ . Hence  $b\mathcal{M}^\perp \subseteq \mathcal{M}$  and so  $bL_a^2 \subseteq \mathcal{M}$ . We will show that the zeros of  $b$  are only in  $D$  and the degree of  $b = \ell$ . Then  $\mathcal{M} = bL_a^2$ . It is clear that the degree of  $b = \ell$ . In this direction, we needed not the condition such that  $(\text{supp}\mu) \cap D$  is a uniqueness set.

If  $a \notin D$ ,  $(z - a)L_a^2$  is dense in  $L_a^2$ . For, assuming  $a \geq 1$  and so  $a = 1$  without a loss of generality, if  $\varepsilon > 0$  then  $(z - 1)L_a^2 = (z - 1)\{z - (1 + \varepsilon)\}^{-1}L_a^2$ . For any  $f \in L_a^2$ , it is easy to see that

$$\int_D \left| \frac{z-1}{z-(1+\varepsilon)} f - f \right|^2 d\mu \rightarrow 0 \quad (\varepsilon \rightarrow 0).$$

This implies that  $(z - 1)L_a^2$  is dense in  $L_a^2$ . Thus all zeros of  $b$  must be in  $D$ . The 'if' part is clear because any point  $a \in D$  gives a bounded evaluation functional. Here we used the condition such that  $(\text{supp}\mu) \cap D$  is a uniqueness set (see [6, (1) of Theorem 8]).

**Proposition 2.** *Suppose that  $(z - a)^{-1}$  does not belong to  $L^2$  for each  $a \in D$ . If  $\mathcal{M}$  is an invariant subspace which contains  $L_a^2$  properly, then the codimension of  $L_a^2$  in  $\mathcal{M}$  is infinite.*

*Proof.* If  $\dim \mathcal{M} \ominus L_a^2 = \ell < \infty$ , by the proof of Proposition 1, there exists a polynomial  $b = \prod_{j=1}^{\ell} (z - a_j)$  such that  $b\mathcal{M} \subseteq L_a^2$  and  $a_j \in D$  ( $1 \leq j \leq \ell$ ). Hence there exists a function  $\phi$  in  $\mathcal{M}$  such that  $\phi \notin L_a^2$  and  $g = b\phi \in L_a^2$ . If  $g(a_k) \neq 0$  for some  $k$ , then  $g/(z - a_k) = \phi \prod_{j \neq k} (z - a_j)$  can not belong to  $L^2$  because  $(z - a_k)^{-1} \notin L^2$ . Hence  $g(a_j) = 0$  for any  $j$ . By [6, the proof in (1) of Theorem 8],  $g \in bL_a^2$  and so  $\phi = g/b$  belongs to  $L_a^2$ . This contradiction implies that  $\dim \mathcal{M} \ominus L_a^2 = \infty$ .

For an invariant subspace  $\mathcal{M}$ , set

$$\mathcal{M}_j = \{f_j \in \mathcal{L}^2; f \in \mathcal{M}, f(z) = \sum_{j=-\infty}^{\infty} f_j(r) e^{ij\theta}\}.$$

Then  $\mathcal{M}_j$  is a subspace in  $\mathcal{L}^2$ ,  $r\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$  and hence  $\dim \mathcal{M}_{j+1} \geq \dim \mathcal{M}_j$ . We call  $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$  the Fourier coefficients of  $\mathcal{M}$ .  $\mathcal{M}_j e^{ij\theta}$  may not belong to  $\mathcal{M}$ . If  $\mathcal{M}_j e^{ij\theta}$  belongs to  $\mathcal{M}$  for any  $j$ , then  $\mathcal{M}$  has the following decomposition :

$$\mathcal{M} = \sum_{j=-\infty}^{\infty} \oplus \mathcal{M}_j e^{ij\theta}.$$

This decomposition is called the Fourier decomposition of  $\mathcal{M}$ . In general,  $\mathcal{M}$  does not have the Fourier decomposition but we can get an extension  $\tilde{\mathcal{M}}$  of  $\mathcal{M}$  which has the Fourier decomposition :

$$\tilde{\mathcal{M}} = \sum_{j=-\infty}^{\infty} \oplus (\text{closure of } \mathcal{M}_j) e^{ij\theta}.$$



**Proposition 3.** *If  $\mathcal{M}$  is an invariant subspace which contains  $L_a^2$  and  $e^{i\theta}\mathcal{M} \subseteq \mathcal{M}$ , then  $\mathcal{M} = \chi_E \bar{q} \mathbf{H}^2 \oplus \chi_{E^c} L^2$  where  $\chi_E$  is a characteristic function in  $\mathcal{L}^2$  and  $q$  is a unimodular function in  $\mathbf{H}^2$ . Hence  $\mathcal{M} \supseteq \mathbf{H}^2$ . If  $\bigcap_{j=0}^{\infty} e^{ij\theta} \mathcal{M} = \{0\}$  then  $\mathcal{M} = \bar{q} \mathbf{H}^2$ .*

*Proof.* Suppose  $S_0 = \mathcal{M} \ominus e^{i\theta} \mathcal{M}$ , then  $\mathcal{M} = (\sum_{j=0}^{\infty} \oplus S_0 e^{ij\theta}) \oplus \mathcal{M}_{-\infty}$  where  $\mathcal{M}_{-\infty} = \bigcap_{j=0}^{\infty} e^{ij\theta} \mathcal{M}$ , and  $rS_0 \subset S_0$  because  $r\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$ . It is well known that  $\mathcal{M}_{-\infty} = \chi_G L^2$  for a characteristic function  $\chi_G$  of some measurable subset in  $D$ . Put  $E = G^c$  then there exists a function  $f$  in  $S_0$  such that

$$|f| > 0 \text{ on } E \text{ and } f = 0 \text{ on } F.$$

Since  $f$  is orthogonal to  $f e^{ij\theta}$  for all  $j \geq 0$ ,  $|f|^2$  belongs to  $\mathcal{L}^1 = L^1(d\sigma) = L^1([0, 1], d\sigma)$  and so  $|f|$  belongs to  $\mathcal{L}^2$ . Hence  $\chi_E$  belongs to  $\mathcal{L}^2$ . Set

$$F(re^{i\theta}) = \begin{cases} f(re^{i\theta})/|f(re^{i\theta})| & \text{if } f \neq 0 \\ 1 & \text{if } f = 0 \end{cases}$$

, then  $F$  is a unimodular function in  $L^2$ . Since  $rS_0 \subseteq S_0$ , we can show that  $\chi_E F$  belongs to  $S_0$  and so  $S_0 = \chi_E F \mathcal{L}^2$ . Hence  $\mathcal{M} \ominus \mathcal{M}_{-\infty} = \chi_E F \mathbf{H}^2$ . Since  $1 \in \mathcal{M}$ ,  $\chi_E \bar{F} \in \mathbf{H}^2$  and  $q = \bar{F}$  belongs to  $\mathbf{H}^2$ .

### Example

(I) For  $0 < \beta < 1$ , put

$$T_\beta = \overline{\text{span}}\{z^n \bar{z}^m ; \beta n \geq m \geq 0\}.$$

Then  $T_\beta$  is an invariant subspace and  $T_\beta \supseteq L_a^2$ . Put  $T_\beta = L_a^2$  for  $\beta = 0$  and  $T_\beta = \mathbf{H}^2$  for  $\beta = 1$ . In general,  $L_a^2 \subseteq T_\beta \subseteq \mathbf{H}^2$  and  $T_\beta (0 \leq \beta < 1)$  has the following Fourier decomposition :

$$T_\beta = \sum_{j=0}^{\infty} \oplus (T_\beta)_j e^{ij\theta}$$

where  $(T_\beta)_j = \text{span}\{r^j p_j(r^2) ; p_j \text{ is a polynomial of degree at most } \beta j / 1 - \beta\}$ . S.Janson and R.Rochberg [2] studied  $H_\phi^{\mathcal{M}}$  when  $\mathcal{M} = (T_\beta)^\perp$ . Then  $(T_\beta)^\perp = e^{i\theta} \mathbf{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{\mathcal{L}^2 \ominus (\bar{T}_\beta)_j\} e^{-ij\theta}$ .

(II) For  $k \geq 0$ , put

$\bar{E}^k = \overline{\text{span}}\{z^m \bar{z}^n ; m = 0, 1, \dots, k ; n = m, m + 1, \dots\}$ .  $\bar{E}^k$  is an invariant subspace and  $L_a^2 \subseteq \bar{E}^k \subseteq \mathbf{H}^2$ .  $\bar{E}^k$  has the following Fourier decomposition :

$$\bar{E}^k = \sum_{j=0}^{\infty} \oplus (\bar{E}^k)_j e^{ij\theta}$$

where  $(\bar{E}^k)_j = \text{span}\{r^j, \dots, r^{j+2k}\}$ . E.Strose [9] studied  $H_\phi^{\mathcal{M}}$  when  $\mathcal{M} = (E^k)^\perp$ . Then  $(E^k)^\perp = e^{i\theta} \mathbf{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{\mathcal{L}^2 \ominus (E^k)_j\} e^{-ij\theta}$ .

(III) Fix a polynomial  $p$  of degree  $k$ , that is,  $p = \sum_{j=0}^k a_j z^j$ . Put

$$Y(p) = \overline{\text{span}}\{z^n, z^m \bar{p}; n \geq 0, m \geq 0\}$$

and

$$Y^k = \overline{\text{span}}\{z^\ell \bar{z}^j; \ell \geq 0, 0 \leq j \leq k\}.$$

Both  $Y(p)$  and  $Y^k$  are invariant subspaces and  $L_a^2 \subseteq Y(p) \subseteq Y^k$ , and  $Y^k$  has the following Fourier decomposition :

$$Y^k = \sum_{j=-k}^{\infty} \oplus (Y^k)_j e^{ij\theta}$$

where  $Y_0^k = \text{span}\{1, r^2, \dots, r^{2k}\}$  and  $(Y^k)_j = r^j(Y_0^k)$  for  $j \geq 0$ , and  $(Y^k)_{-j} = \text{span}\{r^{2\ell-j}; j \leq \ell \leq k\}$  for  $1 \leq j \leq k$ .  $(Y(p))_j \subseteq (Y^k)_j$  for any  $j$  but  $Y(p)$  does not have a Fourier decomposition. If  $a_j \neq 0$  for  $1 \leq j \leq k$ ,  $(Y(p))_j = (Y^k)_j$  for any  $j$  and so  $\hat{Y}(p) = Y^k$ . L.Peng, R.Rochberg and Z.Wu [7] and J.L.-M.Wang and Z.Wu [10] studied  $H_\phi^{\mathcal{M}}$  when  $\mathcal{M} = (\bar{Y}^k)^\perp$ . In general, we can define  $Y(g)$  for any function  $g$  in  $L^2$ . Usually,  $Y(g)$  does not have the Fourier decomposition.

(IV) For a unimodular function  $q$  in  $\mathbf{H}^2$ , put  $\mathcal{M} = \bar{q}\mathbf{H}^2$ . Then  $\mathcal{M}$  is an invariant subspace which contains  $\mathbf{H}^2$ . In general,  $\bar{q}\mathbf{H}^2$  may not have the Fourier decomposition but for  $q = e^{i\ell\theta}$  for some  $\ell \geq 0$ .

$$\mathcal{M} = \sum_{j=-\ell}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}.$$

There are a lot of invariant subspaces between  $\mathbf{H}^2$  and  $e^{-i\ell\theta}\mathbf{H}^2$  even if  $\ell = 1$ .

(V) For arbitrary closed subspaces  $S$  in  $\mathcal{L}^2$ , put  $\mathcal{M} = \mathbf{H}^2 \oplus S e^{-i\theta}$ . Then  $\mathcal{M}$  is an invariant subspace between  $\mathbf{H}^2$  and  $e^{-i\theta}\mathbf{H}^2$ .

### §3. Kronecker's Theorem

In this section, the measure  $\mu$  is an arbitrary finite positive Borel measure such that  $L_a^2$  is closed. We will write

$$\mathcal{M}^\infty = \mathcal{M} \cap L^\infty$$

and for each positive integer  $\ell$

$$\mathcal{M}^{\infty, \ell} = \left\{ \phi \in L^\infty ; \phi(z) = g(z) \prod_{j=1}^{\ell} (z - a_j)^{-1} \quad \text{a.e. } \mu \text{ on } D, \right. \\ \left. g \in \mathcal{M}^\infty \text{ and } a_1, \dots, a_\ell \in D \right\}.$$

Then  $\mathcal{M}^\infty \subseteq \mathcal{M}^{\infty, 1} \subseteq \mathcal{M}^{\infty, 2} \subseteq \dots$ .

Kronecker (cf. [11, p210]) described finite rank Hankel operators on the Hardy space. Theorem 4 describes finite rank intermediate Hankel operators on the (weighted) Bergman space. However the situation is very different from that of Kronecker because  $\mathcal{M}^\infty = \mathcal{M}^{\infty, \ell}$  may happen for some  $\ell > 0$ . See Corollarys 1 and 2.

**Theorem 4.** *Suppose  $\mathcal{M}$  is an invariant subspace which contains  $zL_a^2$ , and  $\phi$  is a function in  $L^\infty$ .  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  if and only if  $\phi$  belongs to  $\mathcal{M}^{\infty, \ell}$ .*

*Proof.* Note that  $\ker H_\phi^{\mathcal{M}} = \{f \in L_a^2 ; \phi f \in \mathcal{M}\}$ . Since  $\mathcal{M}$  is an invariant subspace,  $\ker H_\phi^{\mathcal{M}}$  is also an invariant subspace. Proposition 1 implies the theorem.

**Theorem 5.** *Suppose  $\mathcal{M}$  is an invariant subspace which contains  $L_a^2$ , and  $\phi$  is a function in  $L^\infty$ . Then the following (1)  $\sim$  (4) are equivalent.*

(1) *If  $H_\phi^{\mathcal{M}}$  is of finite rank then  $H_\phi^{\mathcal{M}} = 0$ .*

(2)  *$\mathcal{M}^\infty = \mathcal{M}^{\infty, \ell}$  for any  $\ell > 0$*

(3) *If  $g \in \mathcal{M}^\infty$ ,  $a \in D$  and  $(g(z) - g(a))/(z - a) \in L^\infty$ , then  $(g(z) - g(a))/(z - a)$  belongs to  $\mathcal{M}^\infty$ .*

(4) *If  $\mathcal{M}'$  is an invariant subspace and  $(\mathcal{M}')^\infty \supsetneq \mathcal{M}^\infty$ , then there does not exist a nonzero polynomial  $b$  such that  $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$ .*

*Proof.* By Theorem 4, (1)  $\Leftrightarrow$  (2) is clear. (1)  $\Rightarrow$  (3). If there exists  $g \in \mathcal{M}^\infty$  such that  $(g - g(a))/(z - a) \in L^\infty$  does not belong to  $\mathcal{M}^\infty$ , put  $\phi = (g - g(a))/(z - a)$ , then  $H_\phi^{\mathcal{M}}$  is of rank 1 and  $H_\phi^{\mathcal{M}} \neq 0$ . (3)  $\Rightarrow$  (4). If (4) is not true, there exists  $\psi$  such that  $\psi \notin \mathcal{M}^\infty$ ,  $\psi \in (\mathcal{M}')^\infty$  and  $b\psi \in \mathcal{M}^\infty$  for some polynomial  $b = \prod_{j=1}^{\ell} (z - a_j)$  and  $a_j \in D$  ( $1 \leq j \leq \ell < \infty$ ). We may assume that  $\phi = \psi \prod_{j=1}^{\ell-1} (z - a_j) \notin \mathcal{M}^\infty$  and  $g = (z - a_\ell)\phi \in \mathcal{M}^\infty$ . Then

$$\frac{g - g(a_\ell)}{z - a_\ell} = \phi \in L^\infty \quad \text{but } \phi \notin \mathcal{M}^\infty.$$

(4)  $\Rightarrow$  (1). By Theorem 4, if  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$ , then  $\phi \in \mathcal{M}^{\infty, \ell}$ . If  $\phi \notin \mathcal{M}^\infty$ , suppose  $\mathcal{M}'$  is an invariant subspace generated by  $\phi$  and  $\mathcal{M}$ , then  $(\mathcal{M}')^\infty \supsetneq \mathcal{M}^\infty$  but there does not exist a nonzero polynomial  $b$  such that  $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$ . Since  $\phi \in \mathcal{M}'$ , this contradicts that  $\phi \in \mathcal{M}^{\infty, \ell}$ .

**Corollary 1.** *Suppose  $(\text{supp } \mu) \cap D$  is a uniqueness set for  $\mathcal{H}ol(D)$ . If  $H_\phi^{\text{big}}$  is of finite rank then  $H_\phi^{\text{big}} = 0$ .*

Proof. (3) of Theorem 5 implies the corollary. In fact, if  $g \in L_a^2 \cap L^\infty$  then  $g(z) - g(a) \in (z - a)L_a^2$  by [6, the proof in (1) of Theorem 8]. Thus  $(g(z) - g(a))/(z - a)$  belongs to  $L_a^2 \cap L^\infty$ .

**Corollary 2.** *Suppose  $d\mu = r dr d\theta/\pi$ . Let  $D_0$  be an open subset of  $D$  and  $\mathcal{M} = \{f \in L^2; f \text{ is analytic on } D_0\}$ . Then  $\mathcal{M}$  is an invariant subspace and if  $H_\phi^{\mathcal{M}}$  is of finite rank then  $H_\phi^{\mathcal{M}} = 0$ .*

Proof. It is easy to see that  $\mathcal{M}^\infty$  satisfies (3) of Theorem 5.

**Corollary 3.** *Suppose that if  $H_\phi^{\mathcal{M}}$  is of finite rank then  $H_\phi^{\mathcal{M}} = 0$ . If  $\mathcal{M}'$  is an invariant subspace which contains  $\mathcal{M}$  properly, then the codimension of  $\mathcal{M}$  in  $\mathcal{M}'$  is infinite or  $(\mathcal{M}')^\infty = \mathcal{M}^\infty$ .*

Proof. If  $\dim \mathcal{M}'/\mathcal{M} < \infty$ , as in the proof of Proposition 2, then there exists a nonzero polynomial  $b$  such that  $b\mathcal{M}' \subseteq \mathcal{M}$ . Hence  $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$ . If  $(\mathcal{M}')^\infty \neq \mathcal{M}^\infty$ , by Theorem 5, this contradicts that if  $H_\phi^{\mathcal{M}}$  is of finite rank then  $H_\phi^{\mathcal{M}} = 0$ .

#### §4. General case

In this section, we assume that  $d\mu = d\sigma(r)d\theta$  and  $d\sigma([t, 1]) > 0$  for any  $t > 0$ . Hence we can define the Fourier coefficients  $\{\mathcal{M}_j\}_{j=-\infty}^\infty$  of  $\mathcal{M}$ . We assume  $\mathcal{M} = \tilde{\mathcal{M}}$ , that is,  $\mathcal{M}$  has the Fourier decomposition.

**Theorem 6.** *Suppose  $\mathcal{M}$  is an invariant subspace which contains  $zL_a^2$  and  $\phi = \sum_{j=-\infty}^\infty \phi_j(r)e^{ij\theta}$  is a function in  $L^\infty$ . Then  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  if and only if there exist complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$  and for any integer  $n$*

$$\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r)$$

*belongs to  $\mathcal{M}_n$ . If  $\ell$  is the minimum number of complex numbers  $b_1, \dots, b_\ell$  such that  $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r)$  belongs to  $\mathcal{M}_n$  for all  $n$ , then  $H_\phi^{\mathcal{M}}$  is of rank  $\ell$ .*

Proof. If  $H_\phi^{\mathcal{M}}$  is of rank  $\leq \ell$ , by Theorem 4 there exists a polynomial  $b = \sum_{j=0}^{\ell} b_j z^j$  such that  $b\phi \in \mathcal{M}$ . Then

$$\left( \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta} \right) \left( \sum_{j=0}^{\ell} b_j r^j e^{ij\theta} \right) = \sum_{n=-\infty}^{\infty} \left( \sum_{j=0}^{\ell} \phi_{n-j}(r) b_j r^j \right) e^{in\theta}$$

and so  $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r)$  belongs to  $\mathcal{M}_n$  for any  $n$ . The converse and the second statement are clear by Theorem 4.

**Corollary 4.** Let  $\phi = \phi_t(r)e^{it\theta}$  for some integer  $t$  in Theorem 6. Then  $H_\phi^M$  is of finite rank  $\leq \ell$  if and only if there exist complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$  and for  $t \leq n \leq \ell + t$ ,  $b_{n-t}r^{n-t}\phi_t(r)$  belongs to  $\mathcal{M}_n$ .

Proof. Since  $\phi_j(r) = 0$  for  $j \neq t$ , if  $n < t$  or  $n > \ell + t$  then  $\sum_{j=0}^\ell b_j r^j \phi_{n-j}(r) = 0$ . For  $t \leq n \leq \ell + t$ ,  $\sum_{j=0}^\ell b_j r^j \phi_{n-j}(r) = b_{n-t} r^{n-t} \phi_t(r)$  and so the corollary follows.

**Corollary 5.** Let  $\phi = \sum_{j=1}^\infty a_j z^j + \sum_{j=0}^\infty a_{-j} \bar{z}^j$  in Theorem 6. Then  $H_\phi^M$  is of rank  $\leq \ell$  if and only if there exist complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$  and for any non-positive integer  $n$   $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n}$  belongs to  $\mathcal{M}_n$  and for  $0 < n < \ell$   $\sum_{j=n}^\ell b_j a_{n-j} r^{2j-n}$  belongs to  $\mathcal{M}_n$ .

Proof. If  $n \geq \ell$  and  $n \neq 0$ , then

$$\sum_{j=0}^\ell b_j r^j \phi_{n-j}(r) = \sum_{j=0}^\ell b_j a_{n-j} r^{j+n-j} = \left( \sum_{j=0}^\ell b_j a_{n-j} \right) r^n$$

and hence  $\sum_{j=0}^\ell b_j r^j \phi_{n-j}(r)$  belongs to  $\mathcal{M}_n$  because  $zL_a^2 \subseteq \mathcal{M}$ . Now Theorem 6 implies the corollary.

Theorem 6 does not give an exact relation between the rank of  $H_\phi^M$  and the number  $\ell$  of complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$ . However we can show the following : If  $H_\phi^M$  is of rank  $\ell$ , then there exist complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$ ,  $\sum_{j=0}^\ell b_j r^j \phi_{n-j}(r) \in \mathcal{M}_n$  for any  $n$  and  $b = \sum_{j=0}^\ell b_j z^j$  has just  $\ell$  zeros in  $D$ . That is, if  $\ell = 1$  then  $|b_0| < 1$ .

By Theorem 6,  $H_\phi^M = 0$  if and only if  $\phi_n \in \mathcal{M}_n$  for any  $n$  (that is,  $\phi \in \mathcal{M}$ ). Moreover,  $H_\phi^M$  is of rank  $\leq 1$  if and only if there exist complex numbers  $(b_0, b_1) \neq (0, 0)$  such that  $b_1 = 1$  and  $b_0 \phi_n + b_1 r \phi_{n-1} \in \mathcal{M}_n$  for any  $n$ .

## §5. Big Hankel operator and $\mathcal{M} \subseteq \mathbf{H}^2$

In this section, we assume that  $d\mu = d\sigma(r)d\theta$  and  $d\sigma([t, 1]) > 0$  for any  $t > 0$ . Hence we can define the Fourier coefficients  $\{\mathcal{M}_j\}_{j=-k}^\infty$  of  $\mathcal{M}$  and we assume  $\mathcal{M} = \tilde{\mathcal{M}}$ . In this case,  $H_\phi^M$  is close to  $H_\phi^{big}$ . Recall examples in Section 2, that is,  $T_\beta, \bar{E}^k, Y(p)$  and  $Y^k$ .

**Corollary 6.** Suppose  $\mathcal{M}$  is an invariant subspace between  $zL_a^2$  and  $\mathbf{H}^2$ , and  $\phi = \sum_{j=1}^\infty a_j z^j + \sum_{j=0}^\infty a_{-j} \bar{z}^j$ . Then  $H_\phi^M$  is of finite rank  $\leq \ell$  if and only if  $a_{-n} = 0$  for  $n > \ell$  and there exists complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$  and  $\sum_{j=n}^\ell b_j a_{n-j} r^{2j-n}$  belongs to  $\mathcal{M}_n$  for  $0 \leq n \leq \ell$  and  $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n} = 0$  for  $-\ell < n < 0$ .

Proof. Since  $\mathcal{M} \subseteq \mathbf{H}^2$ , by Corollary 5  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  if and only if there exists complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$  and  $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n} = 0$  for  $n < 0$  and  $\sum_{j=n}^\ell b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$  for  $0 \leq n \leq \ell$ . If  $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n} = 0$  for  $n < 0$ , then  $b_j a_{n-j} = 0$  for  $0 \leq j \leq \ell$  and  $n < 0$ . Hence for each  $j$  ( $0 \leq j \leq \ell$ ),  $b_j a_{-t} = 0$  if  $t > j$ . Thus  $a_{-t} = 0$  if  $t > \ell$ .

**Proposition 7.** Suppose  $\mathcal{M}$  is an invariant subspace between  $zL_a^2$  and  $e^{-ik\theta}\mathbf{H}^2$  where  $k \geq 0$ , and  $\phi = \sum_{j=0}^\infty \phi_{-j}(r)e^{-ij\theta}$  is a function in  $L^\infty$ . Then  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  if and only if

$$\phi(z) = \frac{\sum_{j=-k}^\ell \psi_j(r)e^{ij\theta}}{\sum_{j=0}^\ell b_j r^j e^{ij\theta}}$$

where  $\psi_n = \sum_{j=0}^\ell b_j r^j \phi_{n-j}$  belongs to  $\mathcal{M}_n$  for  $-k \leq n \leq \ell$ . and  $(b_0, \dots, b_\ell) \in \mathbf{C}^\ell$ .

Proof. Note that  $\mathcal{M} \subseteq e^{-ik\theta}\mathbf{H}^2$  and  $\phi_j(r) = 0$  for  $j > 0$ . If  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$ , then by Theorem 6

$$\left(\sum_{j=0}^\ell b_j r^j e^{ij\theta}\right) \left(\sum_{j=0}^\infty \phi_{-j}(r)e^{-ij\theta}\right) = \sum_{n=-k}^\ell \psi_n(r)e^{in\theta}$$

and  $\psi_n = \sum_{j=0}^\ell b_j r^j \phi_{n-j} \in \mathcal{M}_n$  for  $-k \leq n \leq \ell$ . The converse is also a result of Theorem 4.

**Corollary 7.** Suppose  $\mathcal{M}$  is an invariant subspace in Proposition 7. If  $\phi = \phi_+ + \phi_- = \sum_{j=1}^\infty a_j z^j + \sum_{j=0}^\infty a_{-j} \bar{z}^j$  and  $\phi_- \in L^\infty$ , then  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  if and only if

$$\phi(z) = \phi_+ + \frac{\sum_{j=-k}^\ell \psi_j(r)e^{ij\theta}}{\sum_{j=0}^\ell b_j r^j e^{ij\theta}}$$

where  $\psi_n = \sum_{j=0}^\ell b_j a_{n-j} r^{j+|n-j|}$  belongs to  $\mathcal{M}_n$  for  $-k \leq n \leq \ell$  and  $(b_0, \dots, b_\ell) \in \mathbf{C}^\ell$ . If  $(b_0, \dots, b_\ell) = (0, \dots, 0)$ , then  $\psi_n = 0$  and so  $\phi = \phi_+$ .

**Theorem 8.** Suppose  $\mathcal{M}$  is an invariant subspace between  $zL_a^2$  and  $e^{-ik\theta}\mathbf{H}^2$  where  $k \geq 0$ , and  $\phi = \sum_{j=1}^\infty \phi_{-j}(r)e^{ij\theta}$  is a function in  $L^\infty$ . Then,

(1) If  $\mathcal{M}_j \cap r^{j+1}\mathcal{L}^2 = \{0\}$  for any  $j \geq 0$ , then there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$ .

(2) If there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$ , then  $\mathcal{M}_{-(k-j)} \cap r^{j+1}\mathcal{L}^\infty = \{0\}$  for any  $j \geq 0$ .

Proof. (1) If  $H_\phi^{\mathcal{M}}$  is of finite rank  $\ell$ , by Proposition 7

$$\psi_n = \sum_{j=n}^\ell b_j r^j \phi_{n-j} \in \mathcal{M}_n$$

for  $0 \leq n \leq \ell$  because  $\phi_{n-j}(r) = 0$  for  $0 \leq j \leq n-1$ . We may assume  $b_\ell = 1$ . As  $n = \ell-1$ ,  $r^\ell \phi_{-1}(r) \in \mathcal{M}_{\ell-1}$ . Since  $\mathcal{M}_{\ell-1} \cap r^\ell \mathcal{L}^2 = \{0\}$ ,  $\phi_{-1}(r) = 0$ . As  $n = \ell-2$ ,

$$b_{\ell-1} r^{\ell-1} \phi_{-1}(r) + r^\ell \phi_{-2}(r) \in \mathcal{M}_{\ell-2}.$$

Since  $\mathcal{M}_{\ell-2} \cap r^{\ell-1}\mathcal{L}^2 = \{0\}$  and  $\phi_{-1}(r) = 0$ ,  $\phi_{-2}(r) = 0$ . we can get  $\phi_{-j}(r) = 0$  for  $j \leq \ell$ . In Proposition 7,  $\psi_n = 0$  for  $0 \leq n \leq \ell$  and so  $\phi \equiv 0$ .

(2) If  $r^{j+1}g \in \mathcal{M}_{-(k-j)} \cap r^{j+1}\mathcal{L}^\infty$ , then put  $\phi = ge^{-i(k+1)\theta}$ . If  $g \neq 0$  then  $\phi \notin \mathcal{M}$  and

$$z^{j+1}\phi = r^{j+1}g e^{-i(k-j)\theta} \in \mathcal{M}_{-(k-j)}e^{-i(k-j)\theta}.$$

Since  $\mathcal{M}$  has the Fourier decomposition,  $\mathcal{M}_j e^{ij\theta} \subseteq \mathcal{M}$  and so  $z^{j+1}\phi \in \mathcal{M}$ . Theorem 4 gives a contradiction.

We will apply results in this section to Example in Section 2.

### Example.

(I) Suppose  $\mathcal{M} = T_\beta$  ( $0 \leq \beta < 1$ ).

(1) When  $\phi = \sum_{j=1}^{\infty} \phi_{-j}(r)e^{-ij\theta}$  is a function in  $L^\infty$ , there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$  if and only if  $\beta = 0$ .

(2) When  $\phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$  is a function in  $L^\infty$ , there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$  if and only if  $\beta = 0$ .

Proof. Recall that  $T_\beta = \sum_{j=0}^{\infty} \oplus (T_\beta)_j e^{ij\theta}$  and  $(T_\beta)_j = \text{span}\{r^j p_j(r^2)\}$ ;  $p_j$  is a polynomial of degree at most  $\beta j / (1 - \beta)$ . (1) If  $\beta = 0$  then  $(T_\beta)_j \cap r^{j+1}\mathcal{L}^2 = \{0\}$  for any  $j \geq 0$  and if  $\beta \neq 0$  then  $(T_\beta)_j \cap r^{j+1}\mathcal{L}^\infty \neq \{0\}$  for enough large  $j$ . Theorem 8 implies (1). (2) If  $\beta \neq 0$ , then there exists  $n$  such that  $1 - \beta \leq \beta(n - 1)$ . Hence  $(T_\beta)_{n-1} \ni r^{n+1}$ . Suppose  $\phi = \bar{z}$  then  $z^n \phi = r^{n+1} e^{i(n-1)\theta}$  and so  $z^n \phi \in (T_\beta)_{n-1} e^{i(n-1)\theta} \subset T_\beta$ . By Theorem 4  $H_\phi^{\mathcal{M}}$  is of rank  $\leq n$  and  $H_\phi^{\mathcal{M}} \neq 0$ .

(II) Suppose  $\mathcal{M} = \bar{E}^m$  ( $0 \leq m < \infty$ ).

(1) When  $\phi = \sum_{j=1}^{\infty} \phi_{-j}(r)e^{-ij\theta}$ , there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$  if and only if  $m = 0$ .

(2) When  $\phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$  is a function in  $L^\infty$ , there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$  if and only if  $m = 0$  or 1.

Proof. Recall that  $(\bar{E}^m) = \sum_{j=0}^{\infty} \oplus (\bar{E}^m)_j e^{ij\theta}$  and  $(\bar{E}^m)_j = \text{span}\{r^j, \dots, r^{j+2m}\}$ . (1) If  $m = 0$  then  $(\bar{E}^m)_j \cap r^{j+1}\mathcal{L}^2 = \{0\}$  for any  $j \geq 0$  and if  $m \neq 0$  then  $(\bar{E}^m)_j \cap r^{j+1}\mathcal{L}^\infty \neq \{0\}$  for any  $j \geq 0$ . Theorem 8 implies (1). (2) If  $m = 0$ , by (1) there does not exist any finite rank  $H_\phi^{\mathcal{M}}$  except  $H_\phi^{\mathcal{M}} = 0$ . If  $m = 1$ , then  $(\bar{E}^m)_n = \text{span}\{r^n, r^{n+2}\}$  for  $n \geq 0$ . When  $H_\phi^{\mathcal{M}}$  is of finite rank  $\ell$ , by Corollary 6  $a_{-n} = 0$  for  $n > \ell$  and if  $0 \leq n \leq \ell$

$$\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} = cr^n + dr^{n+2}$$

for complex constants  $c, d$ . Hence for  $0 \leq n \leq \ell$

$$b_j a_{n-j} = 0 \quad \text{for } n+2 \leq j \leq \ell.$$

Since  $b_\ell = 1$ ,  $a_{n-\ell} = 0$  for  $0 \leq n \leq \ell$  and so  $a_{-j} = 0$  for  $0 \leq j \leq \ell$ . When  $m \geq 2$ , if  $\phi = \bar{z}$  then  $z\phi = r^2 \in (\bar{E}^m)_0 = \text{span}\{1, r^2, \dots, r^{2m}\}$  and  $z\phi \in \bar{E}^m$  because  $(\bar{E}^m)_0 \subset \bar{E}^m$ . However  $H_\phi^{\mathcal{M}} \neq 0$ .

(III) Suppose  $\mathcal{M} = Y^k$

(1) When  $\phi = \sum_{j=1}^{\infty} \phi_{-j}(r)e^{-ij\theta}$ , there does not exist any finite rank  $H_{\phi}^{\mathcal{M}}$  except  $H_{\phi}^{\mathcal{M}} = 0$  if and only if  $k = 0$ .

(2) When  $\phi = \phi_+ + \phi_- = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$  and  $\phi_+$  are functions in  $L^{\infty}$ , there does not exist any finite rank  $H_{\phi}^{\mathcal{M}}$  except  $H_{\phi}^{\mathcal{M}} = 0$  if and only if  $k = 0$ .

Proof. Since  $H_{\phi}^{\mathcal{M}} = H_{\phi_-}^{\mathcal{M}}$ , it is sufficient to prove (1). Recall that  $Y^k = \sum_{j=-k}^{\infty} (Y^k)_j e^{ij\theta}$  where  $Y_0^k = \text{span}\{1, r^2, \dots, r^{2k}\}$  and  $(Y^k)_j = r^j (Y^k)_0$  for  $j \geq 0$ , and  $(Y^k)_{-j} = \text{span}\{r^{2\ell-j}, j \leq \ell \leq k\}$  for  $1 \leq j \leq k$ . If  $k = 0$  then  $Y^k = L_a^2$ . If  $k \geq 1$ ,  $(Y^k)_{-k} = \text{span}\{r^k\}$ . (2) of Theorem 8 implies that there exists a nonzero finite rank  $H_{\phi}^{\mathcal{M}}$ .

## §6. Small Hankel operator and $\mathcal{M} \supseteq \mathbf{H}^2$

In this section, we assume that  $d\mu = d\sigma(r)d\theta$  and  $d\sigma([t, 1]) > 0$  for any  $t > 0$ . Hence we can define the Fourier coefficients  $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$  of  $\mathcal{M}$ . In this case,  $H_{\phi}^{\mathcal{M}}$  is close to  $H_{\phi}^{\text{small}}$  and far from  $H_{\phi}^{\text{big}}$ . Note that if  $\mathcal{M}'$  is an invariant subspace and  $\mathcal{M}' \subseteq e^{i\theta}\mathbf{H}^2$  then  $\mathcal{M} = (\bar{\mathcal{M}}')^{\perp}$  is an invariant subspace and  $\mathcal{M} \supseteq e^{i\theta}\mathbf{H}^2$ .

**Proposition 10.** *Suppose  $\mathcal{M}$  is an invariant subspace which contains  $e^{ik\theta}\mathbf{H}^2$  for some nonnegative integer  $k$ . If  $\mathcal{M} \neq L^2$ , there exists at least a nonzero finite rank  $H_{\phi}^{\mathcal{M}}$ .*

Proof. If  $\bar{z}^n \in \mathcal{M}$  for all  $n \geq 1$ , then  $z^{\ell}\bar{z}^n \in \mathcal{M}$  for all  $\ell \geq 1$  because  $z\mathcal{M} \subseteq \mathcal{M}$ . Let  $\mathcal{E}$  be the closed linear span of  $\{z^{\ell}\bar{z}^n; n \geq 1, \ell \geq 0\}$ , then  $\mathcal{E} \subseteq \mathcal{M}$  and  $g\mathcal{E} \subseteq \mathcal{E}$  for arbitrary polynomial  $g$  of  $z$  and  $\bar{z}$ . It is well known that  $\mathcal{E} = L^2$ . This contradiction implies that there exists at least  $n$  such that  $\bar{z}^n \notin \mathcal{M}$  and  $n \geq 1$ . If  $\phi = \bar{z}^n$  then  $z^{n+k}\phi \in \mathcal{M}$ . Then  $H_{\phi}^{\mathcal{M}} \neq 0$  but  $H_{\phi}^{\mathcal{M}}$  is of finite rank  $\leq n + k$  by Theorem 4.

**Proposition 11.** *Suppose  $\mathcal{M}$  is an invariant subspace which contains  $e^{ik\theta}\mathbf{H}^2$  for some nonnegative integer  $k$ . The following (1) ~ (3) are valid.*

(1) *If  $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$  is a function in  $L^{\infty}$ , then there exists a function  $\phi'$  in  $L^2$  such that  $\phi' = \sum_{j=0}^{k-1} \phi_j(r)e^{ij\theta} + \sum_{j=1}^{\infty} \phi_{-j}(r)e^{-ij\theta}$  and  $H_{\phi}^{\mathcal{M}} = H_{\phi'}^{\mathcal{M}}$ .*

(2) *If  $\phi = \sum_{j=k}^{\infty} \phi_j(r)e^{ij\theta}$  is a function in  $L^{\infty}$ , then  $H_{\phi}^{\mathcal{M}} = 0$ .*

(3) *If  $\phi = \sum_{j=-\ell}^{\infty} \phi_j(r)e^{ij\theta}$  is a function in  $L^{\infty}$ , then  $H_{\phi}^{\mathcal{M}}$  is of rank  $\leq \ell + k < \infty$ .*

*Conversely if one of (1) or (2) is valid, then  $\mathcal{M}$  contains  $e^{ik\theta}\mathbf{H}^2$ .*

Proof. Both (1) and (2) are clear because  $\mathcal{M} \supseteq e^{ik\theta}\mathbf{H}^2$ . (3) is a result of Theorem 4. The converse is also clear.

We will consider Example in Section 2.



**Example.**

(II) Suppose  $\mathcal{M} = (E^k)^\perp$  ( $0 \leq k < \infty$ ) and  $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$  is a function in  $L^\infty$ .

(1)  $H_\phi^{\mathcal{M}} = 0$  if and only if

$$\int_0^1 \phi_{-j}(r) r^{j+2t} d\sigma = 0 \quad (j \geq 0, 0 \leq t \leq k).$$

(2)  $H_\phi^{\mathcal{M}}$  is of rank  $\leq 1$  if and only if there exist complex numbers  $(b_0, b_1) \neq (0, 0)$  such that

$$b_0 \int_0^1 \phi_{-j}(r) r^{j+2t} d\sigma = -b_1 \int_0^1 \phi_{-j-1}(r) r^{j+2t+1} d\sigma$$

for  $j \geq 0, 0 \leq t \leq k$ .

(3) Suppose  $d\sigma = r dr / 2\pi$ . When  $\phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$ , if  $H_\phi^{\mathcal{M}}$  is of rank  $\leq 1$ , then  $H_\phi^{\mathcal{M}} = 0$ .

Proof. From the remark in the last part of Section 4, (1) and (2) follows. (3) By (2),  $H_\phi^{\mathcal{M}}$  is of rank  $\leq 1$  if and only if there exist complex numbers  $(b_0, b_1) \neq (0, 0)$  such that

$$b_0 a_{-j} \frac{1}{2j+2t+1} = -b_1 a_{-j-1} \frac{1}{2j+2t+3}$$

for  $j \geq 0, 0 \leq t \leq k$ . When  $k \neq 0$ , for each  $j$ , as  $t = 0$

$$b_0 a_{-j} \frac{1}{2j+1} = -b_1 a_{-j-1} \frac{1}{2j+3}$$

and

$$b_0 a_{-j} \frac{1}{2j+3} = -b_1 a_{-j-1} \frac{1}{2j+5}.$$

This implies that  $a_{-j} = a_{-j-1} = 0$  for  $j \geq 0$  and so  $\phi = \sum_{j=1}^{\infty} a_j z^j$ . When  $k = 0$ , Corollary 1 implies (3)

(IV) Suppose  $\mathcal{M} = \bar{q}\mathbf{H}^2$  for some unimodular function  $q$  in  $\mathbf{H}^2$  and  $\phi$  is a function in  $L^\infty$ .  $H_\phi^{\mathcal{M}}$  is of finite rank  $\ell$  if and only if

$$\phi = \bar{q} \sum_{j=-\ell}^{\infty} \psi_j(r) e^{ij\theta}$$

where  $\psi_{-\ell}(r) \neq 0$ .

Proof. If  $\phi = \bar{q} \sum_{j=-\ell}^{\infty} \psi_j(r) e^{ij\theta}$ , then  $z^\ell \phi$  belongs to  $\mathcal{M}$  and so by Theorem 4  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$ . Since  $\psi_{-\ell}(r) \neq 0$ ,  $b\phi \notin \mathcal{M}$  for any polynomial  $b$  of degree  $\leq \ell - 1$  and so  $H_\phi^{\mathcal{M}}$  is of finite rank  $\ell$ . The converse is clear.

(V) Suppose  $\mathcal{M} = \mathbf{H}^2 \oplus S e^{-i\theta}$  and  $S$  is a closed subspace in  $\mathcal{L}^2$ . Let  $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$  be a function in  $L^\infty$ . By Theorems 4 and 6,  $H_\phi^{\mathcal{M}}$  is of finite rank  $\leq \ell$  if

and only if  $\phi_j(r) = 0$  for  $j \leq -(\ell + 2)$  and there exist complex numbers  $b_0, \dots, b_\ell$  such that  $b_\ell = 1$ ,

$$\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0 \quad \text{for } -(\ell + 1) \leq n < -1$$

and

$$\sum_{j=0}^{\ell} b_j r^j \phi_{-1-j}(r) \in S.$$

### §7. Restricted shift operator and $\mathcal{M} \subseteq L_a^2$ .

In this section, we assume  $\mu = r dr d\theta / \pi$  for simplicity. Let  $\mathcal{M}$  be an invariant subspace in  $L_a^2$  and  $\mathcal{K} = L_a^2 \ominus \mathcal{M}$ . For  $\phi$  in  $L_a^\infty = L_a^2 \cap L^\infty$

$$S_\phi^\mathcal{K} f = (I - P^\mathcal{K})(\phi f) \quad (f \in \mathcal{K})$$

where  $P^\mathcal{K}$  is the orthogonal projection from  $L_a^2$  to  $\mathcal{K}$ .  $S_\phi^\mathcal{K}$  is called a restricted shift operator. For any  $\phi$  in  $L_a^\infty$   $S_\phi^\mathcal{K}$  commutes with  $S_z^\mathcal{K}$ . We don't know whether if the bounded linear operator  $T$  on  $\mathcal{K}$  commutes with  $S_z^\mathcal{K}$  then  $T = S_\phi^\mathcal{K}$  for some  $\phi$  in  $L_a^\infty$ . If  $T S_z^\mathcal{K} = S_z^\mathcal{K} T$  and  $\phi = T P^\mathcal{K} 1$  is bounded, then it is easy to see that  $T = S_\phi^\mathcal{K}$  (cf. [5, p784]). In the Hardy space instead of the Bergman space, D.Sarason [8] showed that this is true without any condition and  $\|T\| = \|\phi\|_\infty$ .

We can define the Hankel operator  $H_\phi^\mathcal{M}$  as in Introduction. However  $H_\phi^\mathcal{M}$  is not an intermediate Hankel operator. It is not so difficult to see the following : When  $\mathcal{K} = L_a^2 \ominus \mathcal{M}$  and  $\phi$  in  $L_a^\infty$ ,

$$\|H_\phi^\mathcal{M}\| = \|S_\phi^\mathcal{K}\|.$$

This is known for the Hardy space. In fact, for  $f$  in  $L_a^2$

$$H_\phi^\mathcal{M} f = (I - P^\mathcal{M})\phi f = P^\mathcal{K} \phi P^\mathcal{K} f$$

and so  $H_\phi^\mathcal{M} f = S_\phi^\mathcal{K} P^\mathcal{K} f$  for  $f$  in  $L_a^2$ . Hence  $H_\phi^\mathcal{M}$  is of finite rank  $n$  if and only if  $S_\phi^\mathcal{K}$  is of finite rank  $n$ . It is easy to see that  $S_\phi^\mathcal{K}$  is of finite rank  $\ell \leq n$  if and only if there exists an analytic polynomial  $p$  of degree  $\ell \leq n$  such that  $p(\phi)$  belongs to  $\mathcal{M}^\infty$ . When  $\phi$  is in  $L^\infty$ , Theorems 4 and 6 are true for  $H_\phi^\mathcal{M}$ .

Suppose  $\phi$  is a function in  $L_a^\infty$ .

- (1)  $L_a^2 \supseteq \ker H_\phi^\mathcal{M} \supseteq \mathcal{M}$ .
- (2) When the common zero set  $Z(\mathcal{M})$  of  $\mathcal{M}$  in  $D$  is empty, if  $H_\phi^\mathcal{M}$  is of finite rank then  $H_\phi^\mathcal{M} = 0$ . This is a result of (1) and Proposition 1.
- (3) If  $Z(\mathcal{M})$  is not empty, there exists a nonzero finite rank  $H_\phi^\mathcal{M}$ .

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