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Compact Toeplitz Operators On Bergman Spaces

by

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Abstract. Suppose T_ϕ is a Toeplitz operator on the Bergman space of the open unit disc. When the symbol ϕ is bounded and radial, it is known that T_ϕ is compact if and only if $\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 \phi(r)dr = 0$. In this paper, we study this type theorem for arbitrary bounded symbol ϕ.

§1. Introduction

Let $L_a^2(D)$ be the Bergman space on the open unit disk D. That is, $L_a^2(D)$ consists of analytic functions f in D with

$$\|f\|^2 = \int_D |f(z)|^2 dA(z) < \infty,$$

where dA is the normalized area measure on D. The Bergman projection, denoted P, is then the orthogonal projection from $L^2(D, dA)$ onto $L_a^2(D)$. For ϕ in $L^\infty(D)$ we consider the Toeplitz operator $T_\phi : L_a^2(D) \to L_a^2(D)$ defined by $T_\phi f = P(\phi f), f \in L_a^2(D)$. T_ϕ is clearly bounded. A natural and fundamental question is the following : When is the Toeplitz operator T_ϕ compact on $L_a^2(D)$? A complete answer to the question above is still lacking. Recently, B.Korenblum and K.Zhu [2, Theorem] proved the following which is important in this paper.

Theorem K.Z. Suppose ϕ is bounded and radial, that is , $\phi(re^{i\theta}) = \phi(r)$. Then, T_ϕ is compact if and only if

$$\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 \phi(r)dr = 0.$$

We will try to generalize Theorem K.Z. in two ways. For a function ϕ in $L^\infty(D)$, put

$$\phi_j(r) = \int_0^{2\pi} \phi(re^{i\theta})e^{-ij\theta}d\theta/2\pi$$

for $j = 0, \pm 1, \pm 2, \cdots$. Then each $\phi_j(r)$ belongs to $L^\infty[0,1]$. We call $\{\phi_j(r)\}$ the Fourier coefficients of ϕ. If ϕ is a function in $L^\infty(D)$, then

$$\phi(re^{i\theta}) = \textstyle\sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$$

in the $L^2(D)$-norm and so for $a.e.r$ in [0,1] in the $L^2(\partial D)$-norm. If ϕ is radial, then $\phi(r) = \phi_0(r)$. The following is the first natural question.

Question 1. *Suppose ϕ is a function in $L^\infty(D)$ and $\{\phi_j(r)\}$ are the Fourier coefficients of ϕ. Then, is the following true ? T_ϕ is compact if and only if*

$$\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 \phi_j(r)dr = 0 \quad (j = 0, \pm 1, \pm 2, \cdots).$$

For any function ϕ in $L^1(D)$, if the limit exists, put

$$\Phi(xe^{i\theta}) = \frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr.$$

and if the limit exists as $x \to 1$, put

$$\Phi(e^{i\theta}) = \lim_{x \to 1} \Phi(xe^{i\theta}).$$

If ϕ belongs to $C(\bar{D})$, then $\Phi = \phi$ on ∂D. If ϕ is radial, then Φ is constant when the limit exists. The following is the second question. Then, is the following true ?

Question 2. *Suppose ϕ is a function in $L^\infty(D)$. T_ϕ is compact if and only if*

$$\Phi(e^{i\theta}) = \lim_{x \to 1-} \frac{1}{1-x} \int_x^1 \phi(re^{i\theta}) dr = 0 \quad a.e.\theta.$$

It is easy to see that Question 2 is not true. We don't know whether Question 1 is true or not. In Section 2, we consider Question 1. In Sections 3,4 and 5, we consider Question 2 for some special symbols. As the result, we answer positively Question 1 in some natural cases. The first part of Theorem 1, and Theorems 2,3 are due to Professor Takahiko Nakazi, to whom I am indebted for permission for its use in this paper. The author expresses his sincere thanks to Professor T.Nakazi for his advice and encouragement.

In this paper, $C(X)$ denotes a set of all continuous functions on X and $L^\infty(X)$ denotes a set of all bounded Lebesgue measurable functions on X. For example, $X = [0,1], [0,1), D, \bar{D}$ and ∂D.

§2. Fourier coefficients of symbols

In this section we consider Conjecture 1. We show that the necessary condition is true in general and it is also sufficient under some conditions.

Theorem 1. *Let ϕ be a function in $L^\infty(D)$ and $\phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ where $\{\phi_j(r)\}$ are the Fourier coefficients of ϕ. If T_ϕ is compact, then*

$$\lim_{x \to 1-} \frac{1}{1-x} \int_x^1 \phi_j(r) dr = 0 \quad (j = 0, \pm 1, \pm 2, \cdots).$$

Conversely if $\{\phi_j(r)\}$ satisfy the condition above and one of the following (1) $\sim$ (6), then T_ϕ is compact.

(1) $\bigcup_{j=-\infty}^{\infty}$ supp $\phi_j \subseteq [0, 1-\varepsilon]$ for some $\varepsilon > 0$.

(2) $\sum_{j=-\infty}^{\infty} \|\phi_j\|_\infty < \infty$.

(3) $\phi_j(r) = \sum_{\ell=1}^{m} a_{j\ell} f_\ell(r)$ for $j = 0, \pm 1, \pm 2, \cdots$ and $m < \infty$, where $\sum_{j=-\infty}^{\infty} a_{j\ell} e^{ij\theta}$ is in $C(\partial D)$ and f_ℓ is in $L^\infty[0,1]$.

(4) $\phi_j(r) = \sum_{\ell=1}^{m} a_{j\ell} f_\ell(r)$ for $j = 0, \pm 1, \pm 2, \cdots$ and $m < \infty$, where $\sum_{j=-\infty}^{\infty} a_{j\ell} e^{ij\theta}$ is in $L^\infty(\partial D)$ and f_ℓ is in $C[0,1]$.

(5) For each $j, \phi_j(r)$ is in $C[0,1)$ and $|\phi_j(r)|$ is increasing.

(6) ϕ is real, $\lim_{r\to 1-} \phi_0(r) = \lim_{r\to 1-} \phi_0'(r) = 0$ and $\phi_0''(r)$ is bounded.

Proof. Suppose ϕ is in $L^\infty(D)$ and T_ϕ is compact. Let $\{e_n\}$ be a natural orthonormal basis for $L_a^2(D)$, that is, $e_n = \sqrt{n+1}\, r^n e^{in\theta}$ $(n = 0, 1, 2, \cdots)$. For each j, put

$$\Phi_j(re^{i\theta}) = r^{|j|} e^{-ij\theta} \phi(re^{i\theta}).$$

Then $T_{\Phi_j} = T_{r^{|j|}e^{-ij\theta}} T_\phi$ for $j \geq 0$ and $T_{\Phi_j} = T_\phi T_{r^{|j|}e^{-ij\theta}}$ for $j < 0$. Hence T_{Φ_j} is compact for any j. For each j, if $n \geq 0$, then

$$|\langle T_{\Phi_j} e_n, e_n\rangle| \leq \|T_{\Phi_j} e_n\|_2 \|e_n\|_2 = \|T_{\Phi_j} e_n\|_2.$$

Since T_{Φ_j} is compact for each j and $e_n \longrightarrow 0$ $(n \to \infty)$ weakly, $\|T_{\Phi_j} e_n\|_2 \longrightarrow 0$ $(n \to \infty)$ and so $\langle T_{\Phi_j} e_n, e_n\rangle \longrightarrow 0$ $(n \to \infty)$. For each j,

$$\begin{aligned}
\langle T_{\Phi_j} e_n, e_n\rangle &= \langle \phi r^{|j|} e^{-ij\theta} e_n, e_n\rangle \\
&= (n+1)\int_0^1 2r^{|j|+2n+1} dr \int_0^{2\pi} \phi(re^{i\theta}) e^{-ij\theta} d\theta/2\pi \\
&= 2(n+1)\int_0^1 \phi_j(r) r^{|j|+2n+1} dr \\
&= \frac{2(n+1)}{|j|+2n+2}\,(|j|+2n+2)\int_0^1 \phi_j(r) r^{|j|+2n+1} dr \\
&= \frac{n+1}{\frac{|j|}{2}+n+1}\left(\frac{|j|}{2}+n+1\right)\int_0^1 \phi_j(\sqrt{r}\,) r^{n+\frac{|j|}{2}} dr
\end{aligned}$$

Thus for each j,

$$\lim_{\ell\to\infty}\left(\frac{1}{2}+(\ell+1)\right)\int_0^1 \phi_j(\sqrt{r}\,) r^{\ell+\frac{1}{2}} dr = 0$$

or

$$\lim_{\ell\to\infty}(\ell+1)\int_0^1 \phi_j(\sqrt{r}\,) r^{\ell} dr = 0.$$

Hence

$$\lim_{n\to\infty}(n+1)\int_0^1 \phi_j(\sqrt{r}\,) r^{n} dr = 0$$

because ϕ_j is bounded (see[2, p357]). By Theorem 4 in [2], for each j

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi_j(r) dr = 0.$$

We will show the converse. (1) By hypothesis on $\{\phi_j(r)\}$, supp $\phi \subseteq [0,1-\varepsilon] \times \partial D$ for some $\varepsilon > 0$ and hence it is easy to see that T_ϕ is compact. (2) If $\sum_{j=-\infty}^{\infty} \|\phi_j\|_\infty < \infty$, then $ess \sup_{re^{i\theta}\in D} |\phi(re^{i\theta}) - \sum_{j=-n}^{n} \phi_j(r)e^{ij\theta}| \to 0$ as $n \to \infty$. Since $\lim \frac{1}{1-x}\int_x^1 \phi_j(r)dr = 0$ for each j, by Theorem K.Z. T_{ϕ_n} is compact where $\phi_n(re^{i\theta}) = \sum_{j=-n}^{n} \phi_j(r)e^{ij\theta}$, and so T_ϕ is compact. (5) By the continuity of ϕ_j, for each j

$$\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 \phi_j(r)dr = \phi_j(1) = 0.$$

Since $|\phi_j(r)|$ is increasing, $\phi_j(r) \equiv 0$ and so $\phi(re^{i\theta}) \equiv 0$ *a.e.* on D.

(3) Let $g_\ell(e^{i\theta}) = \sum_{j=-\infty}^{\infty} a_{j\ell}e^{ij\theta}$, then g_ℓ is in $C(\partial D)$ and $\phi(re^{i\theta}) = \sum_{\ell=1}^{m} f_\ell(r)g_\ell(e^{i\theta})$. Let $a_\ell = (\cdots, a_{-1\ell}, a_{0\ell}, a_{1\ell}, \cdots)$ for $1 \le \ell \le m$. If the rank of $\{a_1, \cdots, a_m\}$ is m, then by hypothesis on $\{\phi_j(r)\}$, for $\ell = 1, \cdots, m$

$$\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 f_\ell(r)dr = 0.$$

and so by Theorem K.Z. T_{f_ℓ} is compact. For each ℓ, there exist polynomials $g_{\ell n}(e^{i\theta})$ such that $\|g_\ell - g_{\ell n}\|_\infty \longrightarrow 0$ as $n \to \infty$. Let $\phi_n(re^{i\theta}) = \sum_{\ell=1}^{m} f_\ell(r)g_{\ell n}(e^{i\theta})$, then $\|\phi_n - \phi\|_\infty \to 0$ and T_{ϕ_n} is compact because T_{f_ℓ} is compact for $\ell = 1, \cdots, m$. Hence T_ϕ is compact. If the rank of $\{a_1, \cdots, a_m\}$ is 0, $\phi \equiv 0$ *a.e.* on D and so T_ϕ is compact. If $n =$ the rank of $\{a_1, \cdots, a_m\}$ and $1 \le n \lneq m$, then we can rewrite ϕ as the following : $\phi(re^{i\theta}) = \sum_{\ell=1}^{n} f'_\ell(r)g'_\ell(e^{i\theta})$ and the rank of $\{a'_1, \cdots, a'_n\}$ is n. Thus (3) follows. (4) In the proof of (3), the g_ℓ is in $L^\infty(\partial D)$ but may not be in $C(\partial D)$. Since f_ℓ is in $C[0,1]$, by the proof of (3) $f_\ell(1) = 0$. Hence there exist $f_{\ell n}$ in $C[0,1]$ such that *supp* $f_{\ell n} \subseteq [0, 1-\frac{1}{n}]$ and $\|f_\ell - f_{\ell n}\|_\infty \longrightarrow 0$ as $n \to \infty$. Let $\phi_n(re^{i\theta}) = \sum_{\ell=1}^{m} f_{\ell n}(r)g_\ell(e^{i\theta})$, then $\|\phi_n - \phi\|_\infty \to 0$ and T_{ϕ_n} is compact because *supp* ϕ_n is compact in D. Hence T_ϕ is compact.

(6) We will show that T_ϕ is in the trace class. Since T_ϕ is selfadjoint, it is enough to show that $\sum_{n=1}^{\infty} |\langle T_\phi e_n, e_n\rangle| < \infty$ where $e_n = \sqrt{n+1}\, r^n e^{in\theta}$ for $n = 0, 1, 2, \cdots$. For each $n \ge 0$,

$$\begin{aligned}
\langle T_\phi e_n, e_n\rangle &= (n+1)\int_0^1 2r^{2n+1}\phi_0(r)dr \\
&= 2(n+1)\left\{\left[\phi_0(r)\frac{r^{2n+2}}{2n+2}\right]_0^1 - \int_0^1 \frac{r^{2n+2}}{2n+2}\,\phi_0'(r)dr\right\} \\
&= -\int_0^1 r^{2n+2}\phi_0'(r)dr \\
&= \frac{1}{2n+3}\int_0^1 r^{2n+3}\phi_0''(r)dr.
\end{aligned}$$

Hence for some positive constant γ

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle T_\phi e_n e_n\rangle| &\le \sum_{n=1}^{\infty} \frac{1}{2n+3}\,\frac{1}{2n+4}\sup_r |\phi_0''(r)| \\
&\le \gamma \sum_{n=1}^{\infty} (n+1)^{-2}.
\end{aligned}$$

Let ϕ be a harmonic function in $L^\infty(D)$, then T_ϕ is compact if and only if $\phi \equiv 0$ *a.e.* on D. In fact, if ϕ is a harmonic function in $L^\infty(D)$, then $\phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}$ on D and so $\phi_j(r) = a_j r^{|j|}$ for $j = 0, \pm 1, \pm 2, \cdots$. $\{\phi_j(r)\}$ satisfy the conditions in (5) of Theorem 1. Let ϕ be a function in $L^\infty(D)$ with $\phi(re^{i\theta}) = \phi(e^{i\theta})$, then T_ϕ is compact if and only if $\phi \equiv 0$ *a.e.* on D. This is also a result of (5) of Theorem 1. Later we will show two more general results which contain (3) and (4) of Theorem 1, respectively.

§3. r-continuous symbols

In this section we consider Question 2 under a condition that $\phi(re^{i\theta})$ is r-continuous. Theorem 2 contains the known result about symbols in $C(\bar{D})$. By the proof of Theorem 2, Question 1 is true under the same condition on Theorem 2. If $\phi(re^{i\theta})$ is r-continuous, $\Phi(e^{i\theta}) = \phi(e^{i\theta})$ *a.e.*θ.

Theorem 2. *Suppose ϕ is a function in $L^\infty(D)$ and $\phi(re^{i\theta})$ is r-continuous on [0,1] with respect to ess $\sup_{0 \le \theta < 2\pi} |\phi(re^{i\theta})|$. T_ϕ is compact if and only if*

$$\Phi(e^{i\theta}) = \phi(e^{i\theta}) = 0 \quad a.e.\theta.$$

Proof. For any r, t in [0,1]

$$\begin{aligned} |\phi_j(r) - \phi_j(t)| &\le \int_0^{2\pi} |\phi(re^{i\theta}) - \phi(te^{i\theta})| d\theta/2\pi \\ &\le ess \sup_\theta |\phi(re^{i\theta}) - \phi(te^{i\theta})| \end{aligned}$$

and so $\phi_j(r)$ is continuous because $\phi(re^{i\theta})$ is r-continuous with respect to $ess \sup_\theta |\phi(re^{i\theta})|$. Therefore

$$\phi_j(1) = \lim_{x \to 1-} \frac{1}{1-x} \int_x^1 \phi_j(r) dr \quad (j = 0, \pm 1, \pm 2, \cdots)$$

By Theorem 1, if T_ϕ is compact, then $\phi_j(1) = 0$ for all j and so $\phi(e^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(1) e^{ij\theta} = 0$ *a.e.*θ. Hence by hypothesis, $\lim_{r \to 1} ess \sup_\theta |\phi(re^{i\theta})| = 0$. Since

$$\left| \frac{1}{1-x} \int_x^1 \phi(re^{i\theta}) dr \right| \le \sup_{x \le r \le 1} ess \sup_\theta |\phi(re^{i\theta})|$$

$\Phi(e^{i\theta}) = 0$ *a.e.*θ. Conversely if $\Phi(e^{i\theta}) = \phi(e^{i\theta}) = 0$ *a.e.*θ, by hypothesis $\lim_{r \to 1} ess \sup_\theta |\phi(re^{i\theta})| = 0$. Hence there exist $\{\phi_n\}$ in $L^\infty(D)$ such that the support of ϕ_n is compact in D and $\|\phi_n - \phi\|_\infty \longrightarrow 0$ as $n \to \infty$.

Corollary 1. *Suppose* $\phi(re^{i\theta}) = \sum_{t=1}^{\ell} f_t(r)g_t(e^{i\theta})$ *and* $1 \leq \ell < \infty$, *where* $g_1, \cdots, g_\ell$ *are functions in* $L^\infty(\partial D)$ *and* $f_1, \cdots, f_\ell$ *are functions in* $C[0,1]$. *Then* T_ϕ *is compact if and only if* $\Phi(e^{i\theta}) = \phi(e^{i\theta}) = 0$ *a.e.*θ.

Proof. It is easy to see that $\phi(re^{i\theta})$ is r-continuous with respect to $ess \sup_{0\leq\theta<2\pi} |\phi(re^{i\theta})|$ and so Theorem 2 implies the corollary.

§4. θ-continuous symbols

In this section we consider Question2 under a condition that $\phi(re^{i\theta})$ is θ-continuous. Theorem 3 also contains the known result about symbols in $C(\bar{D})$. By the proof of Theorem 3, Question 1 is true under the same condition on Theorem 3. $\Phi(xe^{i\theta})$ is x-continuous on [0,1) and so if $\phi(re^{i\theta})$ is θ-continuous, then Φ is continuous on D. In this section, Φ is the main tool. $\Phi(e^{i\theta}) \not\equiv \phi(e^{i\theta})$ may happen.

Theorem 3. *Suppose* ϕ *is a function in* $L^\infty(D)$ *and* $\phi(re^{i\theta})$ *is* θ*-continuous with respect to* $ess \sup_{0\leq r<1} |\phi(re^{i\theta})|$. T_ϕ *is compact if and only if* $\Phi(e^{i\theta}) = 0$ *a.e.*θ.

Proof. Let $\{\phi_j(r)\}$ be the Fourier coefficients of ϕ. Set

$$s_n(re^{i\theta}) = \sum_{j=-n}^{n} \phi_j(r)e^{ij\theta}$$

and

$$\sigma_n(re^{i\theta}) = \frac{1}{n} \sum_{\ell=0}^{n-1} s_\ell(re^{i\theta}).$$

When $\phi(re^{i\theta})$ is θ-continuous, it is known ($cf.$[1,p18]) that for each r

$$ess \sup_{0\leq\theta<2\pi} |\phi(re^{i\theta}) - \sigma_n(re^{i\theta})| \longrightarrow 0$$

as $n \to \infty$. If $\phi(re^{i\theta})$ is θ-continuous with respect to $ess \sup_{0\leq r\leq 1} |\phi(re^{i\theta})|$, we can show that $\|\phi - \sigma_n\|_\infty \longrightarrow 0$ as $n \to \infty$ by the proof of [1, p18]. If T_ϕ is compact, then by Theorem 1

$$\lim_{x\to 1-} \frac{1}{1-x} \int_x^1 \phi_j(r)dr = 0 \quad (j = 0, \pm1, \pm2, \cdots)$$

and so for each n

$$\lim_{x\to 1-} \frac{1}{1-x} \int_x^1 \sigma_n(re^{i\theta})dr = 0 \ \ a.e.\theta.$$

Hence

$$\left|\frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr - \frac{1}{1-x}\int_x^1 \sigma_n(re^{i\theta})dr\right|$$
$$\leq \frac{1}{1-x}\int_x^1 |\phi(re^{i\theta}) - \sigma_n(re^{i\theta})|dr \leq \|\phi - \sigma_n\|_\infty$$

and so $\Phi(e^{i\theta}) = 0 \;\; a.e.\theta$. Conversely suppose $\Phi(e^{i\theta}) = 0 \;\; a.e.\theta$. Put

$$\Phi(xe^{i\theta}) = \frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr,$$

then $|\Phi(xe^{i\theta})| \leq \|\Phi\|_\infty \leq \|\phi\|_\infty \;\; a.e.$ on D and $\lim_{x\to 1-} \Phi(xe^{i\theta}) = 0 \;\; a.e.\theta$. Hence by the Lebesgue dominated convergence theorem

$$\lim_{x\to 1-}\int_0^{2\pi} |\Phi(xe^{i\theta})|d\theta/2\pi = 0.$$

Since for $j = 0, \pm 1, \pm 2, \cdots$

$$\int_0^{2\pi}\left(\frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr\right)e^{-ij\theta}d\theta/2\pi = \frac{1}{1-x}\int_x^1 \phi_j(r)dr,$$

$$\left|\frac{1}{1-x}\int_x^1 \phi_j(r)dr\right| \leq \int_0^{2\pi} |\Phi(xe^{i\theta})|d\theta/2\pi$$

and so

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi_j(r)dr = 0.$$

Then it is easy to see that for the Fourier series $\{\sigma_{nj}(r)\}$ of σ_n,

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \sigma_{nj}(r)dr = 0 \quad (j = 0, \pm 1, \pm 2, \cdots)$$

and so T_{σ_n} is compact. Thus T_ϕ is compact because $\|\phi - \sigma_n\|_\infty \longrightarrow 0$ as $n \to \infty$.

Corollary 2. *Let ϕ be a function in $L^\infty(D)$ and $\{\phi_j(r)\}$ the Fourier coefficients of ϕ. Suppose $\sum_{j=-\infty}^{\infty} \|\phi_j\|_\infty < \infty$, then T_ϕ is compact if and only if $\Phi(e^{i\theta}) = 0 \;\; a.e.\theta$.*

Corollary 3. *Suppose $\phi(re^{i\theta}) = \sum_{t=1}^{\ell} f_t(r)g_t(e^{i\theta})$ and $1 \leq \ell < \infty$, where $g_1, \cdots, g_\ell$ are functions in $C(\partial D)$ and $f_1, \cdots, f_\ell$ are functions in $L^\infty[0,1]$. Then T_ϕ is compact if and only if $\Phi(e^{i\theta}) = 0 \;\; a.e.\theta$.*

Proof. It is easy to see that $\phi(re^{i\theta})$ is θ-continuous with respect to *ess* $\sup_{0\leq r<1} |\phi(re^{i\theta})|$. Theorem 3 implies the corollary.

Corollary 4. *Suppose $0 \leq a_\ell < b_\ell < 1$, $a_{\ell+1} \geq b_\ell$, $E_\ell = \{r \in [0,1] \; ; \; 0 \leq a_\ell \leq r < b_\ell \leq 1\}$, $\sup_\ell \|g_\ell\|_\infty < \infty$ and*

$$\phi(re^{i\theta}) = \sum_{\ell=1}^{\infty} g_\ell(e^{i\theta})\chi_{E_\ell}(r).$$

If $\sum_{j=-\infty}^{\infty}(\sup_{\ell}|\hat{g}_\ell(j)|) < \infty$, *then* T_ϕ *is compact.*

Proof. It is a corollary of Corollary 2.

§5. Nonnegative symbols.

When ϕ is a nonnegative function in $L^1(D)$, we know a necessary and sufficient condition for a compact T_ϕ (*see* [3, Corollary 6.2.7]). In this section, we study Question 2 for nonnegative symbols. For nonnegative symbols, the condition of Question 1 is stronger than that of Question 2. In fact, if ϕ is nonnegative, by the proof of Theorem 5

$$\underline{\lim}_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr = 0 \quad a.e.\theta$$

if and only if

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi_0(r)dr = 0.$$

In the previous section, we used Theorem K.Z. in Introduction. Hence we had to assume that the symbol is bounded. In this section, not assuming the boundedness we show Theorem K.Z. when the symbol is nonnegative. It is easy to give an example of symbol ϕ such that T_ϕ is not compact but

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr = 0 \quad a.e.\theta.$$

In fact, let F be a triangle with vertices $e^{i\alpha}, e^{i\beta}$ and $e^{i\gamma}$, and ϕ the characteristic function of F. Then it is easy to see that T_ϕ is not compact and

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr = 0 \quad \text{if } \theta \neq \alpha,\beta,\gamma.$$

Proposition 4. *Suppose ϕ is a nonnegative function in $L^1(D)$ and radial. Then, T_ϕ is compact if and only if*

$$\lim_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi(r)dr = 0.$$

Proof. When ϕ is a nonnegative function, T_ϕ is compact if and only if $\lim_{|z|\to 1-}\hat{\phi}(z) = 0$ where

$$\hat{\phi}(z) = \frac{1}{|S_z|}\int_{S_z}\phi(w)dA(w), \ z \in D.$$

Here $S_z = \{w \in D \; ; \; |z| < |w| < 1, |\arg z - \arg w| < 2\pi(1-|z|)\}$ and $|S_z|$ is the dA - measure of S_z(*see* [1] about this result). Since ϕ is a radial function,

$$\begin{aligned}\hat{\phi}(z) &= \frac{1}{2\pi(1-|z|)^2(1+|z|)} \int_{\arg z-2\pi(1-|z|)}^{\arg z+2\pi(1-|z|)} \int_{|z|}^{1} \phi(re^{i\theta})\frac{r}{\pi} dr d\theta \\ &= \frac{2}{\pi(1-|z|)(1+|z|)} \int_{|z|}^{1} \phi(r) r dr.\end{aligned}$$

Hence $\hat{\phi}(z) \longrightarrow 0$ as $|z| \longrightarrow 1-$ if and only if

$$\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 \phi(r)dr = 0.$$

This implies the proposition.

Theorem 5. *Suppose ϕ is a nonnegative function in $L^1(D)$.*
(1) If T_ϕ is compact and ϕ is bounded, then $\underline{\lim}_{x\to 1-}\Phi(xe^{i\theta}) = 0 \quad a.e.\theta$.
(2) If $\lim\limits_{x\to 1-} ess \sup\limits_{\theta} \Phi(xe^{i\theta}) = 0$, then T_ϕ is compact.

Proof. (1) Since

$$\begin{aligned}&\int_0^{2\pi} d\theta/2\pi \, \frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr \\ &= \frac{1}{1-x}\int_x^1 dr \int_0^{2\pi} \phi(re^{i\theta})d\theta/2\pi = \frac{1}{1-x}\int_x^1 \phi_0(r)dr,\end{aligned}$$

if T_ϕ is compact, then by Theorem 1

$$\underline{\lim}_{x\to 1-} \int_0^{2\pi} d\theta/2\pi \, \frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr = 0.$$

Since $\dfrac{1}{1-x}\displaystyle\int_x^1 \phi(re^{i\theta})dr \geq 0 \quad a.e.\theta$,

$$\underline{\lim}_{x\to 1-}\frac{1}{1-x}\int_x^1 \phi(re^{i\theta})dr = 0 \quad a.e.\theta.$$

(2) Since

$$\begin{aligned}\hat{\phi}(z) &= \frac{1}{2\pi(1-|z|)^2(1+|z|)} \int_{\arg z-2\pi(1-|z|)}^{\arg z+2\pi(1-|z|)} \int_{|z|}^{1} \phi(re^{i\theta})\frac{r}{\pi} dr d\theta \\ &\leq \frac{2}{\pi(1+|z|)} ess \sup_\theta \left(\frac{1}{1-|z|}\int_{|z|}^1 \phi(re^{i\theta})dr\right),\end{aligned}$$

if $\lim\limits_{x\to 1-} ess \sup\limits_{\theta} \Phi(xe^{i\theta}) = 0$, then $\hat{\phi}(z) \to 0$ $(|z| \to 1-)$. By [1], T_ϕ is compact.

§6. Remarks

We don't know whether Question 1 is true or not. It is easy to give a counter example for Question 2 (*see* §5). But we could not give an example of symbol ϕ such that T_ϕ is compact but $\Phi(e^{i\theta}) \not\equiv 0$. The symbol of the triangle example in §5 is not r-continuous or θ-continuous. However we can construct such a r-continuous nonegative symbol using the triangle F in §5. Is Question 2 true when $\phi(re^{i\theta})$ is a θ-continuous function in $L^\infty(D)$ with respect to each $r \in$ [0,1] ? Unfortunately we could not answer it. Moreover if $\lim_{r\to 1-} \phi(re^{i\theta}) = \phi(e^{i\theta})$, then $\Phi(xe^{i\theta})$ is continuous on $\bar{D}$. Hence if $\lim_{x\to 1-} \Phi(xe^{i\theta}) = 0$, then T_Φ is compact. Under some more condition, we can show that T_ϕ is compact.

Proposition 6. *Suppose ϕ is a θ - continuous function in $L^\infty(D)$, $\lim_{r\to 1-} \phi(re^{i\theta}) = \phi(e^{i\theta})$ and*

$$\lim_{x\to 1-} ess \sup_{0\le\theta<2\pi} \frac{1}{1-x}\int_x^1 |\phi(xe^{i\theta}) - \phi(re^{i\theta})|dr = 0.$$

Then T_ϕ is compact if and only if $\Phi(e^{i\theta}) = 0$ a.e.θ.

Proof. It is clear that $\Phi(xe^{i\theta})$ is x -continuous in [0,1). By hypothesis on ϕ,

$$\phi(e^{i\theta}) = \lim_{x\to 1-} \phi(xe^{i\theta}) = \lim_{x\to 1-} \Phi(xe^{i\theta}) = \Phi(e^{i\theta})$$

because

$$|\Phi(xe^{i\theta}) - \Phi(e^{i\theta})| \le \frac{1}{1-x}\int_x^1 |\phi(re^{i\theta}) - \phi(e^{i\theta})|dr.$$

$\Phi(xe^{i\theta})$ is x - continuous at $x = 1$. Since $\phi(e^{i\theta})$ is θ - continuous, $\Phi(xe^{i\theta})$ is θ - continuous and so Φ belongs to $C(\bar{D})$. Between the Fourier coefficients of ϕ and Φ, we have the following relation : for $j = 0, \pm 1, \pm 2, \cdots$

$$\frac{1}{1-x}\int_x^1 \phi_j(r)dr = \int_0^{2\pi} \Phi(xe^{i\theta})\bar{e}^{ij\theta}d\theta/2\pi = \Phi_j(x).$$

If T_ϕ is compact, then by Theorem 1 $\Phi_j(1) = 0$ $(j = 0, \pm 1, \pm 2, \cdots)$ and so $\Phi(e^{i\theta}) = 0$ on ∂D because of continuity of Φ and the relation above. Conversely if $\Phi(e^{i\theta}) = 0$ on ∂D, by Corollary 3 T_Φ is compact because Φ is in $C(\bar{D})$. For any $\varepsilon > 0$, there exists $0 \le t < 1$ such that

$$ess \sup_{|z|\ge t} |\phi(z) - \Phi(z)| < \varepsilon$$

because $\lim_{x\to 1-} ess \sup_\theta |\phi(xe^{i\theta}) - \Phi(xe^{i\theta})| = 0$ by hypothesis on ϕ. Put

$$\phi_t(z) = \begin{cases} \phi(z) & 0 \le |z| < t \\ 0 & t \le |z| < 1 \end{cases}$$

and

$$\Phi_t(z) = \begin{cases} \Phi(z) & 0 \le |z| < t \\ 0 & t \le |z| < 1, \end{cases}$$

then

$$
\begin{aligned}
&\|\phi - \phi_t - \Phi + \Phi_t\|_\infty \\
&= ess \sup_{t \le |z| < 1} |\phi(z) - \Phi(z)| < \varepsilon.
\end{aligned}
$$

Since the Toeplitz operator of the symbol $\phi_t + \Phi - \Phi_t$ is compact, T_ϕ is compact.

Example. Let $E_\ell = [a_\ell, b_\ell], 0 \le a_\ell < a_{\ell+1} = b_\ell < b_{\ell+1} < 1, \bigcup_{\ell=0}^{\infty} E_\ell = [0,1]$ and $d_\ell \in \mathrm{C}$. Suppose

$$\phi(re^{i\theta}) = \textstyle\sum_{\ell=0}^{\infty} e^{i\ell\theta} d_\ell \chi_{E_\ell}(r).$$

Then

$$\phi_j(r) = \begin{cases} d_j \chi_{E_j}(r) & j \ge 0 \\ 0 & j < 0, \end{cases}$$

supp $\phi_j \subseteq [a_j, b_j]$ for $j \ge 0$, $\bigcup_{j=-\infty}^{\infty}$ supp $\phi_j = [0,1]$ and for $j = 0, \pm 1, \pm 2, \cdots$

$$\lim_{x \to 1-} \frac{1}{1-x} \int_x^1 \phi_j(r)dr = 0.$$

and for $a_\ell \le x \le b_\ell$

$$\Phi(xe^{i\theta}) = \frac{1}{1-x}\left\{\sum_{j=\ell+1}^{\infty} e^{ij\theta} d_j(b_j - a_j) + e^{i\ell\theta} d_\ell(b_\ell - x)\right\}$$

Moreover $\phi(re^{i\theta})$ is θ-continuous with respect to each r but not r-continuous. If $|d_\ell| = 1$ for $\ell = 0, 1, 2, \cdots$, then $|\phi(re^{i\theta})| = 1$ on D and $\lim_{x \to 1-} \phi(re^{i\theta})$ does not exist. We don't know whether T_ϕ is compact or not. If $|d_\ell| \longrightarrow 0$ as $\ell \longrightarrow \infty$, then $\phi(re^{i\theta})$ is θ-continuous with respect to $ess \sup_{0 \le r \le 1} |\phi(re^{i\theta})|$ and so by Theorem 2 T_ϕ is compact.

In Theorems 2 and 3, if $\lim_{x \to 1-} \Phi(xe^{i\theta}) = 0$ *a.e.*θ then $\lim_{x \to 1-} ess \sup_\theta |\Phi(xe^{i\theta})| = 0$. However in Theorem 5, $\lim_{x \to 1-} \Phi(xe^{i\theta}) = 0$ *a.e.*θ does not imply $\lim_{x \to 1-} ess \sup_\theta |\Phi(xe^{i\theta})| = 0$. It is easy to see that if $\lim_{r \to 1-} ess \sup_\theta |\phi(re^{i\theta})| = 0$, then T_ϕ is compact. When $\lim_{x \to 1-} ess \sup_\theta |\Phi(xe^{i\theta})| = 0$, we don't know whether T_ϕ is compact or not.

References

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