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On the asymptotic behavior of the prediction error of a stationary process

Akihiko INOUE (Sapporo) and Yukio KASAHARA (Sapporo)

Abstract. We present an example of stationary process with long-time memory for which we can calculate explicitly the prediction error from the finite part of past. The long-time behavior of the prediction error is discussed.

1. Introduction and results

Let $X = (X(t), t \in \mathbb{R})$ be a real, centered, weakly stationary process defined on a probability space (Ω, F, P). For $T \geq 0$, we denote by $P_{[-T,0]}$ the orthogonal projection operator of $L^2(\Omega, F, P)$ to the subspace spanned by $\{X(u) : -T \leq u \leq 0\}$. Similarly, we write $P_{(-\infty,0]}$ for the orthogonal projection operator to the subspace spanned by $\{X(u) : -\infty < u \leq 0\}$. For $T > 0$ and $t > 0$, we define $Q(T,t)$ and $Q(\infty,t)$ by

$$Q(T,t) := E[\{X(t) - P_{[-2T,0]}X(t)\}^2],$$
$$Q(\infty,t) := E[\{X(t) - P_{(-\infty,0]}X(t)\}^2].$$

The reason why we put $2T$ rather than T in the above is that we follow the notation of Dym and McKean [3].

We are concerned with the asymptotic behavior of $Q(T,t) - Q(\infty,t)$ as $T \to \infty$. We are especially interested in the case where the stationary process X has the so called "long-time memory". The main difficulty of this problem comes from that of the calculation of $Q(T,t)$. In this paper, we present an example of X with long-time memory for which we can calculate $Q(T,t)$ explicitly. The authors know no other example of such X.

We write R for the autocovariance function of X: $R(t) = E[X(t)X(0)]$ for $t \in \mathbb{R}$. Let μ be the spectral measure of X: $R(t) = \int_{-\infty}^{\infty} e^{i|t|\gamma}\mu(d\gamma)$. For $\frac{1}{2} < \alpha < 1$, we define the constants $c = c(\alpha)$ and $d = d(\alpha)$ by

$$c := \left\{\pi\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right\}^{\frac{1}{2\alpha}}, \qquad d := \frac{2^{3-2\alpha}\sin(\alpha\pi)}{\Gamma(\alpha-\frac{1}{2})^2}. \tag{1.1}$$

As usual, we write K_ν for the modified Bessel function (cf. Watson [5, 3.7]).

Theorem 1. *Let $T > 0$, $t > 0$ and $\frac{1}{2} < \alpha < 1$. Set $T_1 := T + t$. Let X be a a real, centered, weakly stationary process with spectral measure μ on $\mathbb{R}$ of the form*

$$\mu(d\gamma) = \frac{\sin(\alpha\pi)}{\pi} \cdot \frac{|\gamma|^{1-2\alpha}}{1+\gamma^2} d\gamma. \tag{1.2}$$

Then $Q(T,t) = d\{Q_1(T,t) + Q_2(T,t)\}$ with

$$Q_1(T,t) := \int_T^{T_1} \left\{ \int_s^{T_1} u^{1-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}} K_{-\alpha}(u) du \right\}^2 \frac{T_1^2}{sK_{-\alpha}(s)^2} ds,$$
$$Q_2(T,t) := \int_T^{T_1} \left\{ \int_s^{T_1} u^{2-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}} K_{1-\alpha}(u) du \right\}^2 \frac{ds}{sK_{1-\alpha}(s)^2}.$$

For the stationary process X in the theorem above, we can show that

$$R(t) \sim \frac{1}{\Gamma(2\alpha-1)} t^{-(2-2\alpha)} \quad (t \to \infty) \tag{1.3}$$

(see §4). In other words, X has long-time memory (see Beran [1]).

Theorem 2. *Let α, t and X be as in Theorem 1. Then*

$$Q(T,t) - Q(\infty,t) \sim \sin(\alpha\pi) \left\{ \frac{1}{\Gamma(\alpha-\frac{1}{2})} \int_0^t s^{\alpha-\frac{1}{2}} e^{s-t} ds \right\}^2 \frac{1}{T} \quad (T \to \infty).$$

Let X be as in Theorem 1 and let f be the spectral density of X:

$$f(\gamma) = \frac{\sin(\alpha\pi)}{\pi} \cdot \frac{|\gamma|^{1-2\alpha}}{1+\gamma^2} \qquad (\gamma \in \mathbb{R}). \tag{1.4}$$

We write h for the outer function of X:

$$h(\zeta) := \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+\gamma\zeta}{\gamma-\zeta} \cdot \frac{\log f(\gamma)}{1+\gamma^2} d\gamma \right\} \qquad (\Im z > 0). \tag{1.5}$$

The *canonical representation kernel* F of X is defined by $F := (2\pi)^{-1/2}\hat{h}$, where $\hat{h}$ is the Fourier transform of $h(\cdot) := \text{l.i.m.}_{\eta\downarrow 0} h(\cdot + i\eta) \in L^2(\mathbb{R})$. We have the following relation

$$h(\zeta) = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} e^{i\zeta t} F(t) dt \qquad (\Im z > 0). \tag{1.6}$$

Corollary. *Let $\frac{1}{2} < \alpha < 1$ and $t > 0$. Let X be as in Theorem 1 with autocovariance function R and canonical representation kernel F. Then*

$$Q(\tfrac{1}{2}T,t) - Q(\infty,t) \sim \left(\int_0^t F(s) ds \right)^2 \times \int_T^{\infty} \left\{ \frac{R(u)}{\int_{-u}^{u} R(s) ds} \right\}^2 du \quad (T \to \infty). \tag{1.7}$$

Recently, the authors obtained similar results to the corollary above for the stationary processes with autocovariance function R of the form

$$R(t) = \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbb{R}),$$

where σ is a finite Borel measure on $(0,\infty)$. In them, we assumed that

$$R(t) \sim t^{-p}\ell(t) \qquad (t \to \infty)$$

with $0 < p < \infty$ and ℓ slowly varying. In this case, the index p may be larger than or equal to 1 but the asymptotic relation (1.7) still holds as it is. The proofs of them are quite different from that of the present paper. The first author also obtained some relevant results for the partial autocorrelation coefficients of stationary time series. The details will appear elsewhere.

2. Proof of Theorem 1

In the proof below, we apply the theory of "strings" of Krein as described in [3]. The key is to use Rule 6.9.4 in [3, p. 268].

Step 1. Recall $\frac{1}{2} < \alpha < 1$ and c from (1.1). We write m for the function

$$m(x) := \frac{c^2}{\alpha(1-\alpha)} x^{(1-\alpha)/\alpha} \qquad (0 \le x < \infty).$$

The purpose of this step is to obtain the functions A, B, C, D, K and the measure $d\Delta$ associated with the string specified by m; see [3, Ch. 5] for background.

For $z \in \mathbb{C}$, we consider the following differential equation

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} A(x,z) = -z^2 A(x,z) m'(x) & (0 < x < \infty), \\ A(0+,z) = 1, \quad \dfrac{\partial}{\partial x} A(0+,z) = 0. & \end{cases}$$

The solution to the above is given by

$$A(x,z) := \pi \left\{ \frac{\alpha}{\sin(\alpha\pi)} \right\}^{\frac{1}{2}} z^\alpha x^{\frac{1}{2}} J_{-\alpha}(2czx^{\frac{1}{2\alpha}}) \qquad (0 < x < \infty),$$

where J_ν is the Bessel function of the first kind ([5, 3.1]).

We set, as in [3, p. 172],

$$C(x,z) := A(x,z) \int_0^x \{A(y,z)\}^{-2} dy \qquad (0 < x < \infty,\ \Im z \neq 0).$$

Since

$$\frac{\pi}{2\sin(\alpha\pi)}\cdot\frac{d}{dx}\left\{\frac{J_\alpha(x)}{J_{-\alpha}(x)}\right\}=\frac{1}{xJ_{-\alpha}(x)^2}$$

(cf. [5, 5.11(1)]), we have

$$\frac{\alpha\pi}{\sin(\alpha\pi)}\cdot\frac{d}{dx}\left\{\frac{J_\alpha(2czx^{\frac{1}{2\alpha}})}{J_{-\alpha}(2czx^{\frac{1}{2\alpha}})}\right\}=\frac{1}{xJ_{-\alpha}(2czx^{\frac{1}{2\alpha}})^2},$$

and so

$$C(x,z)=\left\{\frac{\alpha}{\sin(\alpha\pi)}\right\}^{\frac{1}{2}}z^{-\alpha}x^{\frac{1}{2}}J_\alpha(2czx^{\frac{1}{2\alpha}})\qquad(0<x<\infty,\ \Im z\neq 0),$$

where we have used the following asymptotic representation (cf. [5, 3.12]):

$$J_\nu(x)\sim\left(\frac{2}{x}\right)^{\nu}\frac{1}{\Gamma(\nu+1)}\qquad(x\to 0+).\tag{2.1}$$

As usual, we write $H_\nu^{(1)}$ for the Bessel function of the third kind ([5, 3.6]). Let the function D be as in [3, §5.4]. By [3, p. 175] and the asymptotic representations

$$J_\nu(z)=(\pi z/2)^{-\frac{1}{2}}\left[\cos\left\{z-\tfrac{1}{4}\pi(1+2\nu)\right\}+O(z^{-1})\right]\qquad(|z|\to\infty),\tag{2.2}$$

$$H_\nu^{(1)}(z)\sim(\pi z/2)^{-\frac{1}{2}}\exp[i\{z-\tfrac{1}{4}\pi(1+2\nu)\}]\qquad(|z|\to\infty)\tag{2.3}$$

(see [5, 7.2]), we have, for $\Im z\neq 0$,

$$\begin{aligned}D(0,z)&=\lim_{x\to\infty}\frac{C(x,z)}{A(x,z)}=\pi^{-1}z^{-2\alpha}\lim_{x\to\infty}\frac{J_\alpha(2czx^{\frac{1}{2\alpha}})}{J_{-\alpha}(2czx^{\frac{1}{2\alpha}})}\\&=\pi^{-1}z^{-2\alpha}\lim_{x\to\infty}\left\{e^{i\alpha\pi}-i\sin(\alpha\pi)\frac{H_{-\alpha}^{(1)}(2czx^{\frac{1}{2\alpha}})}{J_{-\alpha}(2czx^{\frac{1}{2\alpha}})}\right\}\\&=\pi^{-1}z^{-2\alpha}e^{i\alpha\pi\cdot\mathrm{sgn}(\Im z)}.\end{aligned}$$

Therefore, by [3, p. 175] and [5, 3.7(2)], for $0<x<\infty$ and $\Im z\neq 0$,

$$\begin{aligned}D(x,z)&=D(0,z)A(x,z)-C(x,z)\\&=\left\{\frac{\alpha}{\sin(\alpha\pi)}\right\}^{\frac{1}{2}}z^{-\alpha}x^{\frac{1}{2}}\left\{e^{i\alpha\pi\cdot\mathrm{sgn}(\Im z)}J_{-\alpha}(2czx^{\frac{1}{2\alpha}})-J_\alpha(2czx^{\frac{1}{2\alpha}})\right\}\end{aligned}$$

or, by [5, 3.6(2)],

$$D(x,z)=\begin{cases}i\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}}z^{-\alpha}x^{\frac{1}{2}}H_{-\alpha}^{(1)}(2czx^{\frac{1}{2\alpha}}),\\-i\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}}z^{-\alpha}x^{\frac{1}{2}}H_{-\alpha}^{(2)}(2czx^{\frac{1}{2\alpha}}).\end{cases}$$

In particular, by [5, 3.7(8)],

$$D(x,i)=\frac{2}{\pi}\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}}x^{\frac{1}{2}}K_{-\alpha}(2cx^{\frac{1}{2\alpha}})\qquad(0<x<\infty).\tag{2.4}$$

Recall μ from (1.2). We write $d\Delta$ for the measure $(1+\gamma^2)d\mu(\gamma)$ on $\mathbb{R}$:

$$d\Delta(\gamma) := \frac{\sin(\alpha\pi)}{\pi}|\gamma|^{1-2\alpha}d\gamma.$$

By simple calculation,

$$\int_0^\infty \frac{\gamma^{1-2\alpha}}{\gamma^2 - z^2}d\gamma = \frac{\pi}{2\sin(\alpha\pi)}z^{-2\alpha}\exp\{i\alpha\pi\cdot\operatorname{sgn}(\Im z)\} \qquad (\Im z \neq 0),$$

whence

$$D(0,z) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{d\Delta(\gamma)}{\gamma^2 - z^2} \qquad (\Im z \neq 0).$$

See [3, §5.5] for the implication of this equality.

As in Rule 6.9.4 in [3, p. 268], we set

$$K(x) := -\frac{D(x,i)}{(\partial D/\partial x)(x,i)} \qquad (x > 0).$$

By (2.4) and [5, 3.71(6)],

$$\frac{\partial D}{\partial x}(x,i) = -\frac{2c}{\alpha\pi}\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}}x^{\frac{1-\alpha}{2\alpha}}K_{1-\alpha}(2cx^{\frac{1}{2\alpha}})$$

whence

$$K(x) = c^{-1}\alpha x^{1-\frac{1}{2\alpha}}\frac{K_{-\alpha}(2cx^{\frac{1}{2\alpha}})}{K_{1-\alpha}(2cx^{\frac{1}{2\alpha}})} \qquad (0 < x < \infty).$$

Let B as in [3, §5.7]. Then, for $0 < x < \infty$ and $\gamma \in \mathbb{R}$,

$$B(x,\gamma) = -\frac{1}{\gamma}\frac{\partial}{\partial x}A(x,\gamma) = c\pi\{\alpha\sin(\alpha\pi)\}^{-\frac{1}{2}}\gamma^\alpha x^{\frac{1-\alpha}{2\alpha}}J_{1-\alpha}(2c\gamma x^{\frac{1}{2\alpha}}).$$

Step 2. Recall $T > 0$, $t > 0$ and $T_1 = T + t$. By [3, §6.10] and Rule 6.9.4 in [3, p. 268] applied to the string specified by m, we obtain

$$Q(T,t) = Q_{\text{even}}(T,t) + Q_{\text{odd}}(T,t),$$

where

$$Q_{\text{even}}(T,t) = \pi\int_x^\infty \left[\frac{2}{\pi}\int_0^\infty \cos(\gamma T_1)\{A(y,\gamma) - \gamma K(y)B(y,\gamma)\}\frac{d\Delta(\gamma)}{1+\gamma^2}\right]^2 \frac{dy}{K(y)^2},$$

$$Q_{\text{odd}}(T,t) = \pi\int_x^\infty \left[\frac{2}{\pi}\int_0^\infty \sin(\gamma T_1)\{\gamma A(y,\gamma) + K(y)B(y,\gamma)\}\frac{d\Delta(\gamma)}{1+\gamma^2}\right]^2 dm(y)$$

with $T = \int_0^x \{m'(y)\}^{1/2} dy$, or $x = \{T/(2c)\}^{2\alpha}$. By change of variables $s = 2cy^{\frac{1}{2\alpha}}$,

$$Q_{\mathrm{even}}(T,t) = \frac{2\sin(\alpha\pi)}{\pi} \int_T^\infty \{I_1(s) - I_2(s)\}^2 \frac{ds}{sK_{-\alpha}(s)^2},$$
$$Q_{\mathrm{odd}}(T,t) = \frac{2\sin(\alpha\pi)}{\pi} \int_T^\infty \{I_3(s) - I_4(s)\}^2 \frac{ds}{sK_{1-\alpha}(s)^2},$$

where

$$I_1(s) := sK_{1-\alpha}(s) \int_0^\infty \cos(\gamma T_1) J_{-\alpha}(s\gamma) \frac{\gamma^{1-\alpha}}{1+\gamma^2} d\gamma,$$
$$I_2(s) := sK_{-\alpha}(s) \int_0^\infty \cos(\gamma T_1) J_{1-\alpha}(s\gamma) \frac{\gamma^{2-\alpha}}{1+\gamma^2} d\gamma,$$
$$I_3(s) := sK_{1-\alpha}(s) \int_0^\infty \sin(\gamma T_1) J_{-\alpha}(s\gamma) \frac{\gamma^{2-\alpha}}{1+\gamma^2} d\gamma,$$
$$I_4(s) := sK_{-\alpha}(s) \int_0^\infty \sin(\gamma T_1) J_{1-\alpha}(s\gamma) \frac{\gamma^{1-\alpha}}{1+\gamma^2} d\gamma.$$

Step 3. Recall the constant d from (1.1). In this step, we show that $Q_{\mathrm{even}}(T,t) = d \cdot Q_1(T,t)$. We first note that $I_1 - I_2$ is continuous on $(0,\infty)$ by the asymptotic representations (2.1) and (2.2).

By [4, p. 68], (13.19),

$$\int_0^\infty \frac{x^{\nu+1}}{1+x^2} J_\nu(ax)\cos(xy) d\gamma = \cosh(y) K_\nu(a) \qquad (y < a,\ -1 < \nu < \tfrac{3}{2}),$$

whence $I_1(s) - I_2(s) = 0$ for $s > T_1$ and also for $s = T_1$ by continuity. The calculation of $I_1(s) - I_2(s)$ for $T < s < T_1$ is more tricky. By

$$\frac{d}{dx}\{x^\alpha J_{-\alpha}(x)\} = -x^\alpha J_{1-\alpha}(x), \qquad \frac{d}{dx}\{x^{1-\alpha} J_{1-\alpha}(x)\} = x^{1-\alpha} J_{-\alpha}(x),$$
$$\frac{d}{dx}\{x^\alpha K_{-\alpha}(x)\} = -x^\alpha K_{1-\alpha}(x), \qquad \frac{d}{dx}\{x^{1-\alpha} K_{1-\alpha}(x)\} = -x^{1-\alpha} K_{-\alpha}(x)$$

(see [5, 3.2 and 3.71]), we have

$$\frac{\partial}{\partial s}\{I_1(s) - I_2(s)\} = -sK_{-\alpha}(s) \int_0^{\infty-} \cos(\gamma T_1)\gamma^{1-\alpha} J_{-\alpha}(s\gamma) d\gamma \qquad (T < s < T_1).$$

In fact, by (2.2) and the second integral mean-value theorem ([6, §4.14]), the improper integral on the right converges uniformly in s on each compact subset of (T, T_1), whence we may interchange derivative and integral. By [4, p. 67], (13.13),

$$\frac{\partial}{\partial s}\{I_1(s) - I_2(s)\} = -\frac{2^{1-\alpha}\pi^{\frac{1}{2}} T_1}{\Gamma(\alpha - \frac{1}{2})} s^{1-\alpha} (T_1^2 - s^2)^{\alpha - \frac{3}{2}} K_{-\alpha}(s)$$

for $T < s < T_1$, and so

$$I_1(s) - I_2(s) = \frac{2^{1-\alpha}\pi^{\frac{1}{2}}T_1}{\Gamma(\alpha - \frac{1}{2})} \int_s^{T_1} u^{1-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}} K_{-\alpha}(u)du \quad (T < s < T_1).$$

Thus $Q_{\text{even}}(T,t) = d \cdot Q_1(T,t)$.

Step 4. Finally, we show that $Q_{\text{odd}}(T,t) = d \cdot Q_2(T,t)$. The proof is quite analogous to that for Q_{even} in Step 3. First, $I_3 - I_4$ is continuous on $(0,\infty)$. Next, since

$$\int_0^\infty \frac{x^\nu}{1+x^2} J_\nu(ax)\sin(xy)d\gamma = \sinh(y)K_\nu(a) \qquad (y < a,\ -1 < \nu < \tfrac{5}{2})$$

(see [4, p. 166], (13.20)), $I_3 - I_4$ vanishes on $[T_1, \infty)$. Finally, by [4, p. 164], (13.9), for $T < s < T_1$,

$$\begin{aligned}\frac{\partial}{\partial s}\{I_3(s) - I_4(s)\} &= -sK_{1-\alpha}(s) \int_0^{\infty-} \sin(\gamma T_1)\gamma^{1-\alpha} J_{1-\alpha}(s\gamma)d\gamma \\ &= -\frac{2^{1-\alpha}\pi^{\frac{1}{2}}}{\Gamma(\alpha-\frac{1}{2})} s^{2-\alpha}(T_1^2 - s^2)^{\alpha-\frac{3}{2}} K_{1-\alpha}(s),\end{aligned}$$

whence

$$I_3(s) - I_4(s) = \frac{2^{1-\alpha}\pi^{\frac{1}{2}}}{\Gamma(\alpha - \frac{1}{2})} \int_s^{T_1} u^{2-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}} K_{1-\alpha}(u)du \quad (T < s < T_1).$$

Thus $Q_{\text{odd}}(T,t) = d \cdot Q_2(T,t)$. This completes the proof. □

3. Proof of Theorem 2

Step 1. First we consider $Q_1(T,t)$. By change of variables $s' = s - T$, $u' = u - T$, we obtain

$$Q_1(T,t) = 2^{2\alpha-3} \int_0^t ds \frac{(T+t)^2}{\left\{(T+s)^{\frac{1}{2}} K_{-\alpha}(T+s)\right\}^2}$$

$$\times \left[\int_s^t (T+u)^{\frac{1}{2}} K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha} \left\{T + \tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}} (t-u)^{\alpha-\frac{3}{2}} du\right]^2.$$

By the asymptotic representation

$$x^{\frac{1}{2}} K_\nu(x) = (\pi/2)^{\frac{1}{2}} e^{-x} \left\{1 + \tfrac{1}{8}(4\nu^2 - 1)x^{-1} + O(x^{-2})\right\} \qquad (x \to \infty) \tag{3.1}$$

(see [5, 7.23]), we have, as $T \to \infty$,

$$(T+u)^{\frac{1}{2}} K_{-\alpha}(T+u) = (\pi/2)^{\frac{1}{2}} e^{-T-u} \left\{1 + \tfrac{1}{8}(4\alpha^2 - 1)T^{-1} + O(T^{-2})\right\}.$$

This, together with

$$(1+x)^p = 1 + px + O(x^2) \qquad (x \to 0+), \tag{3.2}$$

gives

$$
\begin{aligned}
&(T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}\\
&\quad=(\pi/2)^{\frac{1}{2}}e^{-T-u}T^{-1}\\
&\qquad\times\left[1+\left\{\tfrac{1}{8}(4\alpha^2-1)+\left(\tfrac{1}{2}-\alpha\right)u+\tfrac{1}{2}(\alpha-\tfrac{3}{2})(t+u)\right\}T^{-1}+O(T^{-2})\right]\\
&\quad=(\pi/2)^{\frac{1}{2}}e^{-T-u}T^{-1}\\
&\qquad\times\left[1+\tfrac{1}{2}\left\{(\alpha^2-\tfrac{1}{4})-2t+(\alpha+\tfrac{1}{2})(t-u)\right\}T^{-1}+O(T^{-2})\right].
\end{aligned}
$$

Hence, as $T\to\infty$,

$$
\begin{aligned}
&\left[\int_s^t(T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\{T+\tfrac{1}{2}(t+u)\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2\\
&\quad=\frac{\pi e^{-2(t+T)}}{2T^2}\left[\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)^2+\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)\right.\\
&\qquad\left.\times\left(\int_0^{t-s}\left\{(\alpha^2-\tfrac{1}{4})-2t+(\alpha+\tfrac{1}{2})u\right\}u^{\alpha-\frac{3}{2}}e^u du\right)T^{-1}+O(T^{-2})\right].
\end{aligned}
$$

Similarly,

$$
\frac{(T+t)^2}{(T+s)K_{-\alpha}(T+s)^2}=\frac{2T^2e^{2(T+s)}}{\pi}\left[1+\left\{2t-(\alpha^2-\tfrac{1}{4})\right\}T^{-1}+O(T^{-2})\right].
$$

Combining,

$$
\begin{aligned}
&\frac{(T+t)^2}{(T+s)K_{-\alpha}(T+s)^2}\\
&\times\left[\int_s^t(T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2\\
&=e^{-2(t-s)}\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)^2\\
&\quad+(\alpha+\tfrac{1}{2})e^{-2(t-s)}\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)\left(\int_0^{t-s}u^{\alpha-\frac{1}{2}}e^u du\right)T^{-1}+O(T^{-2}).
\end{aligned}
$$

Thus

$$
Q_1(T,t)=2^{2\alpha-3}J_0(t)+2^{2\alpha-3}(\alpha+\tfrac{1}{2})J_1(t)T^{-1}+O(T^{-2})\qquad(T\to\infty)\tag{3.3}
$$

with

$$
\begin{aligned}
J_0(t)&:=\int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac{3}{2}}e^u du\right)^2 ds,\\
J_1(t)&:=\int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac{3}{2}}e^u du\right)\left(\int_0^s u^{\alpha-\frac{1}{2}}e^u du\right)ds.
\end{aligned}
$$

Step 2. Next we consider $Q_2(T,t)$. By change of variables $s' = s - T$, $u' = u - T$, we obtain

$$Q_2(T,t) = 2^{2\alpha-3}\int_0^t ds \frac{1}{\left\{(T+s)^{\frac{1}{2}}K_{1-\alpha}(T+s)\right\}^2}$$
$$\times \left[\int_s^t (T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u)(T+u)^{\frac{3}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2.$$

By (3.1), we have, as $T \to \infty$,

$$(T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u) = (\pi/2)^{\frac{1}{2}}e^{-T-u}\left[1+\tfrac{1}{8}\{4(1-\alpha)^2-1\}T^{-1}+O(T^{-2})\right].$$

Therefore,

$$\begin{aligned}&(T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u)(T+u)^{\frac{3}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}\\ &\quad=(\pi/2)^{\frac{1}{2}}e^{-T-u}\\ &\qquad\times\left[1+\tfrac{1}{2}\left\{(1-\alpha)^2-\tfrac{1}{4}+(\alpha-\tfrac{3}{2})(t-u)\right\}T^{-1}+O(T^{-2})\right].\end{aligned}$$

Hence, as $T \to \infty$,

$$\begin{aligned}&\left[\int_s^t (T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u)(T+u)^{\frac{3}{2}-\alpha}\{T+\tfrac{1}{2}(t+u)\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2\\ &=\frac{\pi e^{-2(t+T)}}{2}\left[\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)^2+\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)\right.\\ &\quad\left.\times\left(\int_0^{t-s}\left\{(1-\alpha)^2-\tfrac{1}{4}+(\alpha-\tfrac{3}{2})u\right\}u^{\alpha-\frac{3}{2}}e^u du\right)T^{-1}+O(T^{-2})\right].\end{aligned}$$

Similarly,

$$\frac{1}{(T+s)K_{1-\alpha}(T+s)^2} = \frac{2e^{2(T+s)}}{\pi}\left[1-\left\{(1-\alpha)^2-\tfrac{1}{4}\right)\right\}T^{-1}+O(T^{-2})\right].$$

Combining,

$$\begin{aligned}&\frac{1}{(T+s)K_{1-\alpha}(T+s)^2}\\ &\times\left[\int_s^t (T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u)(T+u)^{\frac{3}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2\\ &=e^{-2(t-s)}\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)^2\\ &+(\alpha-\tfrac{3}{2})e^{-2(t-s)}\left(\int_0^{t-s}u^{\alpha-\frac{3}{2}}e^u du\right)\left(\int_0^{t-s}u^{\alpha-\frac{1}{2}}e^u du\right)T^{-1}+O(T^{-2}).\end{aligned}$$

Thus

$$Q_2(T,t) = 2^{2\alpha-3}J_0(t) + 2^{2\alpha-3}(\alpha - \tfrac{3}{2})J_1(t)T^{-1} + O(T^{-2}) \qquad (T \to \infty). \tag{3.4}$$

Step 3. Since $Q(T,t) \downarrow Q(\infty,t)$ as $T \to \infty$, we obtain, by (3.3) and (3.4),

$$Q(\infty,t) = \frac{2\sin(\alpha\pi)}{\Gamma(\alpha-\frac{1}{2})^2}\int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac{3}{2}}e^u du\right)^2 ds \tag{3.5}$$

and

$$Q(T,t) - Q(\infty,t) = \frac{(2\alpha-1)\sin(\alpha\pi)}{\Gamma(\alpha-\frac{1}{2})^2}J_1(t)T^{-1} + O(T^{-2}). \qquad (T \to \infty).$$

It remains to show that

$$(2\alpha-1)J_1(t) = \left(\int_0^t s^{\alpha-\frac{1}{2}}e^{s-t}ds\right)^2. \tag{3.6}$$

Integrating by parts,

$$(\alpha-\tfrac{1}{2})\int_0^s u^{\alpha-\frac{3}{2}}e^u du = s^{\alpha-\frac{1}{2}}e^s - \int_0^s u^{\alpha-\frac{1}{2}}e^u du,$$

whence the left-hand side of (3.6) is equal to $2\{J_2(t) - J_3(t)\}$, where

$$J_2(t) := \int_0^t s^{\alpha-\frac{1}{2}}e^s\left(\int_0^s u^{\alpha-\frac{1}{2}}e^u du\right)e^{-2s}ds,$$

$$J_3(t) := \int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac{1}{2}}e^u du\right)^2 ds.$$

Again, integrating by parts,

$$J_2(t) = \left(\int_0^t s^{\alpha-\frac{1}{2}}e^{s-t}ds\right)^2 + 2J_3(t) - J_2(t).$$

Thus (3.6) follows. This completes the proof. □

4. Proof of Corollary

First, we prove (1.3). We set

$$\ell(x) := \frac{x^2}{1+x^2}, \qquad k(x) := x^{2\alpha-3}\cos(1/x) \qquad (0 < x < \infty).$$

Then, for $t > 0$,

$$R(t) = \frac{2\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{\gamma^{1-2\alpha}\cos(\gamma t)}{1+\gamma^2}d\gamma = \frac{2\sin(\alpha\pi)}{\pi}\int_0^\infty k(x)\ell(xt)dx.$$

Choose $\delta > 0$ so that $\delta < \min(2-2\alpha, 2\alpha-1)$. Then the improper integrals

$$\int_{0+}^1 x^{-\delta}k(x)dx, \qquad \int_1^{\infty-} x^{-\delta}k(x)dx$$

exist. Therefore, by the Bojanic-Karamata theorem (cf. Bingham, Goldie and Teugels [2, Th. 4.1.5]),

$$\int_0^\infty k(x)\ell(xt)dx \to \int_{0+}^{\infty-} k(x)dx \qquad (t\to\infty).$$

Since

$$\int_{0+}^{\infty-} k(x)dx = \int_0^{\infty-} x^{1-2\alpha}\cos x dx = \frac{\pi}{2\sin(\alpha\pi)\Gamma(2\alpha-1)},$$

(1.3) follows.

By (1.3),

$$\frac{R(u)}{\int_{-u}^{u} R(s)ds} \sim (\alpha-\tfrac{1}{2})u^{-1} \qquad (u\to\infty),$$

whence

$$\int_T^\infty \left\{\frac{R(u)}{\int_{-u}^{u} R(s)ds}\right\}^2 du \sim (\alpha-\tfrac{1}{2})^2 T^{-1} \qquad (T\to\infty). \tag{4.1}$$

Recall f and h from (1.4) and (1.5). By applying Exercises 2.3.4 and 2.7.2 of [3] to the rational functions $1/(1-i\zeta)$ and $-i\zeta/(1-i\zeta)^2$, we obtain

$$\frac{1}{1-i\zeta} = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\gamma\zeta}{\gamma-\zeta}\cdot\frac{\log(1+\gamma^2)^{-1}}{1+\gamma^2}d\gamma\right\} \qquad (\Im z>0),$$

$$-i\zeta = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\gamma\zeta}{\gamma-\zeta}\cdot\frac{\log(\gamma^2)}{1+\gamma^2}d\gamma\right\} \qquad (\Im z>0)$$

(note that $1/(1-i\zeta)$ and $-i\zeta$ are both positive on the upper imaginary axis). Therefore

$$h(\zeta) = \left\{\frac{\sin(\alpha\pi)}{\pi}\right\}^{\frac{1}{2}}\frac{(-i\zeta)^{\frac{1}{2}-\alpha}}{1-i\zeta} \qquad (\Im z>0). \tag{4.2}$$

We set

$$G(t) := \frac{\{2\sin(\alpha\pi)\}^{\frac{1}{2}}}{\Gamma(\alpha-\frac{1}{2})}\int_0^t e^{s-t}s^{\alpha-\frac{3}{2}}ds \qquad (0<t<\infty).$$

Then, by (4.2) and (1.6),

$$\begin{aligned}(2\pi)^{-\frac{1}{2}}\int_0^\infty e^{i\zeta t}G(t)dt &= \left\{\frac{\sin(\alpha\pi)}{\pi}\right\}^{\frac{1}{2}}\frac{(-i\zeta)^{\frac{1}{2}-\alpha}}{1-i\zeta}\\ &= h(\zeta) = (2\pi)^{-\frac{1}{2}}\int_0^\infty e^{i\zeta t}F(t)dt,\end{aligned}$$

whence $F=G$. Therefore,

$$\int_0^t F(s)ds = \frac{\{2\sin(\alpha\pi)\}^{\frac{1}{2}}}{(\alpha-\frac{1}{2})\Gamma(\alpha-\frac{1}{2})}\int_0^t e^{s-t}s^{\alpha-\frac{1}{2}}du. \tag{4.3}$$

This, together with Theorem 2 and (4.1), gives the corollary. $\square$

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