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A two dimensional random crystalline algorithm for Gauss curvature flow

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ABSTRACT

In this paper we propose a two dimensional random crystalline algorithm for Gauss curvature flow.

1. Introduction.

Gauss curvature flow is a mathematical model of the wearing process of stones on beaches (see Firey [3] and also Chow [1] and Tso [10]).

Ishii [7] proposed a discrete time approximation scheme for Gauss curvature flow. Let us briefly introduce the idea. Take $h > 0$. Suppose that we are given a convex stone with a smooth surface at time $t = 0$. At time $t = h$, in every direction of outward normals, the stone is cut off the volume (which converges to zero as $h \to 0$). This process is continued until the extinction time of the stone. As $h \to 0$, the surface of a stone converges to a Gauss curvature flow in Hausdorf metric.

In this paper we propose a two dimensional random model. Roughly speaking, our model is as follows. Take $n \geq 5$ so that $2\pi n^{-1} < 2^{-1}\pi$. Suppose that we are given a convex polygon with n sides with outward normal $(\cos(2\pi i n^{-1}), \sin(2\pi i n^{-1}))$ $(i = 0, \cdots, n-1)$ at time $t = 0$. At a random time $\sigma > 0$, on a randomly chosen side, the set surrounded by the polygon is cut off the deterministic area (which converges to zero as $n \to \infty$). This process is continued until the extinction time. As $n \to \infty$, the polygon converges to a Gauss curvature flow in Hausdorf metric, in probability (see Corollary 2.2 in section 2). Such a model is more realistic in that the stone can not be cut off the portion in every direction at the same time.

In a two dimensional case, Gauss curvature flow is also a mean curvature flow. Our result can be also considered as a random crystalline approximation of a mean curvature flow. A crystalline approximation of a convex curve is useful in numerical analysis (see Girão [4] and also Girão and Kohn [5] and the references therein). In case an initial curve is not convex, Girão's result was generalized by K. Ishii and M. H. Soner (see Ishii and Soner [8] and the references therein for further information on this problem).

Let us introduce Girão's result in a special case since it plays a crucial role in this paper. Take a smooth simple closed convex curve Γ on $\mathbf{R}^2$ and denote by $\{\Gamma(t)\}_{0\le t<T^*}$ on $\mathbf{R}^2$ the mean curvature flow with $\Gamma(0) = \Gamma$. Here T^* denotes the extinction time of $\Gamma(t)$. More precisely speaking, the (inward) normal velocity at a point of $\Gamma(t)$ is equal to the curvature at the point. One can define, as below, polygons with n sides (n-polygon) $\{\Gamma_n(t)\}_{0\le t<T_n^*}$ which approximate $\{\Gamma(t)\}_{0\le t<T^*}$, where T_n^* denotes the extinction time of $\Gamma_n(t)$. For $n \ge 5$, $\Gamma_n(0)$ is a n-polygon of which the i th side is tangent to Γ at a point with outward normal $(\cos(2\pi in^{-1}), \sin(2\pi in^{-1}))$ $(i = 0, \cdots, n-1)$. For $t \in [0, T_n^*)$, the (inward) normal velocity $V_{n,i}(t)$ of the i th side of $\Gamma_n(t)$ is given by the following which is called a discrete curvature:

$$(1.1). \qquad V_{n,i}(t) = 2(\tan(\pi n^{-1}))\ell_{n,i}(t)^{-1},$$

where $\ell_{n,i}(t)$ is a length of the i th side of $\Gamma_n(t)$. From (1.1), it can be shown that $\{\ell_{n,i}(t)\}_{i=0}^{n-1}$ satisfies the follows:

$$(1.2). \qquad \begin{aligned} &d[\ell_{n,i}(t)]/dt = (2\ell_{n,i}(t)^{-1}\cos(2\pi n^{-1}) - \ell_{n,i+1}(t)^{-1} - \ell_{n,i-1}(t)^{-1})(\cos(\pi n^{-1}))^{-2}, \\ &\ell_{n,n}(t) = \ell_{n,0}(t). \end{aligned}$$

Let us recall the definition of Hausdorf metric: for compact sets A and $B \in \mathbf{R}^2$,

$$(1.3). \qquad d_H(A, B) = \max(\max_{p\in A} dist(p, B), \max_{q\in B} dist(q, A)).$$

Denote by $\Omega(t)$ and $\Omega_{\ell,n}(t)$ closed sets surrounded by $\Gamma(t)$ and $\Gamma_n(t)$, respectively. The following was proved in Girão [4].

Theorem 1.1. *For* $t \in [0, T^*)$, $\liminf_{n\to\infty} T_n^* > t$ *and* $\sup_{0\leq s\leq t} d_H(\Omega_{\ell,n}(s), \Omega(s))$ *converges to* 0 *as* $n \to \infty$.

Remark 1.1. Girão's result has not been generalized to a class of closed convex surfaces in $\mathbf{R}^d$ for $d \geq 3$. In Girão [4], they used a result on convergence of discrete curvatures of $\Gamma_n(t)$ to curvatures of $\Gamma(t)$. But Ishii and Soner [8] did not. This is a nice aspect of the viscosity solution approach.

Let us introduce our random model. Take $T(n)$ such that

$$(1.4). \qquad \lim_{n\to\infty} T(n)n^{-5} = \infty,$$

and put

$$(1.5). \qquad \theta_n = 2\pi n^{-1},$$

$$(1.6). \qquad h_n(x) = 2^{-1}(\tan\theta_n)\{-x + (x^2 + 4(\cot\theta_n)T(n)^{-1}\theta_n)^{1/2}\} \quad (x \in \mathbf{R}).$$

For $n \geq 5$, we consider a Markov process $\{(X_{n,i}(t), D_{n,i}(t))\}_{i=0}^{n-1}$ on $\mathbf{R}^{2n}$ such that

$$(1.7). \qquad \{(X_{n,i}(0), D_{n,i}(0))\}_{i=0}^{n-1} = \{(\ell_{n,i}(0), 0)\}_{i=0}^{n-1}$$

and with the generator defined as follows: for a bounded Borel measurable function f from $\mathbf{R}^{2n}$ to $\mathbf{R}$,

$$\begin{aligned}(1.8).\ Lf(x) = T(n)(\sin\theta_n)[\theta_n\cos^2(\theta_n/2)]^{-1}\sum_{i=0}^{n-1} I_{\{\min(x_{2(i-1)},x_{2(i+1)})\sin\theta_n > h_n(x_{2i})\}}(x) \\ \times [f(x + 2(\cot\theta_n)h_n(x_{2i})\mathbf{e}_{2i} - (\sin\theta_n)^{-1}h_n(x_{2i})(\mathbf{e}_{2(i-1)} + \mathbf{e}_{2(i+1)}) \\ + h_n(x_{2i})\mathbf{e}_{2i+1}) - f(x)]\end{aligned}$$

(see Ethier and Kurtz [2], Chap. 4, section 2). Here we put $x = (x_i)_{i=0}^{2n-1}$, $\mathbf{e}_{2n+k} = \mathbf{e}_k$ and $x_{2n+k} = x_k$ ($k \in \mathbf{N}$), and $\{\mathbf{e}_k\}_{k=0}^{2n-1}$ denotes the standard normal base in $\mathbf{R}^{2n}$. (Notice that $h_n(x) > 0$ for $x \in \mathbf{R}$.) Denote by $\Omega_{X,n}(t)$ a closed set surrounded by the n-polygon which has a length $X_{n,k}(t)$ on the k th side with outward normal $(\cos(k\theta_n), \sin(k\theta_n))$ $(k = 0, \cdots, n-1)$. We will show that $\Omega_{X,n}(t)$ converges to $\Omega(t)$ in Hausdorf metric, as $n \to \infty$, in probability (see Corollary 2.2 in section 2).

Let us discuss a heuristic meaning of our model defined above. If the i th side of $\Omega_{X,n}(0)$ is chosen randomly at a random time $\sigma > 0$, and if

$$(1.9). \qquad \min(X_{n,i-1}(0), X_{n,i+1}(0)) \sin\theta_n > h_n(X_{n,i}(0)),$$

then $\Omega_{X,n}(0)(= \Omega_{\ell,n}(0))$ is cut off the isogonal trapezoid with the height $h_n(X_{n,i}(0))$ (and with an area $T(n)^{-1}\theta_n$) on the i th side. Thanks to (1.9), $\partial\Omega_{X,n}(\sigma)$ is again a n-polygon. After this process is repeted finite times< (the volume of $\Omega_{X,n}(0))/(T(n)^{-1}\theta_n)$, it is finished and $\Omega_{X,n}(t)$ remains to be a n-polygon for $t \geq 0$ since $2\pi n^{-1} < 2^{-1}\pi$. Notice that $h_n(x)$ is a positive solution to

$$(1.10). \qquad (x + (\cot\theta_n) h_n(x)) h_n(x) = T(n)^{-1}\theta_n.$$

For $x > 0$, the left hand side of (1.10) is an area of an isogonal trapezoid with a hight $h_n(x)$ and with upper and lower sides of lengths x and $x + 2(\cot\theta_n)h_n(x)$, respectively.

In the proof of our result, we approximate $\Omega_{X,n}(t)$ by $\Omega_{\ell,n}(t)$ and make use of Theorem 1.1.

In section 2 we state our results which will be proved in section 4. Technical lemmas will be stated and proved in section 3.

2. Main results.

Before we show that a random crystalline model converges, in probability, to Gauss curvature flow, let us give some notation.

Put

$$d_{n,i}(t) = \int_0^t 2[\tan(\theta_n/2)]\ell_{n,i}(s)^{-1}ds \quad (i = 0, \cdots, n-1) \tag{2.1.}$$

(see (1.2) and (1.5) for notation).

Put the intersection point of the 0th and 1st side of $\Omega_{\ell,n}(0)$ at the origin. Then the coordinate of the intersection point of the i th and the (i+1) th side of $\Omega_{X,n}(t)$ and $\Omega_{\ell,n}(t)$ can be written as follows, respectively:

$$\begin{aligned}
Y_{n,0}(t) &= (-D_{n,0}(t), D_{n,0}(t)\cot\theta_n - D_{n,1}(t)(\sin\theta_n)^{-1}) \quad (i=0), \\
Y_{n,i}(t) &= (-\sum_{k=1}^{i} X_{n,k}(t)\sin(k\theta_n) - D_{n,0}(t), \sum_{k=1}^{i} X_{n,k}(t)\cos(k\theta_n) \\
&\quad + D_{n,0}(t)\cot\theta_n - D_{n,1}(t)(\sin\theta_n)^{-1}) \quad (1 \le i \le n-1),
\end{aligned} \tag{2.2.}$$

$$\begin{aligned}
y_{n,0}(t) &= (-d_{n,0}(t), d_{n,0}(t)\cot\theta_n - d_{n,1}(t)(\sin\theta_n)^{-1}) \quad (i=0), \\
y_{n,i}(t) &= (-\sum_{k=1}^{i} \ell_{n,k}(t)\sin(k\theta_n) - d_{n,0}(t), \sum_{k=1}^{i} \ell_{n,k}(t)\cos(k\theta_n) \\
&\quad + d_{n,0}(t)\cot\theta_n - d_{n,1}(t)(\sin\theta_n)^{-1}) \quad (1 \le i \le n-1).
\end{aligned} \tag{2.3.}$$

Put for $\delta \in (0, T_n^*)$,

$$\begin{aligned}
C_n(\delta) &= n\min\{\ell_{n,k}(s); 0 \le k \le n-1, 0 \le s \le T_n^* - \delta\}, \\
\tau_{n,\delta} &= \inf\{t > 0; C_n(\delta)/2 \ge n\min\{X_{n,k}(t); 0 \le k \le n-1\}.
\end{aligned} \tag{2.4.}$$

Then the following holds.

Theorem 2.1. *Suppose that (1.4) holds. Then for $\delta \in (0, T^*)$,*

$$\lim_{n\to\infty} n^2 E[\sup_{0 \le t \le \min(T_n^*-\delta, \tau_{n,\delta})} \sum_{i=0}^{n-1} |Y_{n,i}(t) - y_{n,i}(t)|^2] = 0. \tag{2.5.}$$

As a corollary to Theorem 2.1, we obtain the following.

Corollary 2.2. *Suppose that (1.4) holds. Then for any $t \in (0, T^*)$ and $\eta > 0$,*

(2.6).
$$\lim_{n\to\infty} P(\sup_{0\le s\le t} d_H(\Omega_{X,n}(s), \Omega(s)) < \eta) = 1.$$

Let us state the outline of the proof of Theorem 2.1.

(Outline of the proof of Theorem 2.1). Take $n_0 \in \mathbf{N}$ sufficiently large so that $T_n^* > \delta$ for $n \ge n_0$, which is possible from Theorem 1.1.

The following will be proved in section 4: there exists $n_2 \ge n_0$ such that for $n \ge n_2$ and $t \in [0, T_n^* - \delta]$,

(2.7).
$$\begin{aligned}
&E[\sup_{0\le s\le \min(t,\tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2] \\
&\le 2T(n)^{-2}n^7 t|\theta_n([C_n(\delta)2^{-1}]^3 \sin\theta_n)^{-1}|^2 \\
&\quad + 8T(n)^{-1}n^3 t\theta_n(C_n(\delta)^2 \sin\theta_n \cos^2(\theta_n/2))^{-1} \\
&\quad + 32T(n)^{-1}n^2\theta_n(C_n(\delta)^2 \sin\theta_n \cos^2(\theta_n/2))^{-1} \\
&\quad + 6T(n)^{-2}n^3 t|\theta_n(C_n(\delta)\sin\theta_n)^{-1}|^2 \\
&\quad + (4n^2\sin^2\theta_n((\cos\theta_n)C_n(\delta)^2\cos^2(\theta_n/2))^{-1} + 2 + 3) \\
&\quad \times \int_0^t E[\sup_{0\le u\le \min(s,\tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2] ds.
\end{aligned}$$

By Gronwall's inequality, from (1.4), the proof is over, since $\{C_k(\delta)^{-1}\}_{k=n_3}^{\infty}$ is bounded (see (14) of Girão [4]).

Q. E. D.

Next we prove Corollary 2.2 from Theorem 2.1.

(Proof of Corollary 2.2). For any $t \in (0, T^*)$, take $n_1 \in \mathbf{N}$ such that for $n \ge n_1$

(2.8).
$$t < T_n^* - (T^* - t)/2,$$
$$\sup_{0\le s\le t} d_H(\Omega_{\ell,n}(s), \Omega(s)) < \eta/2,$$

which is possible from Theorem 1.1. Put $\delta = (T^* - t)/2$. Then for $n \geq n_1$,

$$(2.9).\quad \begin{aligned} &P(\sup_{0\leq s\leq t} d_H(\Omega_{X,n}(s), \Omega(s)) \geq \eta) \\ &\leq P(\sup_{0\leq s\leq t} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2) \\ &\leq P(\sup_{0\leq s\leq T_n^*-\delta} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2) \\ &\leq P(\tau_{n,\delta} < T_n^* - \delta) + P(\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2). \end{aligned}$$

The first probability in (2.9) can be shown to converge to zero as $n \to \infty$ as follows: by Chebychev's inequality,

$$(2.10).\quad \begin{aligned} &P(\tau_{n,\delta} < T_n^* - \delta) \\ &\leq P(\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |X_{n,k}(s) - \ell_{n,k}(s)| \geq C_n(\delta)(2n)^{-1}) \\ &\leq P(\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)| \geq C_n(\delta)(4n)^{-1}) \\ &\leq (4nC_n(\delta)^{-1})^2 E[\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|^2] \\ &\to 0, \text{ as } n \to \infty \text{ (from Theorem 2.1)}. \end{aligned}$$

Here we used the following:

$$(2.11).\quad |X_{n,k}(s) - \ell_{n,k}(s)| \leq |Y_{n,k}(s) - y_{n,k}(s)| + |Y_{n,k-1}(s) - y_{n,k-1}(s)|.$$

The second probability in (2.9) can be shown to converge to zero as $n \to \infty$ as follows: by Chebychev's inequality,

$$(2.12).\quad \begin{aligned} &P(\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2) \\ &\leq P(\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)| \geq \eta/2) \\ &\leq (\eta/2)^{-2} E[\sup_{0\leq s\leq \min(T_n^*-\delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|^2] \\ &\to 0, \text{ as } n \to \infty \text{ (from Theorem 2.1)}. \end{aligned}$$

Here we used the following:

$$(2.13).\qquad d_H(\Omega_{X,n}(s),\Omega_{\ell,n}(s)) \le \max_{k=0}^{n-1} |Y_{n,k}(s)-y_{n,k}(s)|.$$

Q. E. D.

3. Lemmas.

In this section we state and prove lemmas which will be used in the next section.

Lemma 3.1. *For $n \ge 5$, $i = 0,\cdots,n-1$, and $s \in (0,T_n^*)$,*

$$(3.1).\qquad \begin{aligned} dy_{n,i}(s)/ds &= [\ell_{n,i+1}(s)\cos^2(\theta_n/2)]^{-1}(\sin(i\theta_n),-\cos(i\theta_n)) \\ &\quad - [\ell_{n,i}(s)\cos^2(\theta_n/2)]^{-1}(\sin((i+1)\theta_n),-\cos((i+1)\theta_n)). \end{aligned}$$

(proof). When $i=0$, from (2.1) and (2.3),

$$(3.2).\qquad \begin{aligned} &dy_{n,0}(s)/ds \\ &= (-2\ell_{n,0}(s)^{-1}\tan(\theta_n/2), 2\ell_{n,0}(s)^{-1}\tan(\theta_n/2)\cot\theta_n \\ &\qquad - 2\ell_{n,1}(s)^{-1}\tan(\theta_n/2)(\sin\theta_n)^{-1}) \\ &= [\ell_{n,1}(s)\cos^2(\theta_n/2)]^{-1}(0,-1) - [\ell_{n,0}(s)\cos^2(\theta_n/2)]^{-1}(\sin\theta_n,-\cos\theta_n). \end{aligned}$$

Suppose that (3.1) is true for $i-1(\ge 0)$. Then from (1.2) and (2.3),

$$(3.3).\qquad \begin{aligned} &dy_{n,i}(s)/ds \\ &= dy_{n,i-1}(s)/ds + (2\ell_{n,i}(s)^{-1}\cos\theta_n - \ell_{n,i+1}(s)^{-1} - \ell_{n,i-1}(s)^{-1}) \\ &\qquad \times (\cos^2(\theta_n/2))^{-1}(-\sin(i\theta_n),\cos(i\theta_n)) \\ &= [\cos^2(\theta_n/2)]^{-1}(\ell_{n,i}(s)^{-1}\sin((i-1)\theta_n) - \ell_{n,i-1}(s)^{-1}\sin(i\theta_n) \\ &\qquad - (2\ell_{n,i}(s)^{-1}\cos\theta_n - \ell_{n,i+1}(s)^{-1} - \ell_{n,i-1}(s)^{-1})\sin(i\theta_n), \end{aligned}$$

$$
\begin{aligned}
&- \ell_{n,i}(s)^{-1}\cos((i-1)\theta_n) + \ell_{n,i-1}(s)^{-1}\cos(i\theta_n)\\
&+ (2\ell_{n,i}(s)^{-1}\cos\theta_n - \ell_{n,i+1}(s)^{-1} - \ell_{n,i-1}(s)^{-1})\cos(i\theta_n))\\
= {}& [\ell_{n,i+1}(s)\cos^2(\theta_n/2)]^{-1}(\sin(i\theta_n), -\cos(i\theta_n))\\
&- [\ell_{n,i}(s)\cos^2(\theta_n/2)]^{-1}(\sin((i+1)\theta_n), -\cos((i+1)\theta_n)).
\end{aligned}
$$

Here we used the following:

$$
\begin{aligned}
\sin((i-1)\theta_n) + \sin((i+1)\theta_n) &= 2\cos\theta_n \sin(i\theta_n),\\
\cos((i-1)\theta_n) + \cos((i+1)\theta_n) &= 2\cos\theta_n \cos(i\theta_n).
\end{aligned}
\tag{3.4}
$$

Q. E. D.

Before we state and prove the following lemma, let us give some notation: for $y = (y_i)_{i=0}^{2n-1} \in \mathbf{R}^{2n}$, put

$$
\begin{aligned}
\tilde{L}f(y) = {}& T(n)(\sin\theta_n)[\theta_n\cos^2(\theta_n/2)]^{-1}\\
&\times \sum_{i=0}^{n-1}\{f(y + h_n(\{|y_{2i} - y_{2(i-1)}|^2 + |y_{2i+1} - y_{2i-1}|^2\}^{1/2})\\
&\times (\sin\theta_n)^{-1}([\sin((i-1)\theta_n)]\mathbf{e}_{2(i-1)} - [\cos((i-1)\theta_n)]\mathbf{e}_{2i-1}\\
&\quad - [\sin((i+1)\theta_n)]\mathbf{e}_{2i} + [\cos((i+1)\theta_n)]\mathbf{e}_{2i+1})) - f(y)\}.
\end{aligned}
\tag{3.5}
$$

Put also $\mathbf{Y}_n(t) = (Y_{n,k}(t))_{k=0}^{n-1}$. Then the following can be proved by the Ito formula (see Ikeda and Watanabe [6] or Meyer [9]).

Lemma 3.2. *Suppose that (1.4) holds. Then for any $\delta \in (0, T^*)$, there exists $n_2 \in \mathbf{N}$ such that $\delta < T_n^*$ for $n \geq n_2$ and that for $n \geq n_2$ and any $f \in C_o^2(\mathbf{R}^{2n};\mathbf{R})$*

$$
f(\mathbf{Y}_n(\min(t,\tau_{n,\delta}))) = f(\mathbf{Y}_n(0)) + \int_0^{\min(t,\tau_{n,\delta})} \tilde{L}f(\mathbf{Y}_n(s))ds + M(\min(t,\tau_{n,\delta}))
\tag{3.6}
$$

for $t \geq 0$, P-a.s., where $M(t)$ is a purely discontinuous martingale part.

(proof). Take $n_0 \in \mathbf{N}$ such that $\delta < T_n^*$ for $n \geq n_0$, which is possible from Theorem 1.1. First we show that there exists $n_2 \geq n_0$ such that for $n \geq n_2$ and $k = 0, \cdots, n-1$

$$\min(X_{n,k-1}(t), X_{n,k+1}(t)) \sin\theta_n > h_n(X_{n,k}(t)) \quad \text{for } t \in [0, \tau_{n,\delta}) \text{ P-a.s..} \tag{3.7.}$$

By (14) of Girão [4], $\{C_k(\delta)^{-1}\}_{k=n_0}^{\infty}$ defined in (2.4) is bounded. Therefore there exists $n_2 \geq n_0$ such that for $n \geq n_2$

$$C_n(\delta)(2n)^{-1}\sin\theta_n > (2n)C_n(\delta)^{-1}\theta_n T(n)^{-1}$$

by (1.4). From this, for $0 \leq k,\ i \leq n-1$ and $0 \leq t < \tau_{n,\delta}$

$$\begin{aligned} X_{n,k}(t)\sin\theta_n &> C_n(\delta)(2n)^{-1}\sin\theta_n > (2n)C_n(\delta)^{-1}\theta_n T(n)^{-1} \\ &> X_{n,i}(t)^{-1}\theta_n T(n)^{-1} > h_n(X_{n,i}(t)) \quad \text{(by (1.10)) P-a.s.,} \end{aligned} \tag{3.8.}$$

which implies (3.7).

By (2.2), we have the following:

$$\begin{aligned} \mathbf{Y}_n(s) = &-x_1\sum_{k=0}^{n-1}\mathbf{e}_{2k} + (x_1\cot\theta_n - x_3(\sin\theta_n)^{-1})\sum_{k=0}^{n-1}\mathbf{e}_{2k+1} \\ &+\sum_{i=1}^{n-1}x_{2i}[-(\sin(i\theta_n))\sum_{k=i}^{n-1}\mathbf{e}_{2k} + (\cos(i\theta_n))\sum_{k=i}^{n-1}\mathbf{e}_{2k+1}], \end{aligned} \tag{3.9.}$$

with $x_{2i} = X_{n,i}(s)$ and $x_{2i+1} = D_{n,i}(s)$ $(i = 0, \cdots, n-1)$. Therefore, from (1.8) and (3.7), by the Ito formula, the time- derivative of the continuous bounded variation part of $f(\mathbf{Y}_n(\min(t, \tau_{n,\delta})))$ can be written in the following manner:

$$T(n)(\sin\theta_n)[\theta_n\cos^2(\theta_n/2)]^{-1}\{(I) + (II) + (III) + (IV)\}. \tag{3.10.}$$

Here we put as follows:

$$(3.11).\ (I) = f(\mathbf{Y}_n(s) - h_n(X_{n,0}(s))(\sin\theta_n)^{-1}[-(\sin\theta_n)\sum_{k=1}^{n-1}\mathbf{e}_{2k} + (\cos\theta_n)\sum_{k=1}^{n-1}\mathbf{e}_{2k+1}$$

$$- (\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}]$$

$$- h_n(X_{n,0}(s))\sum_{k=0}^{n-1}\mathbf{e}_{2k} + h_n(X_{n,0}(s))(\cot\theta_n)\sum_{k=0}^{n-1}\mathbf{e}_{2k+1}) - f(\mathbf{Y}_n(s)),$$

$$(3.12).(II) = f(\mathbf{Y}_n(s) + 2(\cot\theta_n)h_n(X_{n,1}(s))[-(\sin\theta_n)\sum_{k=1}^{n-1}\mathbf{e}_{2k} + (\cos\theta_n)\sum_{k=1}^{n-1}\mathbf{e}_{2k+1}]$$

$$- h_n(X_{n,1}(s))(\sin\theta_n)^{-1}[-(\sin(2\theta_n))\sum_{k=2}^{n-1}\mathbf{e}_{2k} + (\cos(2\theta_n))\sum_{k=2}^{n-1}\mathbf{e}_{2k+1}]$$

$$- h_n(X_{n,1}(s))(\sin\theta_n)^{-1}\sum_{k=0}^{n-1}\mathbf{e}_{2k+1}) - f(\mathbf{Y}_n(s)),$$

$$(3.13).\qquad (III) = \sum_{i=2}^{n-2}\{f(\mathbf{Y}_n(s) + 2(\cot\theta_n)h_n(X_{n,i}(s))$$

$$\times [-(\sin(i\theta_n))\sum_{k=i}^{n-1}\mathbf{e}_{2k} + (\cos(i\theta_n))\sum_{k=i}^{n-1}\mathbf{e}_{2k+1}]$$

$$- h_n(X_{n,i}(s))(\sin\theta_n)^{-1}[-(\sin((i-1)\theta_n))\sum_{k=i-1}^{n-1}\mathbf{e}_{2k}$$

$$+ (\cos((i-1)\theta_n))\sum_{k=i-1}^{n-1}\mathbf{e}_{2k+1} - (\sin((i+1)\theta_n))\sum_{k=i+1}^{n-1}\mathbf{e}_{2k}$$

$$+ (\cos((i+1)\theta_n))\sum_{k=i+1}^{n-1}\mathbf{e}_{2k+1}]) - f(\mathbf{Y}_n(s))\},$$

$$(3.14).\qquad (IV) = f(\mathbf{Y}_n(s) + 2(\cot\theta_n)h_n(X_{n,n-1}(s))$$

$$\times [-(\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}]$$

$$- h_n(X_{n,n-1}(s))(\sin\theta_n)^{-1}[-(\sin((n-2)\theta_n))\sum_{k=n-2}^{n-1}\mathbf{e}_{2k}$$

$$+ (\cos((n-2)\theta_n))\sum_{k=n-2}^{n-1}\mathbf{e}_{2k+1}]) - f(\mathbf{Y}_n(s)).$$

Noticing that $X_{n,k}(t) = |Y_{n,k}(t) - Y_{n,k-1}(t)|$, the proof is over by the following (3.15)-(3.18).

On (I), we have the following:

(3.15).
$$\begin{aligned}
&- h_n(X_{n,0}(s))(\sin\theta_n)^{-1}[-(\sin\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k} + (\cos\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k+1} \\
&\quad - (\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}] \\
&\quad - h_n(X_{n,0}(s))\sum_{k=0}^{n-1} \mathbf{e}_{2k} + h_n(X_{n,0}(s))(\cot\theta_n)\sum_{k=0}^{n-1} \mathbf{e}_{2k+1} \\
&= h_n(X_{n,0}(s))(\sin\theta_n)^{-1}[(\sin\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k} - (\cos\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k+1} \\
&\quad + (\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} - (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1} \\
&\quad - (\sin\theta_n)\sum_{k=0}^{n-1} \mathbf{e}_{2k} + (\cos\theta_n)\sum_{k=0}^{n-1} \mathbf{e}_{2k+1}] \\
&= h_n(X_{n,0}(s))(\sin\theta_n)^{-1}[(\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} \\
&\quad - (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1} - (\sin\theta_n)\mathbf{e}_0 + (\cos\theta_n)\mathbf{e}_1].
\end{aligned}$$

On (II), we have the following:

(3.16).
$$\begin{aligned}
&2(\cot\theta_n)h_n(X_{n,1}(s))[-(\sin\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k} + (\cos\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k+1}] \\
&\quad - h_n(X_{n,1}(s))(\sin\theta_n)^{-1}[-(\sin(2\theta_n))\sum_{k=2}^{n-1} \mathbf{e}_{2k} + (\cos(2\theta_n))\sum_{k=2}^{n-1} \mathbf{e}_{2k+1}] \\
&\quad - h_n(X_{n,1}(s))(\sin\theta_n)^{-1}\sum_{k=0}^{n-1} \mathbf{e}_{2k+1} \\
&= h_n(X_{n,1}(s))(\sin\theta_n)^{-1}[-(\sin(2\theta_n))\sum_{k=1}^{n-1} \mathbf{e}_{2k} + 2(\cos^2\theta_n)\sum_{k=1}^{n-1} \mathbf{e}_{2k+1}] \\
&\quad + (\sin(2\theta_n))\sum_{k=2}^{n-1} \mathbf{e}_{2k} - (\cos(2\theta_n))\sum_{k=2}^{n-1} \mathbf{e}_{2k+1} - \sum_{k=0}^{n-1} \mathbf{e}_{2k+1}] \\
&= h_n(X_{n,1}(s))(\sin\theta_n)^{-1}[-(\sin(2\theta_n))\mathbf{e}_2 + (\cos(2\theta_n))\mathbf{e}_3 - \mathbf{e}_1],
\end{aligned}$$

by (3.4).

On (III), we have the following: for $i = 2, \cdots, n-2$,

$$
\begin{aligned}
(3.17). \quad & 2(\cot\theta_n) h_n(X_{n,i}(s))[-(\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k} + (\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,i}(s))(\sin\theta_n)^{-1}[-(\sin((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k+1} - (\sin((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k+1}] \\
= & \, h_n(X_{n,i}(s))(\sin\theta_n)^{-1}[-2\cos\theta_n(\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k} \\
& + 2\cos\theta_n(\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k+1} \\
& + (\sin((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k} - (\cos((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k+1} \\
& + (\sin((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k} - (\cos((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k+1}] \\
= & \, h_n(X_{n,i}(s))(\sin\theta_n)^{-1}[-(\sin((i+1)\theta_n))\mathbf{e}_{2i} + (\cos((i+1)\theta_n))\mathbf{e}_{2i+1} \\
& + (\sin((i-1)\theta_n))\mathbf{e}_{2(i-1)} - (\cos((i-1)\theta_n))\mathbf{e}_{2(i-1)+1}],
\end{aligned}
$$

by (3.4).

On (IV), we have the following in the same way as in (3.17).

$$
\begin{aligned}
(3.18). \quad & 2(\cot\theta_n) h_n(X_{n,n-1}(s))[-(\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} \\
& + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}] \\
& - h_n(X_{n,n-1}(s))(\sin\theta_n)^{-1}[-(\sin((n-2)\theta_n)) \sum_{k=n-2}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((n-2)\theta_n)) \sum_{k=n-2}^{n-1} \mathbf{e}_{2k+1}]
\end{aligned}
$$

$$= h_n(X_{n,n-1}(s))(\sin\theta_n)^{-1}[-(\sin(n\theta_n))\mathbf{e}_{2(n-1)} + (\cos(n\theta_n))\mathbf{e}_{2(n-1)+1}$$
$$+ (\sin((n-2)\theta_n))\mathbf{e}_{2(n-2)} - (\cos((n-2)\theta_n))\mathbf{e}_{2(n-2)+1}].$$

Q. E. D.

4. Proof of Theorem 2.1.

In this section we prove (2.7) to complete the proof of Theorem 2.1. For $n \geq n_2$, by Lemmas 3.1 and 3.2, the following holds: for $t \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$(4.1). \sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2$$
$$= 2\sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), \ell_{n,k+1}(s)^{-1}(\sin(k\theta_n), -\cos(k\theta_n))$$
$$+ \ell_{n,k}(s)^{-1}(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > (\cos^2(\theta_n/2))^{-1} ds$$
$$- 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1} \sum_{k=0}^{n-1} \int_0^t [< y_{n,k-1}(t) - Y_{n,k-1}(s),$$
$$h_n(X_{n,k}(s))(\sin((k-1)\theta_n), -\cos((k-1)\theta_n)) + < y_{n,k}(t) - Y_{n,k}(s)$$
$$, h_n(X_{n,k}(s))(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) >] ds$$
$$+ 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1}(\sin\theta_n) \sum_{k=0}^{n-1} \int_0^t |h_n(X_{n,k}(s))(\sin\theta_n)^{-1}|^2 ds + M(t),$$

where $M(t)$ denotes a purely discontinuous martingale part. Hence

$$(4.2). \quad \sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2$$
$$= 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s),$$
$$(\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1})(\sin(k\theta_n), -\cos(k\theta_n))$$
$$+ (\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1})(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > ds$$
$$+ 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(t) - Y_{n,k}(s), [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1}$$

$$+ X_{n,k+1}(s)^{-1}](\sin(k\theta_n), -\cos(k\theta_n)) + [T(n)h_n(X_{n,k}(s))\theta_n^{-1}$$

$$- X_{n,k}(s)^{-1}](\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > ds$$

$$+ 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1}(\sin\theta_n) \sum_{k=0}^{n-1} \int_0^t |h_n(X_{n,k}(s))(\sin\theta_n)^{-1}|^2 ds$$

$$+ M(t).$$

To complete the proof we show the following: for $t \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$\begin{aligned}
(4.3). \quad & 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), \\
& (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1})(\sin(k\theta_n), -\cos(k\theta_n)) \\
& + (\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1})(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > ds \\
& \le 2\sin^2\theta_n((\cos\theta_n)C_n(\delta)n^{-1}C_n(\delta)(2n)^{-1}\cos^2(\theta_n/2))^{-1} \\
& \times \int_0^t \sup_{0 \le u \le \min(s,\tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds,
\end{aligned}$$

$$\begin{aligned}
(4.4). \quad & 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1} \\
& + X_{n,k+1}(s)^{-1}](\sin(k\theta_n), -\cos(k\theta_n)) + [T(n)h_n(X_{n,k}(s))\theta_n^{-1} \\
& - X_{n,k}(s)^{-1}](\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > ds \\
& \le 2\int_0^t \sup_{0 \le u \le \min(s,\tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds \\
& + 2nt|\theta_n(T(n)[C_n(\delta)(2n)^{-1}]^3 \sin\theta_n)^{-1}|^2,
\end{aligned}$$

$$\begin{aligned}
(4.5). \quad & 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1}(\sin\theta_n) \sum_{k=0}^{n-1} \int_0^t |h_n(X_{n,k}(s))(\sin\theta_n)^{-1}|^2 ds \\
& \le 8T(n)^{-1}n^3 t\theta_n[C_n(\delta)^2 \cos^2(\theta_n/2)\sin\theta_n]^{-1},
\end{aligned}$$

and for $t \in [0, T_n^* - \delta]$,

(4.6). $$\{E[\sup_{0\le s\le \min(t,\tau_{n,\delta})}|M(s)|^2]\}^{1/2}$$
$$\le 32T(n)^{-1}n^2\theta_n(C_n(\delta)^2\sin\theta_n\cos^2(\theta_n/2))^{-1}$$
$$+6T(n)^{-2}n^3t|\theta_n(C_n(\delta)\sin\theta_n)^{-1}|^2$$
$$+3\int_0^t E[\sup_{0\le u\le \min(s,\tau_{n,\delta})}\sum_{k=0}^{n-1}|y_{n,k}(u)-Y_{n,k}(u)|^2]ds.$$

(4.3) is true, since for $s\in[0,\min(T_n^*-\delta,\tau_{n,\delta}))$,

$$\sum_{k=0}^{n-1} < y_{n,k}(s)-Y_{n,k}(s), (\ell_{n,k+1}(s)^{-1}-X_{n,k+1}(s)^{-1})(\sin(k\theta_n),-\cos(k\theta_n))$$
$$+(\ell_{n,k}(s)^{-1}-X_{n,k}(s)^{-1})(-\sin((k+1)\theta_n),\cos((k+1)\theta_n)) >$$
$$=\sum_{k=0}^{n-1} < y_{n,k}(s)-Y_{n,k}(s), (\sin(k\theta_n),-\cos(k\theta_n)) > (\ell_{n,k+1}(s)^{-1}-X_{n,k+1}(s)^{-1})$$
$$+\sum_{k=0}^{n-1} < y_{n,k-1}(s)-Y_{n,k-1}(s)+(\ell_{n,k}(s)-X_{n,k}(s))(-\sin(k\theta_n),\cos(k\theta_n)),$$
$$(-\sin((k+1)\theta_n),\cos((k+1)\theta_n)) > (\ell_{n,k}(s)^{-1}-X_{n,k}(s)^{-1})$$
$$=\sum_{k=0}^{n-1} < y_{n,k}(s)-Y_{n,k}(s), (\sin(k\theta_n)-\sin((k+2)\theta_n),-\cos(k\theta_n)+\cos((k+2)\theta_n)) >$$
$$\times(\ell_{n,k+1}(s)^{-1}-X_{n,k+1}(s)^{-1})$$
$$+\sum_{k=0}^{n-1}(\sin(k\theta_n)\sin((k+1)\theta_n)+\cos(k\theta_n)\cos((k+1)\theta_n))$$
$$\times(\ell_{n,k}(s)-X_{n,k}(s))(\ell_{n,k}(s)^{-1}-X_{n,k}(s)^{-1})$$
$$=\sum_{k=0}^{n-1} < y_{n,k}(s)-Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n),\sin((k+1)\theta_n)) >$$
$$\times(\ell_{n,k+1}(s)^{-1}-X_{n,k+1}(s)^{-1})$$
$$+\sum_{k=0}^{n-1}(\cos\theta_n)(\ell_{n,k}(s)-X_{n,k}(s))(\ell_{n,k}(s)^{-1}-X_{n,k}(s)^{-1})$$
$$\le\sum_{k=0}^{n-1}|y_{n,k}(s)-Y_{n,k}(s)|^2\sin^2\theta_n((\cos\theta_n)\ell_{n,k+1}(s)X_{n,k+1}(s))^{-1}$$
$$\le\sin^2\theta_n((\cos\theta_n)C_n(\delta)n^{-1}C_n(\delta)(2n)^{-1})^{-1}\sum_{k=0}^{n-1}|y_{n,k}(s)-Y_{n,k}(s)|^2$$

(see (2.4)). Here we used the following:

$$
\begin{aligned}
&< y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) > \\
&\qquad \times (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1}) \\
&\le 2|y_{n,k}(s) - Y_{n,k}(s)|(\sin\theta_n)[(\cos\theta_n)\ell_{n,k+1}(s)X_{n,k+1}(s)]^{-1/2} \\
&\qquad \times [(\cos\theta_n)(\ell_{n,k+1}(s)X_{n,k+1}(s))^{-1}]^{1/2}|\ell_{n,k+1}(s) - X_{n,k+1}(s)| \\
&\le |y_{n,k}(s) - Y_{n,k}(s)|^2(\sin^2\theta_n)[(\cos\theta_n)\ell_{n,k+1}(s)X_{n,k+1}(s)]^{-1} \\
&\qquad + (\cos\theta_n)(\ell_{n,k+1}(s)X_{n,k+1}(s))^{-1}(\ell_{n,k+1}(s) - X_{n,k+1}(s))^2.
\end{aligned}
$$

(4.4) is true, since for $s \in [0, \min(T_n^* - \delta, \tau_{n,\delta}))$,

$$
\begin{aligned}
&(\cos^2(\theta_n/2))^{-1}\sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1} \\
&\quad + X_{n,k+1}(s)^{-1}](\sin(k\theta_n), -\cos(k\theta_n)) + [T(n)h_n(X_{n,k}(s))\theta_n^{-1} \\
&\quad - X_{n,k}(s)^{-1}](\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > \\
&\le \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 \\
&\quad + 2^{-1}\sum_{k=0}^{n-1}[-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1} + X_{n,k+1}(s)^{-1}]^2(\cos^2(\theta_n/2))^{-2} \\
&\quad + 2^{-1}\sum_{k=0}^{n-1}[T(n)h_n(X_{n,k}(s))\theta_n^{-1} - X_{n,k}(s)^{-1}]^2(\cos^2(\theta_n/2))^{-2} \\
&\le \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 + \sum_{k=0}^{n-1} |\theta_n(T(n)X_{n,k}(s)^3\sin\theta_n)^{-1}|^2 \\
&\le \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 + n|\theta_n(T(n)[C_n(\delta)(2n)^{-1}]^3\sin\theta_n)^{-1}|^2
\end{aligned}
$$

(see (2.4)). Here we used the following: for $x > 0$,

$$
\begin{aligned}
&x^{-1} - T(n)h_n(x)\theta_n^{-1} \\
&= \theta_n(T(n)x^3\sin\theta_n)^{-1}\{2(1 + (1 + 4x^{-2}T(n)^{-1}\theta_n\cot\theta_n)^{1/2})^{-1}\}^2\cos\theta_n,
\end{aligned}
$$

and

$$\cos\theta_n = \cos^2(\theta_n/2) - \sin^2(\theta_n/2) < \cos^2(\theta_n/2).$$

(4.5) is true, since for $s \in [0, \min(T_n^* - \delta, \tau_{n,\delta}))$,

$$\sum_{k=0}^{n-1} |h_n(X_{n,k}(s))(\sin\theta_n)^{-1}|^2 \leq \sum_{k=0}^{n-1} |\theta_n(T(n)X_{n,k}(s)\sin\theta_n)^{-1}|^2$$
$$\leq n|\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|^2$$

(see (2.4)). Here we used that $h_n(x) < \theta_n(T(n)x)^{-1}$ from (1.10).

(4.6) is true, since

$$\begin{aligned}
M(t) = &\sum_{0\leq s\leq t}\sum_{k=0}^{n-1}\{|y_{n,k}(s) - Y_{n,k}(s)|^2 - |y_{n,k}(s) - Y_{n,k}(s-)|^2\} \\
&- T(n)(\sin\theta_n)[\theta_n\cos^2(\theta_n/2)]^{-1}\sum_{i=0}^{n-1}\int_0^t 2[-h_n(X_{n,i}(s))(\sin\theta_n)^{-1} \\
&\quad \times (< y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) > \\
&+ < y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) >) \\
&+ |h_n(X_{n,i}(s))(\sin\theta_n)^{-1}|^2]ds,
\end{aligned}$$

and henceforth for $t \in [0, T_n^* - \delta]$,

$$
\begin{aligned}
&E[\sup_{0\le s\le \min(t,\tau_{n,\delta})} |M(s)|^2] \\
&\le 4E[|M(\min(t,\tau_{n,\delta}))|^2] \quad \text{(by Doob's inequality)} \\
&= 4T(n)(\sin\theta_n)(\theta_n\cos^2(\theta_n/2))^{-1}\sum_{i=0}^{n-1} E[\int_0^{\min(t,\tau_{n,\delta})} (-2h_n(X_{n,i}(s))(\sin\theta_n)^{-1} \\
&\quad [< y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) > \\
&\quad + < y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) >] \\
&\quad + 2|h_n(X_{n,i}(s))(\sin\theta_n)^{-1}|^2)^2 ds] \\
&\le 16T(n)(\sin\theta_n)(\theta_n\cos^2(\theta_n/2))^{-1}|\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|^2 \\
&\quad \times \sum_{i=0}^{n-1} E[\int_0^{\min(t,\tau_{n,\delta})} (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \\
&\quad + |\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|)^2 ds] \\
&\le \{(16T(n)(\sin\theta_n)(\theta_n\cos^2(\theta_n/2))^{-1}|\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|^2 \\
&\quad + \sum_{i=0}^{n-1} E[\int_0^{\min(t,\tau_{n,\delta})} (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \\
&\quad + |\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|)^2 ds])/2\}^2 \\
&\le \{(16T(n)(\sin\theta_n)(\theta_n\cos^2(\theta_n/2))^{-1}|\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|^2 \\
&\quad + 3nt|\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|^2 \\
&\quad + 6\sum_{i=0}^{n-1} E[\int_0^t |y_{n,i}(\min(s,\tau_{n,\delta})) - Y_{n,i}(\min(s,\tau_{n,\delta}))|^2 ds])/2\}^2.
\end{aligned}
$$

Here we considered as the following. For $s \in [0, \min(t, \tau_{n,\delta}))$,

$$\sum_{i=0}^{n-1}(-2h_n(X_{n,i}(s))(\sin\theta_n)^{-1}$$
$$[< y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) >$$
$$+ < y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) >]$$
$$+ 2|h_n(X_{n,i}(s))(\sin\theta_n)^{-1}|^2)^2$$
$$\leq \sum_{i=0}^{n-1} 4|h_n(X_{n,i}(s))(\sin\theta_n)^{-1}|^2(|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)|$$
$$+ |h_n(X_{n,i}(s))(\sin\theta_n)^{-1}|)^2$$
$$\leq \sum_{i=0}^{n-1} 4|\theta_n(T(n)X_{n,i}(s)\sin\theta_n)^{-1}|^2(|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)|$$
$$+ |\theta_n(T(n)X_{n,i}(s)\sin\theta_n)^{-1}|)^2 \quad \text{(from (1.10))}$$
$$\leq \sum_{i=0}^{n-1} 4|\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|^2(|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)|$$
$$+ |\theta_n(T(n)C_n(\delta)(2n)^{-1}\sin\theta_n)^{-1}|)^2 \quad \text{(from (2.4))}.$$

We also used the following: for $x, y, z \in \mathbf{R}$,

$$xy \leq \{(x+y)/2\}^2, \qquad (x+y+z)^2 \leq 3(x^2+y^2+z^2).$$

Q. E. D.

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