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Essential Spectrum of a Self-Adjoint Operator on an Abstract Hilbert Space of Fock Type and Applications to Quantum Field Hamiltonians

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Abstract

We establish general theorems on locating the essential spectrum of a self-adjoint operator of the form $A \otimes I + I \otimes d\Gamma(S) + H_I$ on the tensor product $\mathcal{H} \otimes \mathcal{F}_b(\mathcal{K})$ of a Hilbert space \mathcal{H} and the abstract Boson Fock space $\mathcal{F}_b(\mathcal{K})$ over a Hilbert space \mathcal{K} , where A is a self-adjoint operator on \mathcal{H} bounded from below, $d\Gamma(S)$ is the second quantization of a nonnegative self-adjoint operator S on \mathcal{K} and H_I is a symmetric operator on $\mathcal{H} \otimes \mathcal{F}_b(\mathcal{K})$. We then apply the theorems to the generalized spin-boson model (A. Arai and M. Hirokawa, *J. Funct. Anal.* **151** (1997), 455–503) and a general class of models of quantum particles coupled to a Bose field including the Pauli-Fierz model in nonrelativistic quantum electrodynamics.

Key words: essential spectrum, Fock space, second quantization, quantum field, spin-boson model, Pauli-Fierz model

1991 MSC: 47B25, 47N50, 81Q10, 81T08

1 Introduction and Main Theorems

Let \mathcal{K} be a Hilbert space and

$$\mathcal{F}_b(\mathcal{K}) := \bigoplus_{n=0}^{\infty} (\otimes_s^n \mathcal{K}) \quad (1.1)$$

be the Boson Fock space over \mathcal{K} , where $\otimes_s^n \mathcal{K}$ is the n -fold symmetric tensor product of \mathcal{K} with convention $\otimes_s^0 \mathcal{K} := \mathbf{C}$ [23, §II.4, Example 2]. Let \mathcal{H} be a Hilbert space. Then one can make a Hilbert space

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b(\mathcal{K}). \quad (1.2)$$

Let S be a nonnegative self-adjoint operator on \mathcal{K} with $\ker S = \{0\}$ and A be a self-adjoint operator on \mathcal{H} bounded from below. We denote by $d\Gamma(S)$ the second quantization of S [23, §VIII.10, Example 2]. Then

$$H_0 := A \otimes I + I \otimes d\Gamma(S) \quad (1.3)$$

is self-adjoint on its natural domain $D(H_0) := D(A \otimes I) \cap D(I \otimes d\Gamma(S))$ and bounded from below, where I denotes identity operator and $D(\cdot)$ operator domain. Let H_I be a symmetric operator on \mathcal{F} such that $D(H_0) \cap D(H_I)$ is dense in \mathcal{F} . Then we are concerned with the symmetric operator

$$H := H_0 + H_I \quad (1.4)$$

on \mathcal{F} with $D(H) = D(H_0) \cap D(H_I)$. This operator is an abstract form of Hamiltonians of various quantum field models (see Sections 3 and 4 below). In this paper, we establish, in the case where H is self-adjoint, theorems on locating the essential spectrum of H and apply them to Hamiltonians of quantum field models.

As is well known, a standard method to locate the essential spectrum of the Hamiltonian of a quantum field model is to construct asymptotic fields (e.g., [15, 19] and references therein) or to discretize the Hamiltonian (e.g., [13] and [5, §3], [7]). As is seen below (Theorems 1.2 and 1.3), the abstract theorems we prove in the present paper are very simple compared with those methods. We regard this as a novelty and an advantage of our theorems. For recent works on spectral analysis of concrete quantum field Hamiltonians, which use different methods from ours, see [6, 8, 9, 10, 12, 14, 16, 17, 18, 21, 26].

To state the theorems, we introduce some symbols. We denote the inner product and the norm of a Hilbert space \mathcal{X} by $(\cdot, \cdot)_{\mathcal{X}}$ (complex linear in the second variable) and $\|\cdot\|_{\mathcal{X}}$ respectively, but, if there is no danger of confusion, then we write them simply as (\cdot, \cdot) and $\|\cdot\|$ respectively.

For convenience, we recall a notion of weak commutator of two operators [3].

Definition 1.1 Let \mathcal{X} be a Hilbert space. Let B and C be densely defined linear operators on \mathcal{X} with the following property: there exist a dense subspace \mathcal{W} of \mathcal{X} and a linear operator L such that $\mathcal{W} \subset D(L) \cap D(B) \cap D(B^*) \cap D(C) \cap D(C^*)$ and

$$(B^*\psi, C\phi) - (C^*\psi, B\phi) = (\psi, L\phi) \quad (1.5)$$

for all $\psi, \phi \in \mathcal{W}$. Then we say that the couple $\langle B, C \rangle$ has the weak commutator on \mathcal{W} , defined by

$$[B, C]_{\mathcal{W}, \mathcal{W}} := L|_{\mathcal{W}},$$

where, for a linear operator T and a subspace $D \subset D(T)$, $T|_D$ denotes the restriction of T to D .

Remark 1.1 (i) A linear operator L satisfying (1.5) is uniquely determined on \mathcal{W} , since \mathcal{W} is dense.

(ii) If the couple $\langle B, C \rangle$ has the weak commutator on \mathcal{W} , then B and C are closable, since $D(B^*)$ and $D(C^*)$ are dense.

(iii) For linear operators X and Y on a Hilbert space \mathcal{X} , their commutator $[X, Y]$ is defined by

$$D([X, Y]) := D(XY) \cap D(YX), \quad [X, Y] := XY - YX. \quad (1.6)$$

If X and Y are bounded with $D(X) = D(Y) = \mathcal{X}$, then, for every dense subspace \mathcal{W} , the couple $\langle X, Y \rangle$ has the weak commutator on \mathcal{W} and $[X, Y]_{\mathcal{W}, \mathcal{W}} = [X, Y]|_{\mathcal{W}}$.

The case where B and C has the weak commutator on \mathcal{W} as in Definition 1.1 is equivalent to the case where the commutator $[B, C]$ in the sense of sesquilinear form on $\mathcal{W} \times \mathcal{W}$ defines a linear operator.

We denote by $a(f)$ the annihilation operator on $\mathcal{F}_b(\mathcal{K})$ with “test vector” $f \in \mathcal{K}$ [24, §X.7]. By definition, $a(f)$ is densely defined, closed and antilinear in f .

We now state our hypotheses.

(H.1) H is self-adjoint and bounded from below.

(H.2) For each $f \in D(S) \cap D(S^{-1/2})$, the couple $\langle H_I, I \otimes a(f)^* \rangle$ has the weak commutator $[H_I, I \otimes a(f)^*]_{\mathcal{W}, D(H)}$ on $D(H)$. Moreover, for all $\Psi \in D(H)$ and all sequences $\{f_n\}_{n=1}^{\infty} \subset D(S) \cap D(S^{-1/2})$ such that $\|f_n\| = 1$, $n \geq 1$ and $w\text{-}\lim_{n \rightarrow \infty} f_n = 0$ ($w\text{-}\lim$ denotes weak limit),

$$\lim_{n \rightarrow \infty} [H_I, I \otimes a(f_n)^*]_{\mathcal{W}, D(H)} \Psi = 0.$$

Remark 1.2 $D(H) \subset D(H_I)$ and, for all $f \in D(S^{-1/2})$, $D(H) \subset D(H_0) \subset D(I \otimes d\Gamma(S)) \subset D(I \otimes a(f)) \cap D(I \otimes a(f)^*)$ (see Lemma 2.1 in Section 2 below).

For a self-adjoint operator T , we denote by $\sigma(T)$ and by $\sigma_{\text{ess}}(T)$ the spectrum and the essential spectrum of T respectively. If T is bounded from below, then we define

$$E_0(T) := \inf \sigma(T), \quad (1.7)$$

the ground state energy of T

Theorem 1.2 Assume (H.1) and (H.2). Then

$$\{E_0(H) + \lambda | \lambda \in \overline{\sigma_{\text{ess}}(S) \setminus \{0\}}\} \subset \sigma_{\text{ess}}(H), \quad (1.8)$$

where $\overline{\sigma_{\text{ess}}(S) \setminus \{0\}}$ is the closure of the set $\sigma_{\text{ess}}(S) \setminus \{0\}$.

As a corollary to this theorem, we have the following.

Theorem 1.3 Assume (H.1) and (H.2). Let

$$\sigma(S) = [0, \infty). \quad (1.9)$$

Then

$$\sigma(H) = \sigma_{\text{ess}}(H) = [E_0(H), \infty). \quad (1.10)$$

Remark 1.3 A non-zero vector in $\ker(H - E_0(H))$ is called a *ground state* of H . Theorems 1.2 and 1.3 improve previously obtained results in [4] where the existence of a ground state of H is additionally assumed.

Remark 1.4 It is an interesting problem to clarify for what kind of classes of A, S and H_I the converse inclusion relation of (1.8) holds, so that

$$\{E_0(H) + \lambda | \lambda \in \overline{\sigma_{\text{ess}}(S) \setminus \{0\}}\} = \sigma_{\text{ess}}(H). \quad (1.11)$$

This should be a Fock space version of the HVZ theorem for N -body Schrödinger operators [25, Theorem XIII.17].

Remark 1.5 In applications to quantum field models (see Sections 3 and 4 below), condition (1.9) is satisfied by *massless* quantum field models. Thus, as far as Hamiltonians of massless quantum field models obeying (H.1), (H.2) and (1.9) are concerned, their spectra are completely identified as (1.10).

The present paper is organized as follows. In Section 2 we prove Theorems 1.2 and 1.3. Sections 3 and 4 are devoted to discussions of applications of these results to Hamiltonians of quantum field models. In Section 3 we consider the generalized spin-boson model introduced in [5]. In Section 4 we define a class of models of N quantum particles interacting with a Bose field in an abstract manner. This class gives a most general form unifying models of nonrelativistic quantum particles coupled to a Bose field including the Pauli-Fierz model without the dipole approximation ([22], [7], [11], [21]) and models of Nelson's type ([20], [27]). We prove self-adjointness of the Hamiltonian of each model in the class and locate the essential spectrum of it.

2 Proof of the Main Theorems

We first recall some basic facts. The following fact is well known (e.g., [2, Lemma 2.1]): For all $f \in D(S^{-1/2})$, $D(d\Gamma(S)^{1/2}) \subset D(a(f)^*) \cap D(a(f))$ and, for all $\Psi \in D(d\Gamma(S)^{1/2})$,

$$\|a(f)\Psi\| \leq \|S^{-1/2}f\| \|d\Gamma(S)^{1/2}\Psi\|, \quad (2.1)$$

$$\|a(f)^*\Psi\|^2 \leq \|S^{-1/2}f\|^2 \|d\Gamma(S)^{1/2}\Psi\|^2 + \|f\|^2 \|\Psi\|^2. \quad (2.2)$$

The operator

$$\widetilde{H}_0 := H_0 - E_0(A) \quad (2.3)$$

is nonnegative and self-adjoint. It is easy to see that $D(I \otimes d\Gamma(S)^{1/2}) \supset D(\widetilde{H}_0^{1/2})$ and, for all $\Psi \in D(\widetilde{H}_0^{1/2})$,

$$\|I \otimes d\Gamma(S)^{1/2}\Psi\| \leq \|\widetilde{H}_0^{1/2}\Psi\|. \quad (2.4)$$

Combining these facts with (2.1) and (2.2), we obtain the following lemma.

Lemma 2.1 For all $f \in D(S^{-1/2})$, $D(\widetilde{H}_0^{1/2}) \subset D(I \otimes a(f)^*) \cap D(I \otimes a(f))$ and, for all $\Psi \in D(\widetilde{H}_0^{1/2})$,

$$\|I \otimes a(f)\Psi\| \leq \|S^{-1/2}f\| \|\widetilde{H}_0^{1/2}\Psi\|, \quad (2.5)$$

$$\|I \otimes a(f)^*\Psi\| \leq \|S^{-1/2}f\| \|\widetilde{H}_0^{1/2}\Psi\| + \|f\| \|\Psi\|. \quad (2.6)$$

Let $\Omega_0 := \{1, 0, 0, \dots\} \in \mathcal{F}_b(\mathcal{K})$ be the Fock vacuum in $\mathcal{F}_b(\mathcal{K})$. For a subspace $\mathcal{D} \subset \mathcal{K}$, we denote by $\mathcal{F}_{0,\text{fin}}(\mathcal{D})$ the subspace algebraically spanned by Ω_0 and all vectors of the form

$$a(f_1)^* \cdots a(f_n)^* \Omega_0, \quad n \geq 1, \quad f_j \in \mathcal{D}, \quad j = 1, \dots, n.$$

If M is dense in \mathcal{K} , then $\mathcal{F}_{0,\text{fin}}(\mathcal{D})$ is dense in $\mathcal{F}_b(\mathcal{K})$.

It is well known (or easy to see) that, for all $f \in D(S)$, $\mathcal{F}_{0,\text{fin}}(D(S)) \subset D(d\Gamma(S)a(f)^*) \cap D(a(f)^*d\Gamma(S))$ and, for all $\Psi \in \mathcal{F}_{0,\text{fin}}(D(S))$

$$[d\Gamma(S), a(f)^*]\Psi = a(Sf)^*\Psi. \quad (2.7)$$

Lemma 2.2 *Let $f \in D(S^{-1/2}) \cap D(S)$. Then the couple $\langle H_0, I \otimes a(f)^* \rangle$ has the weak commutator on $D(H_0)$ and*

$$[H_0, I \otimes a(f)^*]_{w, D(H_0)} = I \otimes a(Sf)^*|_{D(H_0)}. \quad (2.8)$$

Proof. By Lemma 2.1, $D(H_0) \subset D(I \otimes a(f)) \cap D(I \otimes a(f)^*)$. Let $\Psi \in D(A) \otimes_{\text{alg}} \mathcal{F}_{0,\text{fin}}(D(S))$, where \otimes_{alg} means algebraic tensor product. Then, by (2.7), $\Psi \in D(H_0 I \otimes a(f)^*) \cap D(I \otimes a(f)^* H_0)$ and $[H_0, I \otimes a(f)^*]\Psi = I \otimes a(Sf)^*\Psi$. Hence, for all $\Phi \in D(H_0)$, we have

$$(H_0 \Phi, I \otimes a(f)^*\Psi) - (I \otimes a(f)\Phi, H_0 \Psi) = (\Phi, I \otimes a(Sf)^*\Psi). \quad (2.9)$$

Since $D(A) \otimes_{\text{alg}} \mathcal{F}_{0,\text{fin}}(D(S))$ is a core of H_0 and we have Lemma 2.1, (2.9) extends, via a simple limiting argument, to all $\Psi \in D(H_0)$. Thus the desired result follows. \blacksquare

We set

$$\widehat{H} := H - E_0(H) \geq 0. \quad (2.10)$$

Lemma 2.3 *Let $f \in D(S^{-1/2})$. Then $D(\widehat{H}^{1/2}) \subset D(I \otimes a(f)^*)$ and there exists a constant $c > 0$ independent of f such that, for all $\Psi \in D(\widehat{H}^{1/2})$ and*

$$\|1 \otimes a(f)^*\Psi\| \leq c \|S^{-1/2} f\| \|(\widehat{H} + 1)^{1/2} \Psi\| + \|f\| \|\Psi\|.$$

Proof. By (H.1), \widehat{H} is closed on $D(\widehat{H}) = D(H_0) \cap D(H_I)$. Hence, by the closed graph theorem, there exists a constant $c_1 > 0$ such that, for all $\Psi \in D(\widehat{H})$,

$$\|\widetilde{H}_0 \Psi\| + \|H_I \Psi\| \leq c_1 \|(\widehat{H} + 1) \Psi\|.$$

This means that \widetilde{H}_0 is \widehat{H} -bounded. Hence $\widetilde{H}_0^{1/2}$ is $\widehat{H}^{1/2}$ -bounded [24, Theorem X.18(a)]. Namely there exists a constant $c > 0$ such that, for all $\Psi \in D(\widehat{H}^{1/2})$

$$\|\widetilde{H}_0^{1/2} \Psi\| \leq c \|(\widehat{H} + 1)^{1/2} \Psi\|,$$

which, combined with Lemma 2.1, implies the desired result. \blacksquare

The following criterion on essential spectrum (Weyl's criterion) is well known (e.g., [1, Lemma 5.19]).

Lemma 2.4 *Let T be a self-adjoint operator on a Hilbert space. Then $\lambda \in \sigma_{\text{ess}}(T)$ if and only if there exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T)$ such that $\|\psi_n\| = 1$, $w\text{-}\lim_{n \rightarrow \infty} \psi_n = 0$ and $\lim_{n \rightarrow \infty} (T - \lambda)\psi_n = 0$.*

We use also the following fact, a strengthened version of the necessary condition for $\lambda \in \sigma_{\text{ess}}(T)$ in Lemma 2.4.

Lemma 2.5 *Let T be a nonnegative self-adjoint operator on a Hilbert space with $\ker T = \{0\}$ and $a > 0$. Let $\lambda \in \sigma_{\text{ess}}(T) \setminus \{0\}$. Then there exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T) \cap D(T^{-a})$ such that $\|\psi_n\| = 1$, $n \geq 1$, $\sup_{n \geq 1} \|T^{-a}\psi_n\| < \infty$, $w\text{-}\lim_{n \rightarrow \infty} \psi_n = 0$, $\lim_{n \rightarrow \infty} (T - \lambda)\psi_n = 0$ and $\lim_{n \rightarrow \infty} T^{-a}(T - \lambda)\psi_n = 0$.*

Proof. There exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T)$ with the properties described in Lemma 2.4. Since $\lambda > 0$, we can choose $\{\psi_n\}_{n=1}^{\infty}$ such that the support of the measure $\|E_T(\cdot)\psi_n\|^2$ ($E_T(\cdot)$ is the spectral measure of T) is included in an interval $[\lambda - \delta, \lambda + \delta]$ with $\delta > 0$ a constant such that $\lambda > \delta$. Hence

$$\int_{\mathbf{R}} \mu^{-2a} d\|E_T(\mu)\psi_n\|^2 = \int_{[\lambda - \delta, \lambda + \delta]} \mu^{-2a} d\|E_T(\mu)\psi_n\|^2 \leq \frac{1}{(\lambda - \delta)^{2a}} < \infty.$$

Hence $\psi_n \in D(T^{-a})$ and $\sup_{n \geq 1} \|T^{-a}\psi_n\| < \infty$. A similar calculation shows that $\|T^{-a}(T - \lambda)\psi_n\| \leq (\lambda - \delta)^{-a} \|(T - \lambda)\psi_n\| \rightarrow 0$ ($n \rightarrow \infty$). \blacksquare

Proof of Theorem 1.2

We denote by P the spectral measure of \widehat{H} . Let $0 < \varepsilon \leq 1$. Since $0 \in \sigma(\widehat{H})$, it follows that $P([0, \varepsilon)) \neq 0$. Hence there exists a vector Ψ_ε such that $\|\Psi_\varepsilon\| = 1$ and $\Psi_\varepsilon \in \text{Ran}(P([0, \varepsilon)))$, where $\text{Ran}(T)$ denotes the range of an operator T . Let $\Psi \in D(H)$ and $f \in D(S) \cap D(S^{-1/2})$. Then we have by Lemma 2.2 and Hypotheses (H.1) and (H.2)

$$\begin{aligned} & (\widehat{H}\Psi, I \otimes a(f)^*\Psi_\varepsilon) \\ &= \{(\widehat{H}\Psi, I \otimes a(f)^*\Psi_\varepsilon) - (I \otimes a(f)\Psi, \widehat{H}\Psi_\varepsilon)\} + (I \otimes a(f)\Psi, \widehat{H}\Psi_\varepsilon) \\ &= (\Psi, \{I \otimes a(Sf)^* + [H_I, I \otimes a(f)^*]_{w, D(H)}\} \Psi_\varepsilon) + (I \otimes a(f)\Psi, \widehat{H}\Psi_\varepsilon). \end{aligned}$$

Note that $\widehat{H}\Psi_\varepsilon \in D(\widehat{H}) \subset D(H_0)$. Hence $\widehat{H}\Psi_\varepsilon \in D(I \otimes a(f)^*)$, so that $(I \otimes a(f)\Psi, \widehat{H}\Psi_\varepsilon) = (\Psi, I \otimes a(f)^*\widehat{H}\Psi_\varepsilon)$. Hence it follows that $I \otimes a(f)^*\Psi_\varepsilon \in D(\widehat{H})$ and

$$\widehat{H}(I \otimes a(f)^*)\Psi_\varepsilon = \{I \otimes a(Sf)^* + [H_I, I \otimes a(f)^*]_{w, D(H)}\} \Psi_\varepsilon + I \otimes a(f)^*\widehat{H}\Psi_\varepsilon. \quad (2.11)$$

Now let $\lambda \in \sigma_{\text{ess}}(S) \setminus \{0\}$. Then, by Lemma 2.5, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset D(S) \cap D(S^{-1/2})$ such that $\|f_n\| = 1$, $w\text{-}\lim_{n \rightarrow \infty} f_n = 0$, $K := \sup_{n \geq 1} \|S^{-1/2}f_n\| < \infty$, $\lim_{n \rightarrow \infty} (S - \lambda)f_n = 0$ and $\lim_{n \rightarrow \infty} S^{-1/2}(S - \lambda)f_n = 0$. In general, we can show by using the canonical commutation relations of $a(f)$ and $a(g)^*$ ($f, g \in \mathcal{K}$)

$$[a(f), a(g)^*] = (f, g), \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0, \quad (2.12)$$

holding on the subspace of finite particle vectors

$$\mathcal{F}_0(\mathcal{K}) := \{\psi = \{\psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_b(\mathcal{K}) \mid \psi_n = 0 \text{ for all but finitely many } n\text{'s}\} \quad (2.13)$$

of $\mathcal{F}_b(\mathcal{K})$ and a limiting argument that every $\Psi \in D(I \otimes a(f))$ is in $D(I \otimes a(f)^*)$ and

$$\|I \otimes a(f)^* \Psi\|^2 = \|f\|^2 \|\Psi\|^2 + \|I \otimes a(f) \Psi\|^2. \quad (2.14)$$

Hence

$$\|I \otimes a(f)^* \Psi\|^2 \geq \|f\|^2 \|\Psi\|^2. \quad (2.15)$$

Using this inequality, we see that the vector $I \otimes a(f_n)^* \Psi_\varepsilon$ is not zero with

$$C_{n,\varepsilon} := \|I \otimes a(f_n)^* \Psi_\varepsilon\| \geq 1. \quad (2.16)$$

Hence we can define a unit vector

$$\Psi_{n,\varepsilon} := C_{n,\varepsilon}^{-1} I \otimes a(f_n)^* \Psi_\varepsilon.$$

Then we have by (2.11)

$$(\widehat{H} - \lambda) \Psi_{n,\varepsilon} = C_{n,\varepsilon}^{-1} \left\{ I \otimes a((S - \lambda)f_n)^* \Psi_\varepsilon + [H_I, I \otimes a(f_n)^*]_{w,D(H)} \Psi_\varepsilon + I \otimes a(f_n)^* \widehat{H} \Psi_\varepsilon \right\}$$

Hence

$$\begin{aligned} \|(\widehat{H} - \lambda) \Psi_{n,\varepsilon}\| &\leq \|S^{-1/2}(S - \lambda)f_n\| \|I \otimes d\Gamma(S)^{1/2} \Psi_\varepsilon\| \\ &\quad + \|(S - \lambda)f_n\| + \|[H_I, I \otimes a(f_n)^*]_{w,D(H)} \Psi_\varepsilon\| + \|I \otimes a(f_n)^* \widehat{H} \Psi_\varepsilon\|. \end{aligned}$$

We have by Lemma 2.3

$$\begin{aligned} \|I \otimes a(f_n)^* \widehat{H} \Psi_\varepsilon\| &\leq c \|S^{-1/2} f_n\| \|(\widehat{H} + 1)^{1/2} \widehat{H} \Psi_\varepsilon\| + \|f_n\| \|\widehat{H} \Psi_\varepsilon\| \\ &\leq cK \|(\widehat{H} + 1)^{1/2} \widehat{H} \Psi_\varepsilon\| + \|\widehat{H} \Psi_\varepsilon\|. \end{aligned}$$

By the functional calculus, we have

$$\|(\widehat{H} + 1)^{1/2} \widehat{H} \Psi_\varepsilon\| \leq \varepsilon(\varepsilon + 1)^{1/2} \leq \sqrt{2}\varepsilon, \quad \|\widehat{H} \Psi_\varepsilon\| \leq \varepsilon.$$

Hence $\|I \otimes a(f_n)^* \widehat{H} \Psi_\varepsilon\| \leq C\varepsilon$ with $C > 0$ a constant independent of n and ε . Hence

$$\begin{aligned} \|(\widehat{H} - \lambda) \Psi_{n,\varepsilon}\| &\leq \|S^{-1/2}(S - \lambda)f_n\| \|I \otimes d\Gamma(S)^{1/2} \Psi_\varepsilon\| \\ &\quad + \|(S - \lambda)f_n\| + \|[H_I, I \otimes a(f_n)^*]_{w,D(H)} \Psi_\varepsilon\| + C\varepsilon, \end{aligned}$$

which implies that $\limsup_{n \rightarrow \infty} \|(\widehat{H} - \lambda) \Psi_{n,\varepsilon}\| \leq C\varepsilon$. Hence $\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \|(\widehat{H} - \lambda) \Psi_{n,\varepsilon}\| = 0$. This implies that there exist sequences $\{n_k\}_k$ and $\{\varepsilon_k\}_k$ such that $n_k \rightarrow \infty$, $\varepsilon_k \downarrow 0$ as $k \rightarrow \infty$ and

$$\lim_{k \rightarrow \infty} \|(\widehat{H} - \lambda) \Psi_{n_k, \varepsilon_k}\| = 0. \quad (2.17)$$

To show that

$$w\text{-}\lim_{n \rightarrow \infty} \Psi_{n_k, \varepsilon_k} = 0, \quad (2.18)$$

it is sufficient to prove that, for all $\Psi \in \mathcal{H} \otimes_{\text{alg}} \mathcal{F}_{0,\text{fin}}(\mathcal{K})$, $\lim_{n \rightarrow \infty} (\Psi, \Psi_{n_k, \varepsilon_k}) = 0$. So let $\Psi = \psi \otimes a(g_1)^* \cdots a(g_m)^* \Omega_0$ ($\psi \in \mathcal{H}$, $g_j \in \mathcal{K}$, $j = 1, \dots, m$). Then we have

$$\begin{aligned} |(\Psi, \Psi_{n_k, \varepsilon_k})| &\leq \left| \sum_{j=1}^m (g_j, f_{n_k}) (\psi \otimes a(g_1)^* \cdots a(g_{j-1})^* a(g_{j+1})^* \cdots a(g_m)^* \Omega_0, \Psi_{\varepsilon_k}) \right| \\ &\leq \sum_{j=1}^m |(g_j, f_{n_k})| \|\psi \otimes a(g_1)^* \cdots a(g_{j-1})^* a(g_{j+1})^* \cdots a(g_m)^* \Omega_0\| \\ &\rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

It follows from (2.17), (2.18) and Lemma 2.4 that $\lambda \in \sigma_{\text{ess}}(\widehat{H})$. Hence $\sigma_{\text{ess}}(S) \setminus \{0\} \subset \sigma_{\text{ess}}(\widehat{H})$. Since the essential spectrum of a self-adjoint operator is closed, it follows that $\sigma_{\text{ess}}(S) \setminus \{0\} \subset \sigma_{\text{ess}}(\widehat{H})$. Thus (1.8) follows. \blacksquare

Proof of Theorem 1.3

By the present assumption, $\overline{\sigma_{\text{ess}}(S) \setminus \{0\}} = [0, \infty)$. Hence, by Theorem 1.2, $[E_0(H), \infty) \subset \sigma_{\text{ess}}(H)$. On the other hand, it is obvious that $\sigma_{\text{ess}}(H) \subset \sigma(H) \subset [E_0(H), \infty)$. Thus (1.10) follows.

3 The Generalized Spin-Boson Model

In this section we apply Theorems 1.2 and 1.3 to the Hamiltonian of the generalized spin-boson (GSB) model which was introduced in [5] as an abstract unification of some nonrelativistic quantum field models.

We first present an abstract fact on weak commutators.

Lemma 3.1 *Let \mathcal{X} be a Hilbert space and \mathcal{W} be a dense subspace of \mathcal{X} . Let B and C be densely defined linear operators on \mathcal{X} such that $\mathcal{W} \subset D(B) \cap D(B^*) \cap D(C) \cap D(C^*)$. Suppose that there exist a densely defined closed linear operator T on \mathcal{X} and a core D_c of T with the following properties:*

- (i) $D_c \subset \mathcal{W} \subset D(T)$.
- (ii) B and C are T -bounded on D_c .
- (iii) $D_c \subset D(BC) \cap D(CB)$ (so that $[B, C]$ is defined on D_c) and

$$L := [B, C]|_{D_c}$$

is T -bounded on D_c .

- (iv) $D(B^*C^*) \cap D(C^*B^*)$ is dense.

Then L is closable with $D(T) \subset D(\overline{L})$ and the couple $\langle B, C \rangle$ has the weak commutator on \mathcal{W} given by

$$[B, C]_{\mathcal{W}, \mathcal{W}} = \overline{L}|_{\mathcal{W}}.$$

Proof. By property (iii), L is densely defined and $L^* \supset [C^*, B^*]$. By (iv), $D([C^*, B^*])$ is dense. Hence $D(L^*)$ is dense. Therefore L is closable. By the T -boundedness of L on D_c , there exist constants $a, b \geq 0$ such that, for all $\eta \in D_c$,

$$\|L\eta\| \leq a\|T\eta\| + b\|\eta\|. \quad (3.1)$$

Since D_c is a core of T , it follows from a limiting argument that $D(T) \subset D(\bar{L})$ and (3.1) extends to all $\eta \in D(T)$, L being replaced by \bar{L} .

Let $\phi \in \mathcal{W}$. Then, by (i), there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset D_c$ such that $\phi_n \rightarrow \phi$, $T\phi_n \rightarrow T\phi$ ($n \rightarrow \infty$). For all $\psi \in \mathcal{W}$, we have $(B^*\psi, C\phi_n) - (C^*\psi, B\phi_n) = (\psi, L\phi_n)$. The T -boundedness and the closability of B and C imply that $B\phi_n \rightarrow \bar{B}\phi = B\phi$ and $C\phi_n \rightarrow \bar{C}\phi = C\phi$ as $n \rightarrow \infty$. By (3.1) with L replaced by \bar{L} and $\eta \in D(T)$, we have $L\phi_n \rightarrow \bar{L}\phi$ ($n \rightarrow \infty$). Hence we obtain $(B^*\psi, C\phi) - (C^*\psi, B\phi) = (\psi, \bar{L}\phi)$. Thus the desired result follows. \blacksquare

We carry over the notation in Section 1. For each $f \in \mathcal{K}$, the operator

$$\phi(f) := \frac{1}{\sqrt{2}}(a(f)^* + a(f)), \quad (3.2)$$

called the Segal field operator, is essentially self-adjoint on $\mathcal{F}_{0,\text{fin}}(\mathcal{K})$ [24, §X.7, Theorem X.41]. We denote its closure by the same symbol. We consider the case where H_I is of the form

$$H_I = H_I^{\text{GSB}} := \alpha \sum_{j=1}^J B_j \otimes \phi(\lambda_j). \quad (3.3)$$

Here B_j ($j = 1, \dots, J; J < \infty$) are symmetric operators on \mathcal{H} , $\lambda_j \in \mathcal{K}$ ($j = 1, \dots, J$) and $\alpha \in \mathbf{R}$ is a coupling parameter. The Hamiltonian of the model is given by

$$H_{\text{GSB}} := H_0 + H_I^{\text{GSB}} \quad (3.4)$$

We remark that, in [5], the special choice $\mathcal{K} = L^2(\mathbf{R}^\nu)$ ($\nu \in \mathbf{N}$) is made. But, here, we proceed more generally.

We need the following conditions:

(GSB.1) $\lambda_j \in D(S^{-1/2})$, $j = 1, \dots, J$.

(GSB.2) Let

$$\tilde{A} := A - E_0(A) \geq 0.$$

Then $D(\tilde{A}^{1/2}) \subset D(B_j)$, $j = 1, \dots, J$, and there exist constants $a_j \geq 0$, $b_j \geq 0$, $j = 1, \dots, J$, such that, for all $u \in D(\tilde{A}^{1/2})$,

$$\|B_j u\|^2 \leq a_j^2 \|\tilde{A}^{1/2} u\|^2 + b_j^2 \|u\|^2, \quad j = 1, \dots, J.$$

(GSB.3)

$$|\alpha| \left(\sum_{j=1}^J a_j \left\| S^{-1/2} \lambda_j \right\| \right) < 1.$$

It is shown that, under conditions (GSB.1)–(GSB.3), $D(H_0) \subset D(H_I^{\text{GSB}})$, H_{GSB} is self-adjoint on $D(H_0)$ and bounded from below.

For each $f \in \mathcal{K}$, we define

$$L(f) := \frac{\alpha}{\sqrt{2}} \sum_{j=1}^J (\lambda_j, f)_{\mathcal{K}} B_j \otimes I. \quad (3.5)$$

Lemma 3.2 *Assume (GSB.1)–(GSB.3). Then, for all $f \in D(S^{-1/2})$, the couple $\langle H_I^{\text{GSB}}, I \otimes a(f)^* \rangle$ has the weak commutator on $D(H_0)$ and*

$$[H_I^{\text{GSB}}, I \otimes a(f)^*]_{\text{w}, D(H_0)} = L(f)|_{D(H_0)}. \quad (3.6)$$

In particular, for all sequences $\{f_n\}_{n=1}^{\infty} \subset D(S) \cap D(S^{-1/2})$ such that $\|f_n\| = 1$, $n \geq 1$ and $\text{w-lim}_{n \rightarrow \infty} f_n = 0$ and all $\Psi \in D(H_0)$,

$$\lim_{n \rightarrow \infty} [H_I^{\text{GSB}}, I \otimes a(f_n)^*]_{\text{w}, D(H_0)} \Psi = 0. \quad (3.7)$$

Proof. The dense subspace $D_c := D(A) \otimes_{\text{alg}} \mathcal{F}_{0, \text{fin}}(D(S))$ is a core of H_0 . It is easy to see that

$$D_c \subset D(H_I^{\text{GSB}} I \otimes a(f)^*) \cap D(I \otimes a(f)^* H_I^{\text{GSB}}) \cap D(H_I^{\text{GSB}} I \otimes a(f)) \cap D(I \otimes a(f) H_I^{\text{GSB}}).$$

By (2.12), we see that, for all $\Psi \in D_c$,

$$[H_I^{\text{GSB}}, I \otimes a(f)^*] \Psi = L(f) \Psi.$$

The operators $I \otimes a(f)^*$ and H_I^{GSB} are H_0 -bounded (cf. [5]). We can show also that $L(f)$ is H_0 -bounded. Hence we can apply Lemma 3.1 with $B = H_I^{\text{GSB}}$, $C = I \otimes a(f)^*$, $T = H_0$ and $\mathcal{W} = D(H_0)$ to obtain the first half of the lemma.

By (3.5), we have for all $\Psi \in D(H_0)$

$$\|L(f_n) \Psi\| \leq \frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^J |(\lambda_j, f_n)_{\mathcal{K}}| \|B_j \otimes I \Psi\|.$$

Since $(\lambda_j, f_n)_{\mathcal{K}} \rightarrow 0$ ($n \rightarrow \infty$), we have $\|L(f_n) \Psi\| \rightarrow 0$ ($n \rightarrow \infty$). Hence (3.7) follows. ■

Theorem 3.3 *Assume (GSB.1)–(GSB.3). Then*

$$\left\{ E_0(H_{\text{GSB}}) + \lambda \mid \lambda \in \overline{\sigma_{\text{ess}}(S) \setminus \{0\}} \right\} \subset \sigma_{\text{ess}}(H_{\text{GSB}}). \quad (3.8)$$

In particular, if $\sigma(S) = [0, \infty)$, then

$$\sigma(H_{\text{GSB}}) = \sigma_{\text{ess}}(H_{\text{GSB}}) = [E_0(H_{\text{GSB}}), \infty). \quad (3.9)$$

Proof. By Lemma 3.2, we can apply Theorems 1.2 and obtain (3.8). Eq.(3.9) follows from Theorem 1.3. ■

Remark 3.1 Consider the case where $\mathcal{K} = L^2(\mathbf{R}^\nu)$ and S is a multiplication operator by a nonnegative function $\omega : \mathbf{R}^\nu \rightarrow [0, \infty)$ with the following properties:

- (i) ω is continuous on \mathbf{R}^ν and $\lim_{|k| \rightarrow \infty} \omega(k) = \infty$.
- (ii) There exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma(1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbf{R}^\nu.$$

We set

$$m := \inf_{k \in \mathbf{R}^\nu} \omega(k) \geq 0. \quad (3.10)$$

Assume (GSB.1)–(GSB.3). Suppose that A has compact resolvent. Let $m > 0$. Then, the results obtained in [5] imply that H_{GSB} has purely discrete spectrum in $[E_0(H_{\text{GSB}}), E_0(H_{\text{GSB}}) + m)$ and

$$\sigma_{\text{ess}}(H_{\text{GSB}}) = [E_0(H_{\text{GSB}}) + m, \infty), \quad (3.11)$$

see [4]. In the proof of this fact, however, the existence of a ground state of H_{GSB} is used.

Let $m = 0$. Then we have $\sigma(S) = [0, \infty)$. Hence, by Theorem 3.3, we obtain

$$\sigma(H_{\text{GSB}}) = \sigma_{\text{ess}}(H_{\text{GSB}}) = [E_0(H_{\text{GSB}}), \infty). \quad (3.12)$$

Note that, to obtain (3.12), we need neither the additional condition $\lambda_j \in D(S^{-1})$ ($j = 1, \dots, J$) nor a restriction of the coupling constant α into a smaller region (these conditions are needed to prove the existence of a ground state of H_{GSB} in the case $m = 0$ [5]).

4 A class of models of quantum particles coupled to a Bose field

In this section we consider, in an abstract form, a class of models of N quantum particles interacting with a Bose field ($N \in \mathbf{N}$), where the particles move in \mathbf{R}^ν . To treat such a quantum system rigorously, we introduce some notions.

We take the Hilbert space of the system of N quantum particles to be

$$\mathcal{H}_p := L^2(\mathbf{R}^{\nu N}; \mathbf{C}^M), \quad (4.1)$$

where $M \in \mathbf{N}$ is a number related to internal degrees of freedom of the particles (e.g., *spin*). A Hilbert space for the coupled system of the N particles and the Bose field whose one-particle Hilbert space is \mathcal{K} is given by

$$\mathcal{F}_{\text{PF}} := \mathcal{H}_p \otimes \mathcal{F}_b(\mathcal{K}). \quad (4.2)$$

This Hilbert space has a natural identification with

$$L^2(\mathbf{R}^{\nu N}; \oplus^M \mathcal{F}_b(\mathcal{K})) = \int_{\mathbf{R}^{\nu N}}^{\oplus} \oplus^M \mathcal{F}_b(\mathcal{K}) dx, \quad (4.3)$$

the Hilbert space of $\oplus^M \mathcal{F}_b(\mathcal{K})$ -valued Lebesgue square integrable functions on $\mathbf{R}^{\nu N}$ [the constant fibre direct integral with base space $(\mathbf{R}^{\nu N}, dx)$ and fibre $\oplus^M \mathcal{F}_b(\mathcal{K})$]. The identification is given by the unitary correspondence given by

$$\psi \otimes \Psi \rightarrow (\psi_1(x)\Psi, \dots, \psi_M(x)\Psi)$$

($\psi = (\psi_\ell)_{\ell=1}^M \in \mathcal{H}_p$, $\Psi \in \mathcal{F}_b(\mathcal{K})$, $x = (x_1, \dots, x_N)$, $x_j \in \mathbf{R}^\nu$, $j = 1, \dots, N$) [23, §II.4, Theorem II.10(b)]. In what follows we use this identification freely.

Let $g : \mathbf{R}^{\nu N} \rightarrow \mathcal{K}$ be a strongly continuous \mathcal{K} -valued function on $\mathbf{R}^{\nu N}$. Then the operator-valued function $:x \rightarrow \exp[it\phi(g(x))]$ on $\mathbf{R}^{\nu N}$ is strongly continuous for all $t \in \mathbf{R}$ [24, §X.7, Theorem X.41(d)]. Hence it follows that the operator-valued function $:x \rightarrow (\phi(g(x)) + i)^{-1}$ on $\mathbf{R}^{\nu N}$ is strongly continuous. Thus we can define a decomposable self-adjoint operator

$$[\phi](g) := \int_{\mathbf{R}^{\nu N}}^{\oplus} \phi(g(x)) dx \quad (4.4)$$

on \mathcal{F}_{PF} [25, §XIII.16, Theorem XIII.85(a)].

We say that a \mathcal{K} -valued measurable function g on $\mathbf{R}^{\nu N}$ is in $L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$ if

$$\|g\|_\infty := \text{ess.sup}_{x \in \mathbf{R}^{\nu N}} \|g(x)\|_{\mathcal{K}} < \infty, \quad (4.5)$$

where “ess.sup” means essential supremum.

Let A, S and H_0 (resp. B_j) be as in Section 1 (resp. Section 3) with $\mathcal{H} = \mathcal{H}_p$. To define an interaction operator between the particles and the Bose field, we take \mathcal{K} -valued strongly continuous functions g_j and h_j on $\mathbf{R}^{\nu N}$ ($j = 1, \dots, J$), and bounded linear operators V_{jk} , $j, k = 1, \dots, J$, on \mathcal{F}_{PF} such that

$$V_{jk}^* = V_{kj}. \quad (4.6)$$

We define the following operators on \mathcal{F}_{PF} :

$$H_1 := \sum_{j=1}^J \frac{1}{2} (B_j [\phi](g_j) + [\phi](g_j) B_j), \quad (4.7)$$

$$H_2 := \sum_{j,k=1}^J [\phi](h_j) V_{jk} [\phi](h_k). \quad (4.8)$$

The total Hamiltonian of the model is defined by

$$H_{\text{PF}} := H_0 + H_{I,\text{PF}} \quad (4.9)$$

with

$$H_{I,\text{PF}} := H_1 + H_2. \quad (4.10)$$

This model gives a most general abstract unification of Hamiltonians of quantum field models including the Pauli-Fierz model in nonrelativistic quantum electrodynamics without the dipole approximation (e.g., [22, 7, 8, 11, 14, 21, 26]) and models of Nelson's type [20, 27].

4.1 Self-adjointness of H_{PF}

We first show that, under suitable conditions, H_{PF} is self-adjoint on $D(H_0)$ and bounded from below. For this purpose, we first estimate H_1 .

For $\varepsilon > 0$, we define

$$p_\varepsilon := \frac{1}{2} \left(1 + \frac{1}{2\varepsilon} \right). \quad (4.11)$$

Lemma 4.1 *Let $f \in D(S^{-1/2})$ and $\Psi \in D(d\Gamma(S)^{1/2})$. Then, for all $\varepsilon > 0$,*

$$\|\phi(f)\Psi\|^2 \leq (2 + \varepsilon)\|S^{-1/2}f\|^2\|d\Gamma(S)^{1/2}\Psi\|^2 + p_\varepsilon\|f\|^2\|\Psi\|^2. \quad (4.12)$$

Proof. Using (2.1) and (2.2), we have

$$\begin{aligned} \|\phi(f)\Psi\|^2 &\leq \frac{1}{2}(\|a(f)\Psi\|^2 + \|a(f)^*\Psi\|^2) + \|a(f)\Psi\|\|a(f)^*\Psi\| \\ &\leq 2\|S^{-1/2}f\|^2\|d\Gamma(S)^{1/2}\Psi\|^2 + \frac{1}{2}\|f\|^2\|\Psi\|^2 + \|S^{-1/2}f\|\|f\|\|d\Gamma(S)^{1/2}\Psi\|\|\Psi\|. \end{aligned}$$

By the elementary inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad a, b \geq 0, \varepsilon > 0, \quad (4.13)$$

we have

$$\|S^{-1/2}f\|\|f\|\|d\Gamma(S)^{1/2}\Psi\|\|\Psi\| \leq \varepsilon\|S^{-1/2}f\|^2\|d\Gamma(S)^{1/2}\Psi\|^2 + \frac{1}{4\varepsilon}\|f\|^2\|\Psi\|^2.$$

Hence we obtain (4.12). ■

Definition 4.2 Let $\alpha \in \mathbf{R}$. We say that a \mathcal{K} -valued function g on $\mathbf{R}^{\nu N}$ is in the set $L_\alpha^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$ if g is strongly continuous, $g(x) \in D(S^\alpha)$, $x \in \mathbf{R}^{\nu N}$, and $g, S^\alpha g \in L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$.

Lemma 4.3 *Let $g \in L_{-1/2}^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$. Then, $D(d\Gamma(S)^{1/2}) \subset D([\phi](g))$ and, for all $\varepsilon > 0$,*

$$\|[\phi](g)\Psi\|^2 \leq (2 + \varepsilon)\|S^{-1/2}g\|_\infty^2\|d\Gamma(S)^{1/2}\Psi\|^2 + p_\varepsilon\|g\|_\infty^2\|\Psi\|^2. \quad (4.14)$$

Proof. By Lemma 4.1, for all $\Psi \in D(I \otimes d\Gamma(S)^{1/2})$,

$$\|\phi(g(x))\Psi(x)\|^2 \leq (2 + \varepsilon)\|S^{-1/2}g(x)\|^2\|d\Gamma(S)^{1/2}\Psi(x)\|^2 + p_\varepsilon\|g(x)\|^2\|\Psi(x)\|^2$$

for a.e. x . Integrating the both sides with respect to dx , we see that $\Psi \in D([\phi](g))$ and (4.14) holds. ■

In what follows, we assume the following.

(PF.1) The operators A and B_j ($j = 1, \dots, J$) obey condition (GSB.2) in Section 3.

For $g \in L_{-1/2}^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$, $\varepsilon > 0$ and $\varepsilon' > 0$, we define nonnegative constants $c_{\varepsilon, \varepsilon'}^{(j)}(g)$ and $d_{\varepsilon, \varepsilon'}^{(j)}(g)$ ($j = 1, \dots, J$) as follows:

$$\begin{aligned} c_{\varepsilon, \varepsilon'}^{(j)}(g)^2 &:= \left(1 + \frac{\varepsilon}{2}\right) a_j^2 \|S^{-1/2}g\|_\infty^2 \\ &\quad + \varepsilon' \left\{ b_j^2 (2 + \varepsilon) \|S^{-1/2}g\|_\infty^2 + p_\varepsilon a_j^2 \|g\|_\infty^2 \right\}, \end{aligned} \quad (4.15)$$

$$d_{\varepsilon, \varepsilon'}^{(j)}(g)^2 := p_\varepsilon b_j^2 \|g\|_\infty^2 + \frac{1}{4\varepsilon'} \left\{ (2 + \varepsilon) b_j^2 \|S^{-1/2}g\|_\infty^2 + p_\varepsilon a_j^2 \|g\|_\infty^2 \right\}, \quad (4.16)$$

where a_j and b_j are the constants in (GSB.2).

Lemma 4.4 *Assume (PF.1). Let $g \in L_{-1/2}^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$. Then, $D(H_0) \subset \cap_{j=1}^J D([\phi](g)B_j)$ and, for all $\varepsilon > 0$ and $\varepsilon' > 0$ and $\Psi \in D(H_0)$,*

$$\|[\phi](g)B_j\Psi\|^2 \leq c_{\varepsilon, \varepsilon'}^{(j)}(g)^2 \|\widetilde{H}_0\Psi\|^2 + d_{\varepsilon, \varepsilon'}^{(j)}(g)^2 \|\Psi\|^2. \quad (4.17)$$

Proof. By Lemma 4.3, we have

$$\|[\phi](g)B_j\Psi\|^2 \leq (2 + \varepsilon) \|S^{-1/2}g\|_\infty^2 \|d\Gamma(S)^{1/2}B_j\Psi\|^2 + p_\varepsilon \|g\|_\infty^2 \|B_j\Psi\|^2. \quad (4.18)$$

By (GSB.2), we have

$$\|d\Gamma(S)^{1/2}B_j\Psi\|^2 \leq a_j^2 \|\widetilde{A}^{1/2}d\Gamma(S)^{1/2}\Psi\|^2 + b_j^2 \|d\Gamma(S)^{1/2}\Psi\|^2.$$

It is easy to see that

$$\|\widetilde{A}^{1/2}d\Gamma(S)^{1/2}\Psi\|^2 \leq \frac{1}{2} \|\widetilde{H}_0\Psi\|^2.$$

By (4.13), we have

$$\begin{aligned} \|d\Gamma(S)^{1/2}\Psi\|^2 &\leq \|\Psi\| \|d\Gamma(S)\Psi\| \\ &\leq \varepsilon' \|d\Gamma(S)\Psi\|^2 + \frac{1}{4\varepsilon'} \|\Psi\|^2 \\ &\leq \varepsilon' \|\widetilde{H}_0\Psi\|^2 + \frac{1}{4\varepsilon'} \|\Psi\|^2. \end{aligned}$$

Hence we obtain

$$\|d\Gamma(S)^{1/2}B_j\Psi\|^2 \leq \left(\frac{a_j^2}{2} + \varepsilon' b_j^2 \right) \|\widetilde{H}_0\Psi\|^2 + \frac{b_j^2}{4\varepsilon'} \|\Psi\|^2. \quad (4.19)$$

By (GSB.2) and

$$\begin{aligned} \|\widetilde{A}^{1/2}\Psi\|^2 &\leq \|\widetilde{H}_0^{1/2}\Psi\|^2 \leq \|\Psi\| \|\widetilde{H}_0\Psi\| \\ &\leq \varepsilon' \|\widetilde{H}_0\Psi\|^2 + \frac{1}{4\varepsilon'} \|\Psi\|^2, \end{aligned}$$

we have

$$\|B_j\Psi\|^2 \leq a_j^2 \left(\varepsilon' \|\widetilde{H}_0\Psi\|^2 + \frac{1}{4\varepsilon'} \|\Psi\|^2 \right) + b_j^2 \|\Psi\|^2. \quad (4.20)$$

Putting (4.19) and (4.20) into (4.18), we obtain (4.17). \blacksquare

To estimate the operator $B_j[\phi](g)$, we introduce a class of $g \in L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$:

Definition 4.5 We say that a \mathcal{K} -valued function g on $\mathbf{R}^{\nu N}$ is in the set $\mathcal{T}_B(\mathbf{R}^{\nu N}; \mathcal{K})$ if (i) $g \in L_{-1/2}^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$; (ii) $D(H_0) \subset \bigcap_{j=1}^J D(B_j[\phi](g))$; (iii) there exist nonnegative constants $c(B_j, g)$ and $d(B_j, g)$ such that

$$\|[B_j, [\phi](g)]\Psi\| \leq c(B_j, g)\|\widetilde{H}_0\Psi\| + d(B_j, g)\|\Psi\|, \quad \Psi \in D(H_0). \quad (4.21)$$

For the test vectors g_j giving the interaction operator H_1 , we assume the following:

(PF.2) $g_j \in \mathcal{T}_B(\mathbf{R}^{\nu N}; \mathcal{K})$, $j = 1, \dots, J$.

Lemma 4.6 Assume (PF.1) and (PF.2). Then, $D(H_0) \subset D(H_1)$ and, for all $\Psi \in D(H_0)$ and $\varepsilon, \varepsilon' > 0$,

$$\begin{aligned} \|H_1\Psi\| &\leq \sum_{j=1}^J \left\{ c_{\varepsilon, \varepsilon'}^{(j)}(g_j) + \frac{1}{2}c(B_j, g_j) \right\} \|\widetilde{H}_0\Psi\| \\ &\quad + \sum_{j=1}^J \left\{ d_{\varepsilon, \varepsilon'}^{(j)}(g_j) + \frac{1}{2}d(B_j, g_j) \right\} \|\Psi\|. \end{aligned} \quad (4.22)$$

Proof. We can write as $H_1 = \sum_{j=1}^J ([\phi](g_j)B_j + (1/2)[B_j, [\phi](g_j)])$. Then, by Lemma 4.4 and (4.21), we obtain (4.22). \blacksquare

We next estimate H_2 .

Lemma 4.7 Let $\alpha \geq 0$.

(i) Let $f \in D(S^{-1/2})$ and $\Psi \in D(d\Gamma(S)^{1/2+\alpha})$. Then $a(f)\Psi \in D(d\Gamma(S)^\alpha)$ and

$$\|d\Gamma(S)^\alpha a(f)\Psi\| \leq \|S^{-1/2}f\| \|d\Gamma(S)^{1/2+\alpha}\Psi\|. \quad (4.23)$$

(ii) Define

$$c(\alpha) := \begin{cases} 2^{\alpha-1} & \text{for } \alpha \geq 1 \\ 1 & \text{for } 0 \leq \alpha \leq 1 \end{cases} \quad (4.24)$$

Let $f \in D(S^{-1/2}) \cap D(S^\alpha)$ and $\Psi \in D(d\Gamma(S)^{1/2+\alpha})$. Then $a(f)^*\Psi \in D(d\Gamma(S)^\alpha)$ and, for all $\varepsilon > 0$,

$$\|d\Gamma(S)^\alpha a(f)^*\Psi\|^2 \quad (4.25)$$

$$\begin{aligned} &\leq c(\alpha)^2(1 + \varepsilon) \left(\|S^{-1/2}f\|^2 \|d\Gamma(S)^{1/2+\alpha}\Psi\|^2 + \|f\|^2 \|d\Gamma(S)^\alpha\Psi\|^2 \right) \\ &\quad + c(\alpha)^2 \left(1 + \frac{1}{\varepsilon} \right) \left(\|S^{-1/2+\alpha}f\|^2 \|d\Gamma(S)^{1/2}\Psi\|^2 + \|S^\alpha f\|^2 \|\Psi\|^2 \right). \end{aligned} \quad (4.26)$$

Proof. By the spectral theorem of self-adjoint operators, there exist a finite measure space $\langle M, \mu \rangle$, a unitary operator $U : \mathcal{K} \rightarrow L^2(M, d\mu)$ and a real-valued measurable function h on M (μ -a.e. finite) such that (i) $f \in D(S)$ if and only if $h(\cdot)(Uf)(\cdot) \in L^2(M, d\mu)$ and (ii) for all $g \in U(D(S))$, $(USU^{-1}g)(m) = h(m)g(m)$, μ -a.e. $m \in M$ [23, Theorem VIII.4]. It follows that $h(m) > 0$, μ -a.e. $m \in M$. The operator $\Gamma(U) :=$

$\bigoplus_{n=0}^{\infty} \otimes^n U$ with $\otimes^0 U := 1$ is unitary from $\mathcal{F}_b(\mathcal{K})$ onto $\mathcal{F}_b(L^2(M, d\mu))$ (cf. [24, §X.7]). Denote by $\hat{a}(\cdot)$ the annihilation operator on $\mathcal{F}_b(L^2(M, d\mu))$. Then we have

$$\hat{a}(Uf) = \Gamma(U)a(f)\Gamma(U)^{-1}, \quad f \in \mathcal{K}, \quad (4.27)$$

and, for all $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(\hat{a}(v))$ [$v \in L^2(M, d\mu)$] and $n \geq 0$,

$$(\hat{a}(v)\Psi)^{(n)}(m_1, \dots, m_n) = \sqrt{n+1} \int_M v(m)^* \Psi^{(n+1)}(m, m_1, \dots, m_n) d\mu(m), \quad (4.28)$$

cf. [24, p. 209]. We also have

$$d\Gamma(h) = \Gamma(U)d\Gamma(S)\Gamma(U)^{-1}.$$

Let $v \in D(h^{-1/2})$ and $\Psi \in D(d\Gamma(h)^{1/2+\alpha})$. Then

$$\begin{aligned} & |(d\Gamma(h)^\alpha \hat{a}(v)\Psi)^{(n)}(m_1, \dots, m_n)|^2 \\ & \leq \left(\sum_{j=1}^n h(m_j) \right)^{2\alpha} (n+1) \left(\int_M \frac{|v(m)|}{\sqrt{h(m)}} \sqrt{h(m)} |\Psi^{(n+1)}(m, m_1, \dots, m_n)| d\mu(m) \right)^2 \\ & \leq \left(\sum_{j=1}^n h(m_j) \right)^{2\alpha} (n+1) \left(\int_M \frac{|v(m)|^2}{h(m)} d\mu(m) \right) \\ & \quad \times \left(\int_M h(m) |\Psi^{(n+1)}(m, m_1, \dots, m_n)|^2 d\mu(m) \right), \end{aligned}$$

where we have used the Schwarz inequality. Hence

$$\begin{aligned} & \|(d\Gamma(h)^\alpha \hat{a}(v)\Psi)^{(n)}\|^2 \\ & \leq \|h^{-1/2}v\|^2 (n+1) \int_{M^{n+1}} \left(\sum_{j=1}^n h(m_j) \right)^{2\alpha} h(m) \\ & \quad \times |\Psi^{(n+1)}(m, m_1, \dots, m_n)|^2 d\mu(m) d\mu(m_1) \cdots d\mu(m_n) \\ & = \|h^{-1/2}v\|^2 \int_{M^{n+1}} \left(\sum_{j=1}^n h(m_j) \right)^{2\alpha} \left(h(m) + \sum_{k=1}^n h(m_k) \right) \\ & \quad \times |\Psi^{(n+1)}(m, m_1, \dots, m_n)|^2 d\mu(m) d\mu(m_1) \cdots d\mu(m_n) \\ & \leq \|h^{-1/2}v\|^2 \int_{M^{n+1}} \left(h(m) + \sum_{k=1}^n h(m_k) \right)^{2\alpha+1} \\ & \quad \times |\Psi^{(n+1)}(m, m_1, \dots, m_n)|^2 d\mu(m) d\mu(m_1) \cdots d\mu(m_n) \\ & = \|h^{-1/2}v\|^2 \|(d\Gamma(h)^{1/2+\alpha}\Psi)^{(n+1)}\|^2. \end{aligned}$$

From these estimates it follows that $\hat{a}(v)\Psi \in D(d\Gamma(h)^\alpha)$ and

$$\|d\Gamma(h)^\alpha \hat{a}(v)\Psi\| \leq \|h^{-1/2}v\| \|d\Gamma(h)^{1/2+\alpha}\Psi\|, \quad (4.29)$$

which implies (4.23).

As for $\hat{a}(v)^* [v \in L^2(M, d\mu)]$, we have for all $\Psi \in \mathcal{F}_0(L^2(M, d\mu))$,

$$(\hat{a}(v)^*\Psi)^{(n)}(m_1, \dots, m_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n v(m_j) \Psi^{(n-1)}(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_n), \quad n \geq 1,$$

and $(\hat{a}(v)^*\Psi)^{(0)} = 0$ [24, p.209]. Let $v \in D(h^\alpha) \cap D(h^{-1/2})$ and $\Psi \in d\Gamma(h) \cap \mathcal{F}_0(L^2(M, d\mu))$. Then

$$\begin{aligned} & |(d\Gamma(h)^\alpha \hat{a}(v)^*\Psi)^{(n)}(m_1, \dots, m_n)| \\ & \leq \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n h(m_k) \right)^\alpha \sum_{j=1}^n |v(m_j)| |\Psi^{(n-1)}(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_n)|. \end{aligned}$$

It is well known (or easy to see) that, for all $a, b \geq 0$,

$$(a + b)^\alpha \leq c(\alpha) (a^\alpha + b^\alpha).$$

Applying this inequality with $a = h(m_j)$ and $b = \sum_{k \neq j}^n h(m_k)$, we have

$$\begin{aligned} & \left(\sum_{k=1}^n h(m_k) \right)^\alpha |\Psi^{(n-1)}(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_n)| \\ & \leq c(\alpha) h(m_j)^\alpha |\Psi^{(n-1)}(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_n)| \\ & \quad + c(\alpha) |(d\Gamma(h)^\alpha \Psi)^{(n-1)}(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_n)|. \end{aligned}$$

Hence

$$|(d\Gamma(h)^\alpha \hat{a}(v)^*\Psi)^{(n)}| \leq c(\alpha) \left\{ (\hat{a}(h^\alpha |v|)^* |\Psi|)^{(n)} + (\hat{a}(|v|)^* |d\Gamma(h)^\alpha \Psi|)^{(n)} \right\},$$

where $|\Psi| := \{|\Psi^{(n)}|\}_{n=0}^\infty$. Therefore, using (2.2) and the elementary inequality

$$(a + b)^2 \leq (1 + \varepsilon)a^2 + \left(1 + \frac{1}{\varepsilon}\right)b^2, \quad a, b \geq 0, \varepsilon > 0, \quad (4.30)$$

we have

$$\begin{aligned} & \|d\Gamma(h)^\alpha \hat{a}(v)^*\Psi\|^2 \\ & \leq c(\alpha)^2 \left\{ \left(1 + \frac{1}{\varepsilon}\right) \|\hat{a}(h^\alpha |v|)^* |\Psi|\|^2 + (1 + \varepsilon) \|\hat{a}(|v|)^* |d\Gamma(h)^\alpha \Psi|\|^2 \right\} \\ & \leq c(\alpha)^2 \left(1 + \frac{1}{\varepsilon}\right) \left(\|h^{-1/2+\alpha} v\|^2 \|d\Gamma(h)^{1/2} \Psi\|^2 + \|h^\alpha v\|^2 \|\Psi\|^2 \right) \\ & \quad + c(\alpha)^2 (1 + \varepsilon) \left(\|h^{-1/2} v\|^2 \|d\Gamma(h)^{1/2+\alpha} \Psi\|^2 + \|v\|^2 \|d\Gamma(h)^\alpha \Psi\|^2 \right), \end{aligned}$$

which implies (4.26). ■

For $\varepsilon, \varepsilon_j > 0, j = 1, 2, 3$ and $f \in D(S^{1/2}) \cap D(S^{-1/2})$, we define

$$q_\varepsilon := \left(2 + \varepsilon + \frac{1}{\varepsilon}\right)^{1/2}, \quad (4.31)$$

$$\begin{aligned}
E(\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3; f) &:= \left(1 + \frac{\varepsilon}{2} + \sqrt{1 + \varepsilon} + \varepsilon_1 + \varepsilon_2\right) \|S^{-1/2} f\|^2 \\
&\quad + \frac{q_\varepsilon^2 \varepsilon_3}{4\varepsilon} \left(2 + \frac{1}{\varepsilon_1}\right) \|f\|^2, \tag{4.32}
\end{aligned}$$

$$\begin{aligned}
F(\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3; f) &:= \frac{1}{2} \left(1 + \frac{1}{\varepsilon}\right) \left(1 + \frac{1}{2\varepsilon_3}\right) \|S^{1/2} f\|^2 \\
&\quad + \frac{q_\varepsilon^2}{4} \left(2 + \frac{1}{\varepsilon_1}\right) \frac{1}{4\varepsilon_3} \|f\|^2. \tag{4.33}
\end{aligned}$$

Lemma 4.8 *Let $f \in D(S^{1/2}) \cap D(S^{-1/2})$. Then $D(d\Gamma(S)) \subset D(d\Gamma(S)^{1/2}\phi(f))$ and, for all $\Psi \in D(d\Gamma(S))$ and $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$,*

$$\|d\Gamma(S)^{1/2}\phi(f)\Psi\|^2 \leq E(\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3; f) \|d\Gamma(S)\Psi\|^2 + F(\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3; f) \|\Psi\|^2. \tag{4.34}$$

Proof. Using Lemma 4.7 with $\alpha = 1/2$, we have

$$\begin{aligned}
&\|d\Gamma(S)^{1/2}\phi(f)\Psi\|^2 \\
&\leq \frac{1}{2} (\|d\Gamma(S)^{1/2}a(f)^*\Psi\|^2 + \|d\Gamma(S)^{1/2}a(f)\Psi\|^2) \\
&\quad + \|d\Gamma(S)^{1/2}a(f)^*\Psi\| \|d\Gamma(S)^{1/2}a(f)\Psi\| \\
&\leq \left(1 + \frac{\varepsilon}{2} + \sqrt{1 + \varepsilon}\right) \|S^{-1/2} f\|^2 \|d\Gamma(S)\Psi\|^2 + \frac{q_\varepsilon^2}{2} \|f\|^2 \|d\Gamma(S)^{1/2}\Psi\|^2 \\
&\quad + \frac{1}{2} \left(1 + \frac{1}{\varepsilon}\right) \|S^{1/2} f\|^2 \|\Psi\|^2 + q_\varepsilon \|S^{-1/2} f\| \|f\| \|d\Gamma(S)\Psi\| \|d\Gamma(S)^{1/2}\Psi\| \\
&\quad + \sqrt{1 + \frac{1}{\varepsilon}} \|S^{-1/2} f\| \|S^{1/2} f\| \|d\Gamma(S)\Psi\| \|\Psi\|.
\end{aligned}$$

Note that

$$\begin{aligned}
q_\varepsilon \|S^{-1/2} f\| \|f\| \|d\Gamma(S)\Psi\| \|d\Gamma(S)^{1/2}\Psi\| &\leq \varepsilon_1 \|S^{-1/2} f\|^2 \|d\Gamma(S)\Psi\|^2 \\
&\quad + \frac{q_\varepsilon^2}{4\varepsilon_1} \|f\|^2 \|d\Gamma(S)^{1/2}\Psi\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{1 + \frac{1}{\varepsilon}} \|S^{-1/2} f\| \|S^{1/2} f\| \|d\Gamma(S)\Psi\| \|\Psi\| \\
&\leq \varepsilon_2 \|S^{-1/2} f\|^2 \|d\Gamma(S)\Psi\|^2 + \frac{1}{4\varepsilon_2} \left(1 + \frac{1}{\varepsilon}\right) \|S^{1/2} f\|^2 \|\Psi\|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|d\Gamma(S)^{1/2}\Psi\|^2 &\leq \|\Psi\| \|d\Gamma(S)\Psi\| \\
&\leq \varepsilon_3 \|d\Gamma(S)\Psi\|^2 + \frac{1}{4\varepsilon_3} \|\Psi\|^2.
\end{aligned}$$

From these estimates, we obtain (4.34). ■

We assume the following:

(PF.3) For each $j, k = 1, \dots, J$, V_{jk} is bounded with (4.6), leaves $D(d\Gamma(S)^{1/2})$ invariant and there exist constants $v_{jk}, w_{jk} \geq 0$ such that, for all $\Psi \in D(d\Gamma(S)^{1/2})$ and $j, k = 1, \dots, J$,

$$\| [d\Gamma(S)^{1/2}, V_{jk}] \Psi \| \leq v_{jk} \| d\Gamma(S)^{1/2} \Psi \| + w_{jk} \| \Psi \|. \quad (4.35)$$

As for $h_j, j = 1, \dots, J$, we assume the following:

(PF.4) For each $j = 1, \dots, J$, $h_j \in L_{-1/2}^\infty(\mathbf{R}^{\nu N}; \mathcal{K}) \cap L_{1/2}^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$.

For $s, t > 0$, we define

$$G_{jk}(s, t) := (2+s)(1+t) \| S^{-1/2} h_j \|_\infty^2 (v_{jk} + \| V_{jk} \|^2), \quad (4.36)$$

$$H_{jk}(s, t) := (2+s) \left(1 + \frac{1}{t} \right) \| S^{-1/2} h_j \|_\infty^2 w_{jk}^2 + p_s \| h_j \|_\infty^2 \| V_{jk} \|^2. \quad (4.37)$$

Lemma 4.9 *Assume (PF.3) and (PF.4). Let $s, t, r, \varepsilon, \varepsilon_\ell > 0, \ell = 1, 2, 3$ and $\Psi \in D(d\Gamma(S))$. Then, $\Psi \in D([\phi](h_j) V_{jk} [\phi](h_k))$ ($j, k = 1, \dots, J$) and*

$$\begin{aligned} & \| [\phi](h_j) V_{jk} [\phi](h_k) \Psi \|^2 \\ & \leq G_{jk}(s, t) \left(\sup_{x \in \mathbf{R}^{\nu N}} E(\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3; h_k(x)) \right) \| d\Gamma(S) \Psi \|^2 \\ & \quad + H_{jk}(s, t) (2+r) \| S^{-1/2} h_k \|_\infty^2 \| d\Gamma(S)^{1/2} \Psi \|^2 \\ & \quad + \left\{ G_{jk}(s, t) \left(\sup_{x \in \mathbf{R}^{\nu N}} F(\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3; h_k(x)) \right) + H_{jk}(s, t) p_r \| h_k \|_\infty^2 \right\} \| \Psi \|^2. \end{aligned} \quad (4.38)$$

Proof. By Lemma 4.3, we have

$$\begin{aligned} \| [\phi](h_j) V_{jk} [\phi](h_k) \Psi \|^2 & \leq (2+s) \| S^{-1/2} h_j \|_\infty^2 \| [d\Gamma(S)^{1/2} V_{jk} [\phi](h_k) \Psi] \|^2 \\ & \quad + p_s \| h_j \|_\infty^2 \| V_{jk} \|^2 \| [\phi](h_k) \Psi \|^2. \end{aligned} \quad (4.39)$$

By (PF.3), we have

$$\begin{aligned} & \| d\Gamma(S)^{1/2} V_{jk} [\phi](h_k) \Psi \| \\ & \leq \| [d\Gamma(S)^{1/2}, V_{jk}] [\phi](h_k) \Psi \| + \| V_{jk} d\Gamma(S)^{1/2} [\phi](h_k) \Psi \| \\ & \leq (v_{jk} + \| V_{jk} \|) \| d\Gamma(S)^{1/2} [\phi](h_k) \Psi \| + w_{jk} \| [\phi](h_k) \Psi \|. \end{aligned}$$

Using (4.30), we obtain

$$\begin{aligned} \| d\Gamma(S)^{1/2} V_{jk} [\phi](h_k) \Psi \|^2 & \leq (1+t)(v_{jk} + \| V_{jk} \|^2) \| d\Gamma(S)^{1/2} [\phi](h_k) \Psi \|^2 \\ & \quad + \left(1 + \frac{1}{t} \right) w_{jk}^2 \| [\phi](h_k) \Psi \|^2. \end{aligned}$$

Putting this estimate into (4.39) yields

$$\| [\phi](h_j) V_{jk} [\phi](h_k) \Psi \|^2 \leq G_{jk}(s, t) \| d\Gamma(S)^{1/2} [\phi](h_k) \Psi \|^2 + H_{jk}(s, t) \| [\phi](h_k) \Psi \|^2. \quad (4.40)$$

Using Lemmas 4.3 and 4.8 to estimate $\|[\phi](h_k)\Psi\|^2$ and $\|d\Gamma(S)^{1/2}[\phi](h_k)\Psi\|^2$ respectively, we obtain (4.38). \blacksquare

We are now ready to prove self-adjointness of H_{PF} . Let

$$C := \sum_{j=1}^J \left\{ a_j \|S^{-1/2}g_j\|_\infty + \frac{1}{2}c(B_j, g_j) \right\} + \sum_{j,k=1}^J 2(v_{jk} + \|V_{jk}\|) \|S^{-1/2}h_j\|_\infty \|S^{-1/2}h_k\|_\infty. \quad (4.41)$$

Theorem 4.10 *Assume (PF.1)–(PF.4). Suppose that $C < 1$. Then H_{PF} is self-adjoint on $D(H_{\text{PF}}) = D(H_0)$, essentially self-adjoint on every core of H_0 and bounded from below.*

Proof. Lemmas 4.6 and 4.9 imply that $D(H_0) \subset D(H_1) \cap D(H_2)$ and that, for each $\varepsilon > 0$, there exists a constant $D_\varepsilon \geq 0$ such that, for all $\Psi \in D(H_0)$,

$$\|(H_1 + H_2)\Psi\| \leq (C + \varepsilon)\|\widetilde{H}_0\Psi\| + D_\varepsilon\|\Psi\|.$$

It is obvious that H_{PF} is symmetric on $D(H_0)$. By condition $C < 1$, we can take an $\varepsilon > 0$ such that $C + \varepsilon < 1$. Hence an application of the Kato-Rellich theorem (e.g., [24, Theorem X.12]) gives the desired result. \blacksquare

4.2 Essential spectrum of H_{PF}

We apply Theorems 1.2 and 1.3 to locate the essential spectrum of H_{PF} . To do that, we have to estimate the weak commutator of $a(f)^*$ with $H_1 + H_2$ on $D(H_0)$ ($f \in D(S^{-1/2}) \cap D(S)$).

For $f \in \mathcal{K}$ and $g \in L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$, we denote by $M_g(f)$ the multiplication operator on $L^2(\mathbf{R}^{\nu N}; \mathbf{C}^M)$ by the function: $x \rightarrow (g(x), f)_\mathcal{K}$ ($x \in \mathbf{R}^{\nu N}$). It follows that $M_g(f)$ is bounded with

$$\|M_g(f)\| \leq \|g\|_\infty \|f\|_\mathcal{K}. \quad (4.42)$$

Definition 4.11 We say that a \mathcal{K} -valued function g on $\mathbf{R}^{\nu N}$ is in the set $\widehat{\mathcal{T}}_B(\mathbf{R}^{\nu N}; \mathcal{K})$ if the following conditions are satisfied:

- (i) $g \in \mathcal{T}_B(\mathbf{R}^{\nu N}; \mathcal{K})$.
- (ii) For all $f \in D(S) \cap D(S^{-1/2})$, $M_g(f)D(A) \subset \bigcap_{j=1}^J D(B_j)$ and $[B_j, M_g(f)]|D(A)$ is bounded with

$$\|[B_j, M_g(f)]|D(A)\| \leq c_j(g)\|f\|, \quad j = 1, \dots, J,$$

where $c_j(g)$ is a constant independent of f .

- (iii) If $w\text{-}\lim_{n \rightarrow \infty} f_n = 0$ ($f_n \in D(S) \cap D(S^{-1/2})$), then

$$\lim_{n \rightarrow \infty} [B_j, M_g(f_n)]u = 0$$

for all $u \in D(A)$.

Lemma 4.12 Assume (PF.1). Let $g_j \in \hat{\mathcal{T}}_B(\mathbf{R}^{\nu N}; \mathcal{K})$, $j = 1, \dots, J$ and $f \in D(S) \cap D(S^{-1/2})$. Then the couple $\langle H_1, a(f)^* \rangle$ has the weak commutator on $D(H_0)$ given by

$$[H_1, a(f)^*]_{w, D(H_0)} = \frac{1}{2\sqrt{2}} \sum_{j=1}^J (M_{g_j}(f)B_j + B_jM_{g_j}(f)) |D(H_0). \quad (4.43)$$

Moreover, if $w\text{-}\lim_{n \rightarrow \infty} f_n = 0$ ($f_n \in D(S) \cap D(S^{-1/2})$), then

$$\lim_{n \rightarrow \infty} [H_1, a(f_n)^*]_{w, D(H_0)} \Psi = 0 \quad (4.44)$$

for all $\Psi \in D(H_0)$.

Proof. Let $\Psi, \Phi \in D(H_0)$. It is easy to see that $[B_j, a(f)]\Psi = 0$, $[B_j, a(f)^*]\Phi = 0$. Using this fact, we can write

$$\begin{aligned} & (H_1\Psi, a(f)^*\Phi) - (a(f)\Psi, H_1\Phi) \\ &= \frac{1}{2} \sum_{j=1}^J \{([a(f), [\phi](g_j)]\Psi, B_j\Phi) + (B_j\Psi, [[\phi](g_j), a(f)^*]\Phi)\} \\ &= \frac{1}{2\sqrt{2}} \sum_{j=1}^J \{(M_{g_j}(f)^*\Psi, B_j\Phi) + (B_j\Psi, M_{g_j}(f)\Phi)\}. \end{aligned}$$

Hence (4.43) follows.

By (4.43), we have

$$\begin{aligned} & \| [H_1, a(f_n)^*]_{w, D(H_0)} \Psi \| \\ & \leq \frac{1}{2\sqrt{2}} \sum_{j=1}^J (2\|M_{g_j}(f_n)B_j\Psi\| + \|[B_j, M_{g_j}(f_n)]\Psi\|). \end{aligned}$$

It is easy to see that, for all $\Phi \in \mathcal{F}_{PF}$ and $g \in L^\infty(\mathbf{R}^{\nu N}; \mathcal{K})$,

$$\lim_{n \rightarrow \infty} \|M_g(f_n)\Phi\| = 0. \quad (4.45)$$

Since $g_j \in \hat{\mathcal{T}}_B(\mathbf{R}^{\nu N}; \mathcal{K})$, it follows that

$$\lim_{n \rightarrow \infty} \|[B_j, M_{g_j}(f_n)]\Psi\| = 0.$$

Thus we obtain (4.44). ■

Lemma 4.13 Assume (PF.3) and (PF.4). Suppose that, for all $f \in D(S) \cap D(S^{-1/2})$, $[V_{jk}, a(f)^*] = 0$ on $D(H_0)$, $j, k = 1, \dots, J$. Then, for all $f \in D(S) \cap D(S^{-1/2})$, the couple $\langle H_2, a(f)^* \rangle$ has the weak commutator on $D(H_0)$ given by

$$[H_2, a(f)^*]_{w, D(H_0)} = \frac{1}{\sqrt{2}} \sum_{j,k=1}^J \{[\phi](h_j)V_{jk}M_{h_k}(f) + M_{h_j}(f)V_{jk}[\phi](h_k)\}. \quad (4.46)$$

Moreover, if $w\text{-}\lim_{n \rightarrow \infty} f_n = 0$ ($f_n \in D(S) \cap D(S^{-1/2})$), then

$$\lim_{n \rightarrow \infty} [H_2, a(f_n)^*]_{w, D(H_0)} \Psi = 0 \quad (4.47)$$

for all $\Psi \in D(H_0)$.

Proof. Let $\Psi, \Phi \in D(H_0)$. Then we can write

$$\begin{aligned} & (H_2\Psi, a(f)^*\Phi) - (a(f)\Psi, H_2\Phi) \\ &= \sum_{j,k=1}^J \{(V_{jk}[\phi](h_k)\Psi, [[\phi](h_j), a(f)^*]\Phi) - ([[\phi](h_j), a(f)]\Psi, V_{jk}[\phi](h_k)\Phi)\} \\ &= \sum_{j,k=1}^J \frac{1}{\sqrt{2}} \{(V_{jk}[\phi](h_k)\Psi, M_{h_j}(f)\Phi) + (M_{h_j}(f)^*\Psi, V_{jk}[\phi](h_k)\Phi)\} \end{aligned}$$

Hence (4.46) follows.

By Lemmas 4.3 and (PF.3), we have

$$\|[\phi](h_j)V_{jk}M_{h_k}(f_n)\Psi\| \leq c_1\|M_{h_k}(f_n)d\Gamma(S)^{1/2}\Psi\| + c_2\|M_{h_k}(f_n)\Psi\|,$$

where $c_j, j = 1, 2$, are constants independent of f_n . Hence, by (4.45) and (4.46), we see that (4.47) holds. \blacksquare

We now state and prove a theorem on the essential spectrum of H_{PF} :

Theorem 4.14 *Let $g_j \in \hat{T}_B(\mathbf{R}^{\nu N}; \mathcal{K})$, $j = 1, \dots, J$. Assume (PF.1), (PF.3) and (PF.4) and that $C < 1$. Suppose that for all $f \in D(S) \cap D(S^{-1/2})$, $[V_{jk}, a(f)^*] = 0$ on $D(H_0)$, $j, k = 1, \dots, J$. Then*

$$\left\{ E_0(H_{\text{PF}}) + \lambda \mid \lambda \in \overline{\sigma_{\text{ess}}(S) \setminus \{0\}} \right\} \subset \sigma_{\text{ess}}(H_{\text{PF}}). \quad (4.48)$$

In particular, if $\sigma(S) = [0, \infty)$, then

$$\sigma(H_{\text{PF}}) = \sigma_{\text{ess}}(H_{\text{PF}}) = [E_0(H_{\text{PF}}), \infty). \quad (4.49)$$

Proof. By Theorem 4.10, Lemmas 4.12 and 4.13, $H = H_{\text{PF}}$ satisfies the basic hypotheses (H.1) and (H.2) in Section 1 with $H_I = H_{I, \text{PF}}$. Hence we can apply Theorems 1.2 and 1.3 to obtain the desired result. \blacksquare

Remark 4.1 Most of the Hamiltonians of concrete models of nonrelativistic particles coupled to a Bose field satisfy the assumption of Theorem 4.14, except for that $C < 1$, which depends on the coupling constants of the Hamiltonians.

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