Generic knots in contact 3-manifolds

Jiro Adachi

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Generic knots in contact 3-manifolds

Jiro ADACHI

Abstract

The notion of the *generic* knots in contact 3-manifolds is introduced, in this paper, as an extension of that of the transversal knots. We show that any generic knots in contact 3-manifolds are constructed by perturbations of some Legendrian knots. As for the classification problem, the situation is completely different from the cases of the transversal and Legendrian knots. Two generic knots in a contact 3-manifold are generically isotopic if and only if they have the same number of non-transversal points and belong to the same topological knot class. We treat, in this paper, not only trivial knots in tight contact 3-manifolds but also non-trivial knots and those in overtwisted contact 3-manifolds.

1 Introduction

There are two well known notions of knots in contact 3-manifolds. The notion of Legendrian knots, which are tangent to the contact structure, is one of them. That of transversal knots, which are transversal to the contact structure, is the other. In this article, we treat generic knots which are simply tangent to the contact structure at a finite number of isolated points.

The classification problem, in each case, is more complicated than the topological knot theory. Even for topologically trivial knots are endowed by the contact structure with rich structures. Ya. Eliashberg and M. Fraser classified topologically trivial transversal and Legendrian knots in tight contact 3-manifolds in [E2] and [EF]. But, with respect to overtwisted contact structures, characteristic foliation may have limit cycles. The transversal isotopies along the characteristic foliation cannot pass through limit cycles. (see the definition in this section.) If we consider the non-standard contact structure on S^3 obtained by the Lutz modification, the characteristic foliation on any Seifert disc of a transversal knot which links to the Lutz tube always has a limit cycle. Thus, there

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cannot develop trees which consist of leaves of the characteristic foliation on Seifert discs. Therefore, in this case, we cannot use the method in his article.

We note that the general case, in which knots may be topologically non-trivial, is argued in [FT]. It is proved that topologically isotopic Legendrian and transversal knots cannot be classified by finite order Vassiliev invariants such as Bennequin and Maslov numbers for Legendrian knots and the self-linking number for transversal knots. (see [B], [E2] for definition.)

Although, the situation is completely different in the generic case. In this article, we classify generic knots in contact 3-manifolds, even non-trivial knots, and in overtwisted contact manifolds.

First, recall several notions used in this paper.

A *contact structure* on a 3-manifold M is a completely non-integrable tangent plane field ξ. In other words, a contact structure ξ is defined, at least locally, as a kernel of a differential 1-form α on M which satisfies $\alpha \wedge d\alpha \neq 0$ everywhere. This 1-form α is called the *contact form.*

For an embedded surface F in a contact manifold (M, ξ), the contact structure ξ traces a singular foliation on F. That is called the *characteristic foliation* on F with respect to ξ, and we denote it by F_ξ. At singular points of F_ξ, ξ is tangent to F. When ξ and F are oriented, a singular point is called *positive* or *negative* depending on whether the orientation of them coincide at the point or not. Generically, singular points of F_ξ are isolated, finite, and the indices of them are ± 1. A singular point $p \in F$ is called *elliptic* if its index is $+1$, and *hyperbolic* if it is -1.

A contact structure ξ is called *tight* if for any embedded disc D in (M, ξ) D_ξ never have limit cycle. A contact structure is called *overtwisted* if there exists an embedded disc, on which the characteristic foliation has a limit cycle and a singular point inside it. It is proved by Ya. Eliashberg in [E1] that these two notion form two complementary classes of contact structure on 3-manifolds.

Knots are embeddings $f : S^1 \to (M, \xi)$. We also call the image $f(S^1) \subset M$ a knot. In this article, we consider knots to be oriented. A knot $f : S^1 \to (M, \xi)$ is called *Legendrian* if the pull-back of a contact form α by f, which we denote by $f^*\alpha$, vanishes for any point of S^1. A knot f is called *transversal* if $f^*\alpha$ never vanishes on S^1. Let θ be a coordinate of S^1 which gives the positive orientation. When ξ is cooriented, a knot f is called *positively* (resp., *negatively*) transversal if $f^*\alpha(\partial/\partial\theta)$ is positive (resp., negative).

Now, we define the generic knots. In this paper, we call a knot f *generic* if $f^*\alpha$ vanishes on finite points of S^1 and they are "simple" zero, that is, $f^*\alpha(\partial/\partial\theta) = 0$ has a finite number of simple roots. We call a point of a generic knot $\gamma = f(S^1)$ the *non-transversal point* if γ is not transversal to the contact structure ξ at the point. Let $f, f' : S^1 \to (M, \xi)$ be generic knots. We call f and f' are *generically isotopic* if there exists a smooth family

$f_t : S^1 \to (M, \xi)$, $0 \leq t \leq 1$, of generic knots with $f_0 = f$ and $f_1 = f'$. In this paper, we treat only non-transversal generic knots with non-transversal points.

The main results of this article are the following.

Theorem A *Let Γ, Γ' be generic knots in a contact 3-manifold (M, ξ). Γ is generically isotopic to Γ' if and only if they are isotopic in M and have the same number of non-transversal points.*

Moreover, we consider the construction of generic knots.

Theorem B *For any isotopy classes of knots in M and even integer $2k$ ($k \in \mathbb{Z}$), there exists a generic knot which belongs to the given knot class and has $2k$ non-transversal points.*

As a consequence, we obtain generic knots for all generic isotopy classes of generic knots in a contact 3-manifold.

In the following section, we study generic isotopies along embedded surfaces. There are the characteristic foliations on the surfaces. We study the relation between generic isotopies and the characteristic foliations. Isotopies of curves transversal to the characteristic foliations are transversal isotopies. In the generic case, we must check the simplicity of non-transversal points. We use the local classification of hypersurfaces in contact manifolds, which is given by M. Ya. Zhitomirskii in [Zh].

Section 3 is devoted to the proofs of Theorems. First, we construct generic knots for any isotopy class of knots and with any even number of non-transversal points. At last, the generic isotopies between given two generic knots with the same number of non-transversal points are constructed. As a consequence, we obtain the complete list of generic knots in contact 3-manifolds.

2 Generic isotopies along embedded surfaces

We observe which perturbations of curves along embedded surfaces can be realized as generic isotopies, in relation to characteristic foliations.

Let (M, ξ) be a contact 3-manifold and F an embedded surface in M. Let γ be a curve in F. A curve γ is a transversal curve if γ is transversal to leaves of the characteristic foliation F_ξ. If γ is generic, it may be tangent to a leaf of F_ξ or pass through a singular point, at its non-transversal point. It is immediate from the simplicity of non-transversal points that passing through a non-transversal point, the curve changes from positively (resp., negatively) transversal to negatively (resp., positively) transversal. In other words, if the curve γ and the characteristic foliation F_ξ are oriented, the orientation defined by positively oriented tangent vectors of γ and a leaf of F_ξ changes. On account of Darboux's theorem, it is sufficient that we observe local standard models. We use the local classification of surfaces in contact 3-manifolds given in [Zh] by M. Ya. Zhitomirskii.

(1) At non-singular points.

First, we give the local classification of surfaces in contact 3-manifolds at non-singular points of the characteristic foliations, where surfaces are transverse to the contact structures by definition.

Theorem 2.1 **([L], [Zh])** *Any germ of a surface in the standard contact 3-space $(\mathbb{R}^3, \xi_0)$ at a non-singular point is reducible to the germ $\{y = 0\}$ by a contact diffeomorphism. Where ξ_0 is the kernel of the standard contact form $\alpha_0 = dz - y \cdot dx$.*

Let $F := \{y = 0\} \subset (\mathbb{R}^3, \xi_0)$ be the above standard form of surface at a non-singular point $(0,0,0)$. In this case, the characteristic foliation on F is $F_{\xi_0} = \{y = 0, z = c\}_{c\in\mathbb{R}}$. Let $\gamma(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ be a germ of a curve which passes through the origin (i.e., $\gamma(0) = (0,0,0)$). The pull back of the standard contact form α_0 by γ is

$$\gamma^*\alpha_0 = (\dot{z}(t) - y(t)\dot{x}(t))dt,$$

where "$\dot{}$" means $\partial/\partial t$. We set $\varphi(t) := \dot{z}(t) - y(t)\dot{x}(t)$. This germ of a curve γ is simply tangent to ξ_0 at the origin if and only if

$$(2.1) \qquad \begin{cases} \varphi(0) = \dot{z}(0) - y(0) \cdot \dot{x}(0) = 0 \\ \dot{\varphi}(0) = \ddot{z}(0) - \dot{y}(0)\dot{x}(0) - y(0)\ddot{x}(0) \neq 0. \end{cases}$$

When γ is contained in (x, z)-plane $F = \{y = 0\}$, the equation(2.1) is reduced to

$$\begin{cases} \dot{z}(0) = 0 \\ \ddot{z}(0) \neq 0. \end{cases}$$

Then we have the following proposition.

Proposition 2.2 *The curve γ in an embedded surface F in a contact 3-manifold (M, ξ) is simply tangent to the contact structure ξ at a non-singular point of F_ξ, if and only if γ is tangent to a leaf of the characteristic foliation F_ξ with at most second order.*

According to the above proposition, a smooth family $\{\gamma_t\}$ of curves in an embedded surface, which are tangent to leaves of the characteristic foliation with at most second order, defines a generic isotopy of a curve. (see Fig.1 (a).)

(2) At singular points.

Next, we consider singular points, where surfaces are tangent to the contact structure by definition.

Theorem 2.3 ([L], [Zh]) *Any germ of a generic embedded surface in the standard contact 3-space* $(\mathbb{R}^3, \xi_0)$ *at a singular point is reducible to the germ*

$$F = \left\{ z - \frac{1}{2}xy + G = 0 \right\}$$

at the origin by a contact diffeomorphism.

Where $G = G(x, y)$ *is the normal form of the quadratic Hamiltonians in* $(\mathbb{R}^2, dx \wedge dy)$. *(see [A].)*

In this case $G(x,y) = -\mu \cdot xy$ or $G(x,y) = \pm(1/2)(\theta x^2 + y^2)$, where μ, θ are non-negative real numbers.

- **Case 1** $G(x,y) = -\mu \cdot xy$

First, we investigate the type of singular points. The restriction of contact form α_0 to $F = \{z - (1/2 + \mu)xy = 0\}$ is

$$\alpha_0|_F = \left(\mu - \frac{1}{2}\right) ydx + \left(\mu + \frac{1}{2}\right) xdy.$$

The vector field generating the characteristic foliation F_{ξ_0} near the origin is

$$X = \left(\mu + \frac{1}{2}\right) x\frac{\partial}{\partial x} - \left(\mu - \frac{1}{2}\right) y\frac{\partial}{\partial y}.$$

The eigenvalues of X-linearization are $\lambda = 1/2 \pm \mu$. Then the singular point is elliptic if $(1/2 + \mu)(1/2 - \mu) < 0$, that is, $0 \leq \mu < 1/2$, and hyperbolic if $(1/2 + \mu)(1/2 - \mu) > 0$, that is, $1/2 < \mu$. When $\mu = 1/2$, the surfaces are non-generic.

A germ of an embedded curve $\gamma(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$, which satisfies $\gamma(0) = (0,0,0)$, is simply tangent to ξ_0 at the origin if and only if

$$\begin{cases} \dot{z}(0) = 0 \\ \ddot{z}(0) - \dot{y}(0)\dot{x}(0) \neq 0. \end{cases}$$

When γ is contained in $F = \{z - (1/2 + \mu)xy = 0\}$, this condition is reduced to the following condition,

$$\ddot{z}(0) - \dot{y}(0)\dot{x}(0) = 2\mu\dot{x}(0)\dot{y}(0) \neq 0,$$

then, we have

$$\dot{x}(0) \neq 0, \ \dot{y}(0) \neq 0, \ \text{and } \mu \neq 0. \tag{2.2}$$

Then, the curve $\gamma(t)$ in F is simply tangent to ξ_0 at the origin if and only if it suffices the condition (2.2). (see Fig.1 (b), (c), (d).)

- **Case 2** $G(x,y) = \pm\frac{1}{2}(\theta x^2 + y^2)$

The vector field generating the characteristic foliation on $F = \{z-(1/2)xy\pm(1/2)(\theta x^2+y^2) = 0\}$ near the origin is

$$X = \left(\frac{1}{2}xy \pm y\right)\frac{\partial}{\partial x} + \left(\frac{1}{2}y \mp \theta x\right)\frac{\partial}{\partial y}.$$

The eigenvalues of X-linearization are $\lambda = 1/2 \pm \sqrt{-\theta}$. Then the singular point is elliptic for any $\theta \geq 0$.

If the embedded curve γ which is contained in F passes through the origin $(0,0,0) = \gamma(0)$, then we have

$$\text{(2.3)} \qquad \begin{cases} \dot{z}(0) = \dfrac{1}{2}\{\dot{x}(0)y(0) + x(0)\dot{y}(0) \pm 2(\theta x(0)\dot{x}(0) + y(0)\dot{y}(0))\} = 0 \\ \ddot{z}(0) - \dot{y}(0)\dot{x}(0) = \pm(\theta\dot{x}(0)^2 + \dot{y}(0)^2) \neq 0. \end{cases}$$

As the curve γ is an embedding, $(\dot{x}(0), \dot{y}(0)) \neq (0,0)$. Then, if $\theta > 0$, the curve γ is simply tangent to the contact structure ξ_0. If $\theta = 0$, γ is simply tangent to ξ_0 when $\dot{y}(0) \neq 0$. (see Fig.1 (e), (f).)

The hyperbolic points of the characteristic foliation appear only in the Case 1 with $1/2 < \mu$. Then there are generic isotopies of generic curves, which pass through the hyperbolic points if at the hyperbolic points they are not tangent to separatrices of the hyperbolic points. (see Fig.1 (b).)

There are 4 types of the elliptic points. For the type in the Case 1 with $\mu = 0$, generic curves cannot pass through this elliptic points. Though, we may change the type to another one by a small perturbation of the embedded surface. So, there are generic isotopies of generic curves, which pass through the elliptic point if they suffice each condition (2.2) (2.3) at each type of elliptic points. We may suppose that they suffice these conditions, by using generic isotopy if necessary. (see Fig.1 (c), (d), (e), (f).)

As a consequence, we obtain the following.

Proposition 2.4 *A smooth family of curves $\{\gamma_t\}_{t\in[0,1]}$ in an embedded surface F defines a generic isotopy if it suffice the following conditions.*

(i) *The curves γ_t are transversal to the characteristic foliation F_ξ except a finite number of points of each γ_t.*

(ii) *non-transversal points are the following 3 types.*

- *The points where a leaf of F_ξ and γ_t are tangent with at most second order.*
- *The hyperbolic points of F_ξ where γ_t is not tangent to separatrices of the hyperbolic points.*
- *The elliptic points.*

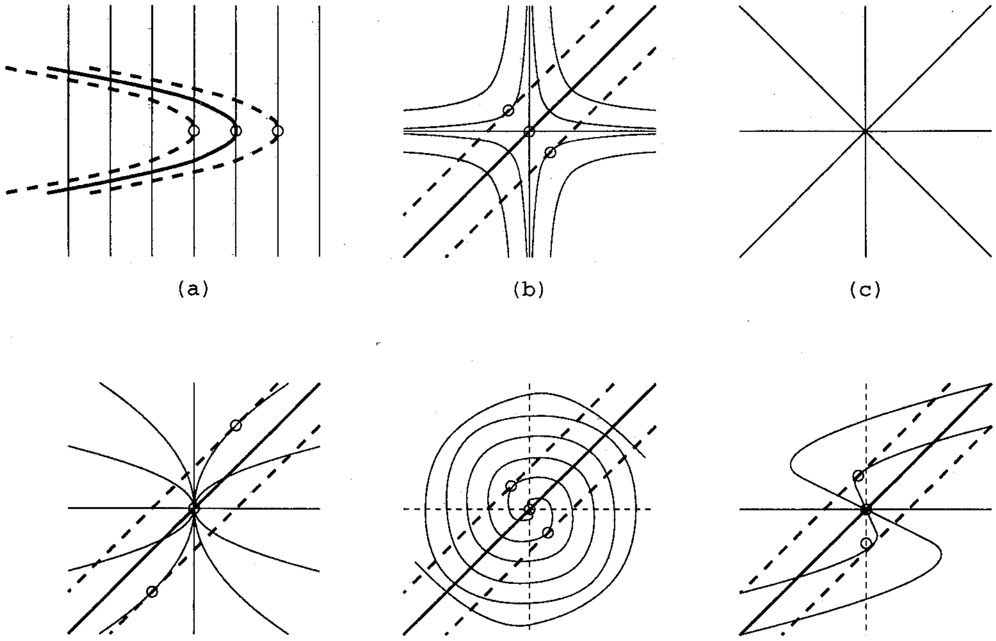

Figure 1:

3 Proofs of Theorems

3.1 Fundamental operations.

First, we prepare fundamental operation of generic knots by generic isotopies. To do this, we need the following Elimination lemma which is a basic tool for the study of characteristic foliations. It is a result due to E. Giroux (see [G], [E1]) in an improved version by D. Fuchs. Let F be an embedded surface in a contact 3-manifold (M, ξ).

Lemma 3.1 (Elimination Lemma) *Let $p, q \in F$ be elliptic and hyperbolic points of F_ξ which have the same sign. If there is a trajectory γ of F_ξ joining p and q, then there exists a C^0-small isotopy $h_t : F \to (M, \xi)$, $t \in [0,1]$, which has the following properties.*

(1) *It is fixed on γ and outside of a neighborhood U of γ.*

(2) $h_0 = id$.

(3) *The characteristic foliation $\tilde{F}_\xi$ on $\tilde{F} := h_1(F)$ has no singular point in $\tilde{F} \cap U$.*

(see Figure 2.)

The inverse of this lemma is easily shown (see [E1]). That is to say, it is easy to create singular points.

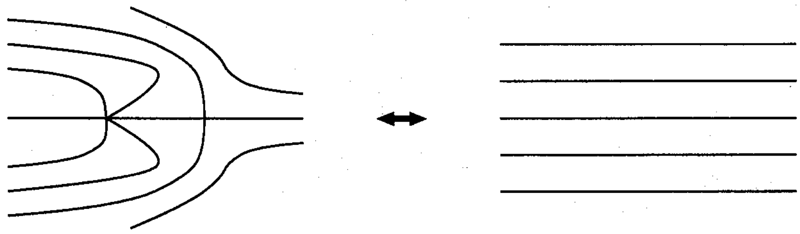

Figure 2:

Lemma 3.2 *By a C^0-small isotopy of the embedded surface near a non-singular point, a pair of elliptic and hyperbolic singular points of F_ξ, which have the same sign, can be created.*

Then we prepare the fundamental operations as follows. Let γ be a generic knot in a contact 3-manifold (M, ξ), and F an embedded surface in M which contains γ. We suppose that non-transversal points are regular points of F_ξ. In other words, γ is tangent to a leaf of F_ξ at each of its non-transversal point. A non-transversal point is defined to be *positive* with respect to F if the orientation of γ and a leaf tangent to γ are coincide at the point, and *negative* if they are inverse.

Proposition 3.3 *A pair of consecutive non-transversal points of a generic curve γ with the same sign for an embedded surface F near generic point of F_ξ can be changed the sign by a generic isotopy.*

Proof. We suppose that non-transversal points are positive with respect to F. Applying Lemma 3.2, we obtain a hyperbolic point $h \in F$ near non-transversal points by C^0-small isotopy of F as in Fig.3. We denote the perturbed surface by $\tilde{F}$. According to Proposition 2.4, γ is moved by a generic isotopy along $\tilde{F}$ so that a non-transversal point passes through a separatrix of h. Then there is a generic isotopy of γ, by which another non-transversal point is moved through the hyperbolic point h. The sign of this non-transversal point is changed at the time. We do the inverse operations to each of the other non-transversal points. Then we obtain a pair of negative non-transversal points. Last of all, we eliminate singular points which are created at the beginning of this proof. □

Remark *We note that the above generic isotopy is not performed along the surface F.*

3.2 Construction of generic knots of each class

Theorem B is proved in this section.

The following fact is a well known property of non-degenerate distributions.

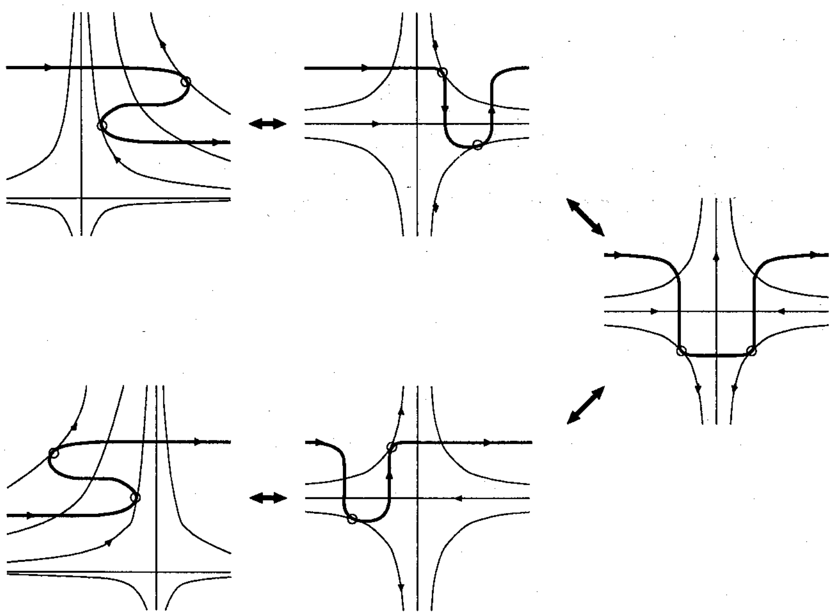

Figure 3:

Lemma 3.4 **([E2], [B])** *Any embedded curve Γ in a contact 3-manifold can be made Legendrian by C^0-small isotopy.*

Then, for each topological class of knots, there exist Legendrian knots because C^0-small isotopy does not change the knot class.

The following is also well known.

Lemma 3.5 **([E2], [B])** *Let Γ be a Legendrian knot in a contact 3-manifold (M, ξ). The knots $\Gamma^{\pm}$, obtained by pushing slightly in positive (resp., negative) normal direction of Γ in ξ, is a positively (resp., negatively) transversal knot.*

In a similar fashion, we have the generic version of the above lemma.

Lemma 3.6 *Let L be a Legendrian knot. We may make L be a generic knot with any even number of non-transversal points, by C^0-small perturbation.*

Proof. Let $\varphi : S^1 \to (M, \xi)$ be a Legendrian embedding which image is the given Legendrian knot L. There is a transversal narrow band $\psi : S^1 \times [-1, 1] \to M$ which is an extension of φ ; $\psi|_{S^1 \times \{0\}} = \varphi$. We set $S := \psi(S^1 \times [-1, 1])$. Then the characteristic foliation S_ξ is as in Fig.4. By pushing the Legendrian knot $L = \Phi(S^1 \times \{0\})$ as in Fig.4, or making a "pleat", we obtain a generic knot with two non-transversal points. Making k pleats, we can create $2k$ non-transversal points.

We note that such generic knots are independent of the construction up to generic isotopy if the numbers of non-transversal points coincide, according to Proposition 3.3.

□

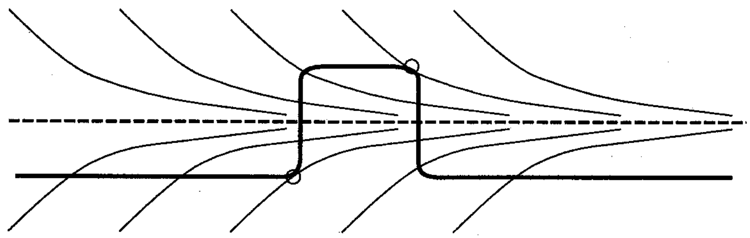

Figure 4:

Thus we can construct generic knots with any even number of non-transversal points from Legendrian knots which belong to all topological knot class. This completes the proof of Theorem B.

3.3 Generic isotopy between two generic knots.

In this section, we prove Theorem A.

Let Γ_0, Γ_1 be generic knots in a contact 3-manifold (M, ξ). If they are generically isotopic, it is clear that they have the same number of non-transversal points. Let us suppose, on the contrary, that they have the same number $2k$ of non-transversal points. Moreover, we suppose that they are isotopic in M as (topological) knots. That is, there is a smooth map $F : S^1 \times [0,1] \to M$ which satisfies $F_0(S^1) = \Gamma_0$, $F_1(S^1) = \Gamma_1$, where $F_t(x) := F(x,t)$ is an embedding for each $t \in [0,1]$. We may assume that there are a finite number of numbers $0 = t_0 < \ldots < t_m = 1$, for which the restriction of F to annuli $S^1 \times [t_{i-1}, t_i]$ are embeddings. In fact, it is known in topological knot theory that if two knots are isotopic, then their regular presentations are moved each other by finite numbers of the Reidemeister moves. The Reidemeister moves correspond to isotopies $S^1 \times [0,1] \to M$ which are embeddings of an annulus (see [K]). Each embedded annulus can be taken to be generic. Moreover, we may assume that each $\Gamma_{t_i} = F_{t_i}(S^1)$ is generic with $2k$ non-transversal points on account of Lemma 3.4 and 3.6. Then the following is a key lemma for the proof of Theorem A.

Lemma 3.7 *Let A be an embedded annulus in contact 3-manifold (M, ξ), which is bounded by two generic knots γ_0 and γ_1 with the same number of non-transversal points. Then γ_0 is generically isotopic to γ_1.*

Proof. We may suppose that there is a Legendrian knot L in A, which is isotopic to each boundary γ_0 and γ_1 along A by means of Lemma 3.4 and 3.5. Moreover, we may suppose that there are only a finite number of positive elliptic and negative hyperbolic points of A_ξ on L if there exist singular points on L. The annulus A is divided by L

into two annuli A_0 and A_1, each of which is bounded by γ_i and L $(i = 0, 1)$. There are transversal knots L_i in A_i $(i = 0, 1)$ arbitrary near L, which is isotopic to L.

Now, we consider moving each γ_i, by a generic isotopy, sufficiently close to L. If the characteristic foliation $(A_i)_\xi$ has limit cycles, we apply Lemma 3.2 to these closed leaves. Then we may suppose that $(A_i)_\xi$ has no closed leaf. On account of Proposition 2.4, γ_i can be moved by a generic isotopy along A_i, through all singular points of the foliation $(A_i)_\xi$. We denote the obtained generic knot by $\bar{\gamma}_i$. Let $\tilde{A}_i \subset A$ be a sub-annulus bounded by $\bar{\gamma}_i$ and L_i. Then the characteristic foliation $(\tilde{A}_i)_\xi$ has no singular point. Applying the non-singular case of Lemma 2.4, the generic knot $\bar{\gamma}_i$ can be moved by a generic isotopy along A_i so that its non-transversal points are tangent to leaves which come from L_i. We also denote the resultant generic knot by $\bar{\gamma}_i$. Then there exists a generic isotopy of $\bar{\gamma}_i$ along $\tilde{A}_i$ which contracts $\bar{\gamma}_i$ arbitrary close to L_i. We continue to denote the obtained generic knot by $\bar{\gamma}_i$, and the sub-annulus bounded by $\bar{\gamma}_i$ and L_i by $\tilde{A}_i$.

By fixing a point $x_0 \in \bar{\gamma}_i$ at which $\bar{\gamma}_i$ is transversal to ξ, the generic isotopy class of the generic knot $\bar{\gamma}_i$ in $\tilde{A}_i$ is represented by a sequence of signs of non-transversal points with respect to $\tilde{A}_i$ (such as $[++-\cdots+-+]$). There are even number of non-transversal points, and number of negative signs among even-th signs and that among odd-th are coincide, because $\bar{\gamma}_i$ is embedded into $\tilde{A}_i$ and tangent to leaves of $(\tilde{A}_i)_\xi$ coming from L_i at non-transversal points.

As a corollary of Proposition 3.3 we obtain the following lemma.

Lemma 3.8 (1) *The subsequence* $[++]$ *and* $[--]$ *can be replaced with each other by a generic isotopy of* $\bar{\gamma}_i$.
(2) *The subsequence* $[-++]$ *can be substituted for* $[++-]$ *by a generic isotopy of* $\bar{\gamma}_i$.

Proof. We obtain (1) as a direct result of Proposition 3.3. Then we show (2). Applying (1) to the subsequence $[++]$ of $[-++]$, we obtain a subsequence $[---]$. We can change it to a subsequence $[++-]$ by applying (1) to the subsequence $[--]$ of $[---]$ which consists of the left two signs. □

From the condition about the number of negative signs, there exists, side by side, a pair of non-transversal points with the same sign. On account of the above Lemma 3.8, we can increase the number of positive signs, and gather negative signs to the right side. After all, we can change all the signs positive because of the condition about the number of negative signs. In other words, $\bar{\gamma}_i$ is generically isotopic to $\tilde{\gamma}_i \subset A_i$ arbitrary close to L, all of which non-transversal points are positive with respect to A_i.

The generic knot $\tilde{\gamma}_0$ is moved by a generic isotopy along A into a sub-annulus bounded by L and $\tilde{\gamma}_1$ in the following way. If L is a limit cycle, we create a pair of positive elliptic and hyperbolic points by Lemma 3.2. We note that the Legendrian knot L consists of a finite number of singular points and stable separatrices of hyperbolic points. There is a generic isotopy which moves a non-transversal point of $\tilde{\gamma}_i$ through a separatrix in L

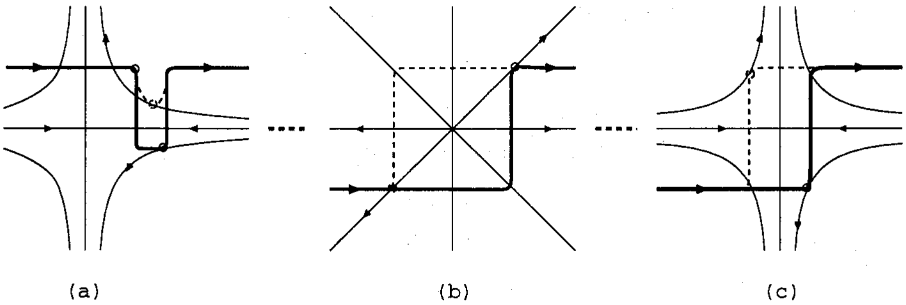

Figure 5:

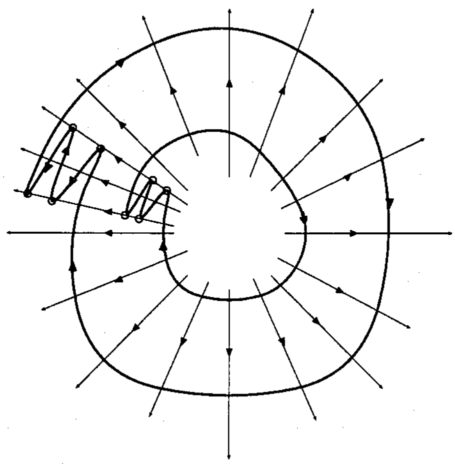

Figure 6:

(see Fig.5-(a)). This non-transversal point is moved by a generic isotopy along A through hyperbolic and elliptic points in turns (see Fig.5-(b),(c)). After all, there is a generic isotopy which moves all of the other non-transversal points through a separatrix. Then $\tilde{\gamma}_0$ is moved to another side of L. We note that all of non-transversal points of $\tilde{\gamma}_0$ are still positive with respect to A. We also denote the resultant generic knot by $\tilde{\gamma}_0$.

Let $\tilde{A}$ be a sub-annulus of A bounded by $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. All non-transversal points of $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are positive. The characteristic foliation $\tilde{A}_\xi$ has no singular point and each of its leaves intersects both of $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. On account of non-singular case of Proposition 2.4, $\tilde{\gamma}_0$ coincides with $\tilde{\gamma}_1$ by a generic isotopy along $\tilde{A}$.

This completes the proof of Lemma 3.7. $\square$

Let us continue the proof of Theorem A. Applying Lemma 3.7 to each $F(S^1 \times [t_{i-1}, t_i])$, we have that the generic knots $\Gamma_{t_{i-1}} = F_{t_{i-1}}(S^1)$ and $\Gamma_{t_i} = F_{t_i}(S^1)$ are generically isotopic for $i = 1, 2, \ldots, m$. Consequently, $\Gamma_{t_0} = \Gamma_0$ is generically isotopic to $\Gamma_{t_m} = \Gamma_1$.

This complete the proof of Theorem A.

Remark The knots constructed in Section 3.2 give the complete list of generic knots in contact 3-manifold. Indeed, there exist generic knots for all isotopy class of knots and any even number of non-transversal points.

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Department of Mathematics,
Osaka University,
Toyonaka Osaka 560-0043, Japan
e-mail : adachi@math.sci.osaka-u.ac.jp

Department of Mathematics,
Hokkaido University,
Sapporo 060-0810, Japan
e-mail : j-adachi@math.sci.hokudai.ac.jp