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by

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Abstract. Let α and β be bounded measurable functions on the unit circle T. The singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha Pf + \beta Qf \quad (f \in L^2(T))$ where P is an analytic projection and Q is a co-analytic projection. In the previous paper, the norm of $S_{\alpha,\beta}$ was calculated in general, using α, β and $\alpha\bar{\beta} + H^\infty$ where H^∞ is a Hardy space in $L^\infty(T)$. In this paper, the essential norm $\|S_{\alpha,\beta}\|_e$ of $S_{\alpha,\beta}$ is calculated in general, using $\alpha\bar{\beta} + H^\infty + C$ where C is a set of all continuous functions on T. Hence if $\alpha\bar{\beta}$ is in $H^\infty + C$ then $\|S_{\alpha,\beta}\|_e = \max(\|\alpha\|_\infty, \|\beta\|_\infty)$. This gives a known result when α, β are in C.

§1. Introduction and results

Let m denote the normalized Lebesgue measure on the unit circle T. For $1 \leq p \leq \infty, L^p = L^p(T, m)$ denotes the usual Lebesgue space on T and H^p denotes the usual Hardy space on T. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi i} \int_T \frac{f(\eta)}{\eta - \zeta} d\eta \quad (a.e.\ \zeta \in T)$$

where the integral is understood in the sense of Cauchy's principal value ($cf.$ [3, Vol I, p12]). If f is in L^1 then $(Sf)(\zeta)$ exists for almost everwhere ζ on T. Let an analytic projection and a co-analytic projection be

$$P = (I + S)/2 \text{ and } Q = (I - S)/2,$$

where I denotes the identity operator. If $\alpha, \beta \in L^\infty$, then the singular integral operator $S_{\alpha,\beta}$ on L^2 is defined by

$$S_{\alpha,\beta} f = \alpha P f + \beta Q f \quad (f \in L^2).$$

The following inequality is well known and not difficult to establish

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|S_{\alpha,\beta}\| \leq \|\sqrt{|\alpha|^2 + |\beta|^2}\|_\infty.$$

In the previous paper, the author and T.Yamamoto [4] showed the following theorem. This implies that $\|S_{\alpha,\beta}\| = \max(\|\alpha\|_\infty, \|\beta\|_\infty)$ when $\alpha\bar{\beta} \in H^\infty$.

Norm Theorem. *Let $\alpha, \beta \in L^\infty$. Then*

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .$$

The essential norm $\|S_{\alpha,\beta}\|_e$ is the distance to $\mathcal{K}(L^2)$, the set of all compact operators on L^2. It is known [3, Vol II, p219] that $\|S_{\alpha,\beta}\|_e = \max(\|\alpha\|_\infty, \|\beta\|_\infty)$ when α, β are in C, a set of all continuous functions on T. In this paper, we show the following theorem using Norm Theorem. This implies the above result.

Essential Norm Theorem. *Let $\alpha, \beta \in L^\infty$. Then*

$$\|S_{\alpha,\beta}\|_e^2 = \inf_{k \in H^\infty + C} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .$$

Corollary. *If* $\alpha\bar{\beta} \in H^\infty + C$, *then* $\|S_{\alpha,\beta}\|_e = \max(\|\alpha\|_\infty, \|\beta\|_\infty)$.

The norms and essential norms of Toeplitz operators and Hankel operators are very well known. If $\phi \in L^\infty$, then the Toeplitz operator T_ϕ is defiened by $T_\phi f = P(\phi f)$ for $f \in H^2$. The Hankel operator H_ϕ is defined by $H_\phi f = Q(\phi f)$ for $f \in H^2$. $\|T_\phi\| = \|\phi\|_\infty$ (*cf.* [2]) and $\|H_\phi\| = \|\phi + H^\infty\|$ by the Nehari theorem [6] (*cf.* [5]). $\|T_\phi\|_e = \|\phi\|_\infty$ (*cf.* [2]) and $\|H_\phi\|_e = \|\phi + H^\infty + C\|$ (*cf.* [5]).

§2. The proof of Essential Norm Theorem

Put $U = S_{z,\bar{z}}$, then $U^n = S_{z^n,\bar{z}^n}$ and $U^{*n}U^n = I$ for any positive integer n. Let K be arbitrary compact operator on L^2, that is, $K \in \mathcal{K}(L^2)$. Then

$$\begin{aligned}
&\|S_{\alpha,\beta} + K\| \\
&\geq \|(S_{\alpha,\beta} + K)U^n\| \geq \|S_{z^n\alpha,\bar{z}^n\beta}\| - \|KU^n\| \\
&\geq \|S_{z^n\alpha,\bar{z}^n\beta}\| - (\|PKU^nP\| + \|PKU^nQ\| \\
&+ \|QKU^nP\| + \|QKU^nQ\|)
\end{aligned}$$

Here $PKU^nQ = PKQ\bar{z}^nQ, QKU^nP = QKPz^nP, PKU^nP = PKPz^nP$ and $QKU^nQ = QKQ\bar{z}^nQ$. Since $(Pz^nP)^* \to 0$ and $(Q\bar{z}^nQ)^* \to 0$ as $n \to \infty$ in the strong opertor topology.

$$\lim \|PKU^nQ\| = \lim \|QKU^nP\| = 0.$$

and

$$\lim \|PKU^nP\| = \lim \|QKU^nQ\| = 0$$

because PKQ, QKP, PKP and QKQ are all compact.
By Norm Theorem in §1,

$$\|S_{z^n\alpha,\bar{z}^n\beta}\|^2 = \inf_{k \in \bar{z}^{2n}H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty$$

and hence

$$\begin{aligned}
&\lim_{n\to\infty} \|S_{\bar{z}^n\alpha,\bar{z}^n\beta}\|^2 \\
&= \inf_{k \in H^\infty + C} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty
\end{aligned}$$

because $H^\infty + C$ is closed. Thus

$$\|S_{\alpha,\beta}\|_e^2 \geq \inf_{k \in H^\infty + C} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .$$

Put $P_1 = zP\bar{z}$, then P_1 is an orthogonal projection from L^2 to zH^2 and $P_1 = zP\bar{z}(P+Q) = zP\bar{z}P$. Simitarly, $Q_1 = \bar{z}Qz$ is an orthogonal projection from L^2 to $\bar{z}^2\bar{H}^2$ and $Q_1 = \bar{z}QzQ$. Since $P - P_1$ and $Q - Q_1$ are rank one projections, $\|\alpha P + \beta Q\|_e = \|\alpha P_1 + \beta Q_1\|_e$. Since $\beta\bar{z}Q\bar{z}P$ and $\alpha zPzQ$ are finite rank operators,

$$\begin{aligned}
&\|(\alpha zP + \beta\bar{z}Q)(\bar{z}P + zQ)\|_e \\
&= \|\alpha zP\bar{z}P + \beta\bar{z}QzQ + \beta\bar{z}Q\bar{z}P + \alpha zPzQ\|_e \\
&= \|\alpha P_1 + \beta Q_1\|_e = \|\alpha P + \beta Q\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|S_{\alpha,\beta}\|_e &\leq \|S_{z\alpha,\bar{z}\beta}\|_e \|\bar{z}P + zQ\|_e \\
&\leq \|S_{z\alpha,\bar{z}\beta}\|_e \\
&= \|S_{z\alpha,\bar{z}\beta}\|
\end{aligned}$$

because

$$\begin{aligned}
\|\bar{z}P + zQ\|_e &= \|P\bar{z}P + QzQ\|_e \\
&= \max\{\|P\bar{z}P\|_e, \|QzQ\|_e\} = 1
\end{aligned}$$

Therefore

$$\begin{aligned}
\|S_{\alpha,\beta}\|_e &\leq \lim_{n\to\infty} \|S_{z^n\alpha,\bar{z}^n\beta}\| \\
&= \inf_{k \in H^\infty + C} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .
\end{aligned}$$

This completes the proof of Essential Norm Theorem.

References

1. S.Axler, I.D.Berg, N.Jewell, and A.Shields, Approximation by compact operators and the space $H^\infty + C$, Ann. of Math. 109(1979), 601-612.

2. R.G.Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.

3. I.Gohberg and N.Krupnik, One-dimensional linear singular integral equations, Vols. I,II, Birkhäuser Verlag, Basel, 1992.

4. T.Nakazi and T.Yamamoto, Norms of some singular integral operators and their inverse operators, J.Operator Th. 40(1998), 185-207.

5. N.K.Nikol'skii, Treatise on the Shift Operator, Springer Verlag, Berlin, 1986.

6. Z.Nehari, On bounded bilinear forms, Ann. of Math. 65(1957), 153-162.