THE MEROMORPHIC SOLUTIONS OF THE BRUSCHI-CALOGERO EQUATION

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ABSTRACT. We give all the meromorphic functions defined near the origin $0 \in \mathbb{C}$ satisfying a functional equation investigated by Bruschi and Calogero [1], [2].

0. Introduction.

It is an important problem to find a Lax pair $\mathbf{L}$ and $\mathbf{M}$ whose equations of motion are equivalent to the Lax equation [10], [11], [12]. In order to prove their complete integrability it is convenient to use a Lax representation.

The systems of Calogero-Sutherland type, which describe one-dimensional n-particle dynamics, are defined by the following Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + U(q_1, \dots, q_n),$$

where the potential U has the form

$$U(q_1, \dots, q_n) = g^2 \sum_{j<k}^{n} v(q_j - q_k).$$

Lax pairs for the system above were originated by Calogero [3] and Moser [7], and are given by the matrices

$$L_{jk} = p_j \delta_{jk} + \sqrt{-1} g (1 - \delta_{jk}) x(q_j - q_k),$$
$$M_{jk} = g[\delta_{jk} \sum_{l \neq j} z(q_j - q_l) - (1 - \delta_{jk}) y(q_j - q_k)].$$

Substituting these matrices in the Lax equation $\sqrt{-1}\dot{\mathbf{L}} = [\mathbf{M}, \mathbf{L}]$ and requiring that this equation is equivalent to the Hamiltonian equations, we get a certain functional equation for the functions $x(\xi)$ and $z(\xi)$. This functional equation has

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been solved in a series of papers including [4], [9]. The solutions are expressed in terms of elliptic functions, trigonometric functions or rational functions.

Later, Ruijsenaars and Schneider [15] have introduced a class of integrable dynamical systems characterized by the equations of motion

$$\ddot{q}_j = \sum_{\substack{k=1\\k\neq j}}^{n} \dot{q}_j\, \dot{q}_k\, v(q_j - q_k), \quad q_j = q_j(t), \quad j = 1, 2, \ldots, n. \tag{0.1}$$

Bruschi and Calogero [1] discovered a representation of the equations of motion of the system (0.1) in the Lax form

$$\dot{\mathbf{L}} = [\mathbf{L}, \mathbf{M}],$$

where $\mathbf{L}$ and $\mathbf{M}$ are the $n \times n$ matrices,

$$L_{jk} = \delta_{jk}\dot{q}_j + (1-\delta_{jk})(\dot{q}_j\,\dot{q}_k)^{1/2}\alpha(q_j - q_k),$$
$$M_{jk} = \delta_{jk}\sum_{\substack{m=1\\m\neq j}}^{n} \dot{q}_m\beta(q_j - q_m) + (1-\delta_{jk})(\dot{q}_j\,\dot{q}_k)^{1/2}\gamma(q_j - q_k).$$

Here the function $\alpha(x)$ is a solution of the following functional equation of addition type

$$\alpha(x)\alpha'(y) - \alpha'(x)\alpha(y) = (\alpha(x+y) - \alpha(x)\alpha(y))\,(\eta(x) - \eta(y))\,, \tag{0.2}$$

which we call the Bruschi-Calogero equation. The function $v(x)$ is given by

$$v(x) = \frac{d}{dx}\log(\alpha(x)\alpha(-x) - 1)\,.$$

Bruschi and Calogero [1], [2] have investigated general analytic solutions of this functional equation (0.2). They have obtained some solutions α expressed by elliptic functions in the most general case, and they had some trigonometric and rational solutions by degenerating the periods of the elliptic functions.

The main purpose of the present paper is to solve the functional equation (0.2) in the most rigorous way. More precisely, we shall give all meromorphic solutions of the functional equation (0.2) defined near the origin $0 \in \mathbb{C}$.

Theorem 0.1. *Let α be a holomorphic function defined on a small punctured disk $\{x \in \mathbb{C}; 0 < |x| < r\}$ for some $r > 0$. If there exists a holomorphic function η defined on the punctured disk $\{x \in \mathbb{C}; 0 < |x| < r\}$ such that α and η satisfy the functional equation (0.2), then α is equal to one of the following meromorphic functions defined on the whole plane $\mathbb{C}$.*

(0) $\qquad \alpha(x) = Ce^{\rho x}, \quad (C, \rho \in \mathbb{C})$

(I) $$\alpha(x) = e^{\rho x}\frac{\sigma(\mu;\tau_1,\tau_2)\sigma(\lambda x+\nu;\tau_1,\tau_2)}{\sigma(\nu;\tau_1,\tau_2)\sigma(\lambda x+\mu;\tau_1,\tau_2)},$$
$$(\rho,\mu,\nu\in\mathbb{C},\ \lambda,\tau_1,\tau_2\in\mathbb{C}\setminus\{0\}, \operatorname{Im}\tau_2/\tau_1 > 0)$$

(II) $$\alpha(x) = e^{\rho x}\frac{a(e^{2x/\lambda}-1)+b}{c(e^{2x/\lambda}-1)+b},$$
$$(\lambda,\rho,a,b,c\in\mathbb{C},\ \lambda\neq 0,\ b(a-c)\neq 0)$$

(III) $$\alpha(x) = e^{\rho x}\frac{ax+b}{cx+b}, \quad (\rho,a,b,c\in\mathbb{C},\ b(a-c)\neq 0)$$

Here $\sigma(x;\tau_1,\tau_2)$ is the elliptic sigma function.

It should be remarked that a meromorphic function defined near the origin $0 \in \mathbb{C}$ is holomorphic on a small punctured disk $\{x \in \mathbb{C}; 0 < |x| < r\}$ for some $r > 0$. Hence our result covers all the meromorphic solutions defined near the origin $0 \in \mathbb{C}$.

The methods we use in this paper are quite different from those of Bruschi and Calogero [1], [2]. We should note that some trigonometric solutions listed above are not included in Bruschi and Calogero [1], [2].

The outline to get all meromorphic solutions is as follows. First, we shall show that the solution η is the logarithmic derivative of some meromorphic function φ, and that the set of zeroes of φ is a discrete subgroup of $\mathbb{C}$. As is known, such a subgroup is isomorphic to $\mathbb{Z}^2$, $\mathbb{Z}$ or $\{0\}$. If this subgroup is isomorphic to $\mathbb{Z}^2$, we find out that η and α are expressed by elliptic functions by a standard argument. In other two cases, the key tool to obtaining the explicit form of the solution α is the great Picard theorem (see, for example, [5]). As a result, we shall show that α is expressed by trigonometric functions or rational functions.

After the first draft of this paper was completed, we were informed that Ochiai, Oshima and Sekiguchi [8], [13] have studied all the completely integrable systems with the invariance under the action of the Weyl groups. In their papers, they solved the functional differential equations of the potential function. We should note that these functional differential equations are of addtion type also.

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1. Equivalence of two functional equations.

In this section we prove that any holomorphic solutions of the equation (0.2) defined on a small punctured disk $\{x \in \mathbb{C}; 0 < |x| < r\}$ for some $r > 0$ extend themselves to meromorphic functions defined near the origin $0 \in \mathbb{C}$. Moreover we show that the equation (0.2) and the equation

$$\alpha(x+y) - \alpha(x)\alpha(y) = \varphi(x)\varphi(y)\psi(x+y). \tag{1.1}$$

for meromorphic functions α, φ and ψ are equivalent to each other. Before we begin proving them, we have to distinguish a certain kind of solutions from other ones.

Lemma 1.1. *Let $\alpha = \alpha(x)$ be a holomorphic function defined on a small punctured disk $\{x \in \mathbb{C}; 0 < |x| < r\}$ for some $r > 0$. Suppose α satisfies one of the equations*

(1) $\quad \alpha(x)\alpha'(y) - \alpha'(x)\alpha(y) = 0,$

(2) $\quad$ *α is holomorphic at $0 \in \mathbb{C}$, and* $\quad \alpha(0)\alpha(x+y) - \alpha(x)\alpha(y) = 0,$

(3) $\quad \alpha(x+y) - \alpha(x)\alpha(y) = 0,$

where x, y and $x+y$ run over the defining domain. Then there exist C and $\rho \in \mathbb{C}$ such that $\alpha(x) = Ce^{\rho x}$.

Proof. When $\alpha = 0$, the lemma is trivial. So we may assume $\alpha \neq 0$. The equation (3) implies (2). In fact, choose a point x in $\{0 < |x| < r\}$. Then $\lim_{y\to 0}\alpha(y) = \lim_{y\to 0}\alpha(x+y)/\alpha(x) = 1$. So we have $\alpha(0) = 1 \neq \infty$. Thus we obtain (2).

The equation (2) implies (1). If we substitute $x = t$ and $y = s - t$ into (2), then we have $\alpha(0)\alpha(s) - \alpha(t)\alpha(s-t) = 0$. Differentiate it by the variable t. Then we have $\alpha(t)\alpha'(s-t) - \alpha'(t)\alpha(s-t) = 0$, which is equivalent to (1).

From the equation (1) we have $\alpha'(y)/\alpha(y) = \alpha'(x)/\alpha(x)$ for any x and y in the defining domain. In other words, α'/α is a constant function. Hence there exist C and $\rho \in \mathbb{C}$ such that $\alpha(x) = Ce^{\rho x}$. $\square$

The exponential function $\alpha(x) = Ce^{\rho x}$ for any C and $\rho \in \mathbb{C}$ satisfies both of the equations (0.2) and (1.1). Throughout this paper we call such a exponential solution *obvious*.

Next we prove that the solutions have no essential singularities at $0 \in \mathbb{C}$.

Lemma 1.2. *Let $\alpha = \alpha(x)$ and $\eta = \eta(x)$ be holomorphic functions defined on a small punctured disk $\{x \in \mathbb{C}; 0 < |x| < r\}$ for some $r > 0$. Suppose α and η satisfy the functional equation* (0.2). *Then the functions α and η extend themselves to meromorphic functions defined near the origin $0 \in \mathbb{C}$.*

Proof. In the case where the solutions are obvious, the lemma is trivial. So we may assume that they are not obvious. Hence we have $\eta(x) - \eta(y) \neq 0$, and $\alpha(x+y) - \alpha(x)\alpha(y) \neq 0$, as functions in (x, y).

Fix an arbitrary point t_0 in the punctured disk. Then (0.2) implies

$$\alpha(s) = \alpha(t_0)\alpha(s-t_0) + \frac{\alpha(t_0)\alpha'(s-t_0) - \alpha'(t_0)\alpha(s-t_0)}{\eta(t_0) - \eta(s-t_0)}$$

for any s, $0 < |s| < r - |t_0|$. The right-hand side extends itself to a meromorphic function near $s = 0$. Hence $\alpha(s)$ is meromorphic at $s = 0$.

Fix a point y_0 in the punctured disk. From (0.2) again, we have

$$\eta(x) = \eta(y_0) + \frac{\alpha(x)\alpha'(y_0) - \alpha'(x)\alpha(y_0)}{\alpha(x+y_0) - \alpha(x)\alpha(y_0)}$$

for any x, $0 < |x| < r - |y_0|$. The right-hand side is meromorphic at $x = 0$, and so is the function $\eta(x)$. $\square$

Thus, in what follows, we may assume that the solutions are meromorphic functions defined near the origin $0 \in \mathbb{C}$.

Now we can prove the equivalence of the two equations (0.2) and (1.1).

Proposition 1.3. *The two functional equations* (0.2) *and* (1.1) *for meromorphic functions defined on the whole plane $\mathbb{C}$ (or defined near the origin $0 \in \mathbb{C}$) are equivalent to each other. Moreover the solutions φ and η are related to each other as follows:*

$$\eta(x) = \varphi'(x)/\varphi(x).$$

Proof. We may assume a solution α is *not obvious*. Hence we have $\alpha(x+y) - \alpha(x)\alpha(y) \neq 0$ and $\alpha(x)\alpha'(y) - \alpha'(x)\alpha(y) \neq 0$ from Lemma 1.1.

Introduce a coordinate system (s, t) on the plane $\mathbb{C}^2$ by

$$(1.2) \qquad \begin{cases} s := x + y \\ t := x. \end{cases}$$

Then the equation (0.2) is equivalent to

$$\frac{\partial}{\partial t}\log\left(\alpha(s)-\alpha(t)\alpha(s-t)\right)=\eta(t)-\eta(s-t), \tag{1.3}$$

and the equation (1.1) to

$$\alpha(s)-\alpha(t)\alpha(s-t)=\varphi(t)\varphi(s-t)\psi(s). \tag{1.4}$$

The equation (1.3) is derived from the equation (1.4) by differentiating it by the variable t.

In order to prove that the equation (1.4) is derived from (1.3), we need the following.

Lemma 1.4. *If meromorphic functions α and η satisfy the equation* (1.4) *and α is not obvious, then we have* $\operatorname{ord}_p \eta = -1$ *and* $\operatorname{Res}_p \eta \in \mathbb{Z}$ *for any pole $p \in \mathbb{C}$ of the function η.*

Proof of Lemma 1.4. Let $\lambda \in \mathbb{C}$ be a pole of the function η. Choose a generic $s_0 \in \mathbb{C}$. Then the singular part of $\eta(t)-\eta(s_0-t)$ at $t=\lambda$ coincides with that of $\eta(t)$. On the other hand, the equation (1.4) implies that the function $\eta(t)-\eta(s_0-t)$ is equal to the logarithmic derivative of the function $\alpha(s_0)-\alpha(t)\alpha(s_0-t)$. Thus the order of $\eta(t)$ at the point λ is -1, and its residue is equal to the order of $\alpha(s_0)-\alpha(t)\alpha(s_0-t)$ at the point $t=\lambda$. This means the residue is an integer, as was to be shown. □

Now suppose meromorphic functions α and η defined on the whole plane $\mathbb{C}$ satisfy the equation (1.3). Lemma 1.4 means that the differential equation

$$\eta(x)=\varphi'(x)/\varphi(x)$$

has a local meromorphic solution φ near any point of the whole plane $\mathbb{C}$. Since the plane $\mathbb{C}$ is simply connected, such local meromorphic solutions extend themselves to the global solution φ defined on the whole $\mathbb{C}$. Then the equation (1.3) implies

$$\frac{\partial}{\partial t}\log\left(\frac{\alpha(s)-\alpha(t)\alpha(s-t)}{\varphi(t)\varphi(s-t)}\right)=0,$$

so that there exists a meromorphic function $\psi(s)$ defined on the whole plane $\mathbb{C}$ such that $\alpha(s)-\alpha(t)\alpha(s-t)=\varphi(t)\varphi(s-t)\psi(s)$ for any s and $t \in \mathbb{C}$. Substituting (1.2) into this equation, we obtain the equation (1.4). As for local solutions, the situation is simpler. This completes the proof of Proposition 1.3. □

2. Behavior of solutions near the origin.

In what follows, we confine ourselves to *non-obvious* meromorphic solutions α, η, φ and ψ of the functional equations (0.2) and (1.1). We remark $\eta(x)-\eta(y) \neq 0$ and $\varphi \neq 0$, since α is not obvious. We consider local solutions, i.e., solutions defined near the origin, and global solutions, i.e., solutions defined on the whole plane $\mathbb{C}$, simultaneously.

Lemma 2.1.
(1) *Let $p \in \mathbb{C}$ be a point in the defining domain of solutions α, η, ϕ and ψ. Then p is a pole of the function φ, if and only if p is a pole of the function α.*
(2) *If $p \in \mathbb{C}$ is a pole of φ, then the orders of α and φ at p are equal to each other, i.e., $\mathrm{ord}_p\, \varphi = \mathrm{ord}_p\, \alpha$. Especially we have $(\alpha/\varphi)(p) \neq 0, \infty$.*
(3) *A point $p \in \mathbb{C}$ is a zero of the function φ, if and only if the equation $\alpha(s) = \alpha(p)\alpha(s-p)$ holds for all s in the defining domain of α.*

Proof. Choose a generic $s_0 \in \mathbb{C}$ such that $\alpha(s_0-p)$, $\alpha(s_0)$, $\varphi(s_0-p)$ and $\psi(s_0) \neq 0$, ∞. Then we have

$$\mathrm{ord}_p\,(\alpha(s_0)-\alpha(t)\alpha(s_0-t)) = \mathrm{ord}_p(\varphi(t)\varphi(s_0-t)) = \mathrm{ord}_p\,\varphi, \tag{2.1}$$

since $(\alpha(s_0)-\alpha(t)\alpha(s_0-t))\varphi(t)^{-1}\varphi(s_0-t)^{-1} = \psi(s_0) \neq 0, \infty$, and $\varphi(s_0-p) \neq 0, \infty$.

Now suppose $\varphi(p) = \infty$. Then $\alpha(s_0) - \alpha(p)\alpha(s_0-p) = \infty$. We have $\alpha(p) = \infty$, since $\alpha(s_0-p)$, $\alpha(s_0) \neq 0, \infty$. Moreover $\mathrm{ord}_p(\varphi) = \mathrm{ord}_p\,(\alpha(s_0)-\alpha(t)\alpha(s_0-t)) = \mathrm{ord}_p\,(\alpha(t)\alpha(s_0-t)) = \mathrm{ord}_p\,\alpha$. Conversely suppose $\alpha(p) = \infty$. Then $\alpha(s_0) - \alpha(p)\alpha(s_0-p) = \infty$, so that $\varphi(p) = \infty$ by (2.1).

Finally suppose $\varphi(p) = 0$. Then (2.1) implies $\alpha(s_0)-\alpha(p)\alpha(s_0-p) = 0$ for generic s_0. Hence $\alpha(s) = \alpha(p)\alpha(s-p)$ for all s in the defining domain of α. Conversely we assume $\alpha(s) = \alpha(p)\alpha(s-p)$ for all s in the defining domain. Then (2.1) implies $\varphi(p) = 0$.

This completes the proof of Lemma 2.1. □

Our main purpose in this section is to prove the following.

Lemma 2.2. *Let α, η, φ and ψ be non-obvious meromorphic solutions of* (0.2) *and* (1.1) *defined near the origin* $0 \in \mathbb{C}$. *Then we have $\varphi(0) = 0$, $\alpha(0) = 1$, $\varphi'(0) \neq 0$ and*

$$\psi(s) = \left(\frac{\alpha'(s)}{\alpha(s)} - \frac{\alpha'(0)}{\alpha(0)}\right)\frac{\alpha(s)}{\varphi'(0)\varphi(s)}. \tag{2.2}$$

Proof. We begin by proving that φ and α is holomorphic near the origin $0 \in \mathbb{C}$.

Assume $\varphi(0) = \infty$. Then $\alpha(0) = \infty$ and $(\alpha/\varphi)(0) \neq 0, \infty$ from Lemma 2.1. The equation (1.1) implies

$$\varphi(t)^{-1}\alpha(s)\varphi(s-t)^{-1} - (\alpha/\varphi)(t)(\alpha/\varphi)(s-t) = \psi(s).$$

We obtain $\psi(s) = -(\alpha/\varphi)(0)(\alpha/\varphi)(s)$ as $t \to 0$. Hence we have

$$-(\alpha/\varphi)(0)(\alpha/\varphi)(s)\frac{1}{\alpha(s)} = \frac{\psi(s)}{\alpha(s)} = \varphi(t)^{-1}\varphi(s-t)^{-1}\left(1 - \frac{\alpha(t)\alpha(s-t)}{\alpha(s)}\right).$$

and $\varphi(t)^{-1}\varphi(-t)^{-1} = 0$ as $s \to 0$. This means $\varphi = 0$, and contradicts the assumption α is not obvious. Therefore we find out $\varphi(0) \neq \infty$. From Lemma 2.1 we have $\alpha(0) \neq \infty$.

Thus φ and α are holomorphic near the origin. Especially we may substitute $x = 0$ into the derivatives $\alpha'(x)$ and $\varphi'(x)$. Differentiate the equation (1.4) by the variable t, and substitute $t = 0$ into it. Then we obtain

$$\alpha(0)\alpha'(s) - \alpha'(0)\alpha(s) = \psi(s)(\varphi'(0)\varphi(s) - \varphi(0)\varphi'(s)). \tag{2.3}$$

Next, in order to prove $\varphi(0) = 0$, assume $\varphi(0) \neq 0$. Substituting $t = 0$ into the equation (1.1), we obtain

$$\psi(s) = \frac{(1-\alpha(0))\alpha(s)}{\varphi(0)\varphi(s)}.$$

This, together with the equation (2.3), implies

$$\alpha(0)\frac{\alpha'(s)}{\alpha(s)} - \alpha'(0) = (1-\alpha(0))\left(\frac{\varphi'(0)}{\varphi(0)} - \frac{\varphi'(s)}{\varphi(s)}\right).$$

Hence we have

$$\alpha(0)\frac{\alpha'(x)}{\alpha(x)} + (1-\alpha(0))\frac{\varphi'(x)}{\varphi(x)} = \alpha'(0) + (1-\alpha(0))\frac{\varphi'(0)}{\varphi(0)} = \alpha(0)\frac{\alpha'(y)}{\alpha(y)} + (1-\alpha(0))\frac{\varphi'(y)}{\varphi(y)}$$

and so

$$\begin{aligned}(1-\alpha(0))(\eta(x)-\eta(y)) =&(1-\alpha(0))\left(\frac{\varphi'(x)}{\varphi(x)} - \frac{\varphi'(y)}{\varphi(y)}\right) = \alpha(0)\left(\frac{\alpha'(y)}{\alpha(y)} - \frac{\alpha'(x)}{\alpha(x)}\right)\\ =&\alpha(0)\left(\frac{\alpha(x+y)}{\alpha(x)\alpha(y)} - 1\right)(\eta(x)-\eta(y)).\end{aligned}$$

Here recall $\eta(x) - \eta(y) \neq 0$, since α is not obvious. Therefore we have $\alpha(0)\alpha(x+y) = \alpha(x)\alpha(y)$. This means α is obvious from Lemma 1.1, and contradicts our assumption.

Therefore we obtain $\varphi(0) = 0$. From Lemma 2.1 (3) we have $\alpha(s) - \alpha(0)\alpha(s) = 0$. Hence $\alpha(0) = 1$. Now the formula (2.3) turns out to be

$$\alpha'(s) - \alpha'(0)\alpha(s) = \psi(s)\varphi'(0)\varphi(s).$$

Since α is not obvious, we have $\varphi'(0) \neq 0$. Hence we have $\psi(s) = (\alpha'(s)/\alpha(s) - \alpha'(0)/\alpha(0))\alpha(s)/\varphi'(0)\varphi(s)$, as was to be shown. □

The formula (2.2) means

$$\frac{\alpha(s) - \alpha(t)\alpha(s-t)}{\varphi(t)\varphi(s-t)} = \frac{\alpha'(s) - \alpha'(0)\alpha(s)}{\varphi'(0)\varphi(s)}, \tag{2.4}$$

which plays an important role throughout this paper.

As a consequence of the relation (2.2), we obtain

Proposition 2.3. *Any meromorphic solutions of the equations* (0.2) *and* (1.1) *defined near the origin extend themselves to those defined on the whole plane* $\mathbb{C}$.

Proof. As for obvious solutions, the proposition is trivial. So we may assume α, η, φ and ψ are non-obvious meromorphic solutions of the equations (0.2) and (1.1) defined on the small disk $\{x \in \mathbb{C}; |x| < r\}$ for some $r > 0$.

From (0.2) we have

$$\alpha(x+y) = \alpha(x)\alpha(y) + \frac{\alpha(x)\alpha'(y) - \alpha'(x)\alpha(y)}{\eta(x) - \eta(y)},$$

which implies α extends itself to $\{x \in \mathbb{C}; |x| < 2r\}$. From the equation (1.1)

$$\psi(x+y) = \frac{\alpha(x+y) - \alpha(x)\alpha(y)}{\varphi(x)\varphi(y)}$$

and the equation (2.2)

$$\varphi(s) = \left(\frac{\alpha'(s)}{\alpha(s)} - \frac{\alpha'(0)}{\alpha(0)}\right)\frac{\alpha(s)}{\varphi'(0)\psi(s)},$$

ψ and φ extend themselves to $\{x \in \mathbb{C}; |x| < 2r\}$. Recall η is equal to φ'/φ. Such extensions satisfy the functional equations by means of the permanence of functional relations.

Consequently these solutions extend themselves to $\{x \in \mathbb{C}; |x| < 2^n r\}$ for all $n \geq 1$, and so to the whole plane $\mathbb{C}$. This completes the proof. $\square$

3. Discrete subgroups.

In view of Proposition 2.3 we may confine ourselves to non-obvious meromorphic solutions defined on the whole plane $\mathbb{C}$.

We denote the zeroes of φ, the poles of φ and the zeroes of α by Λ_+, Λ_- and Λ^α, respectively. Clearly these subsets Λ_+, Λ_- and Λ^α are all discrete subsets in the plane $\mathbb{C}$. In this section we study these discrete subsets.

By Lemma 2.1 (3) we have

$$\Lambda_+ = \{u \in \mathbb{C}; \alpha(t+u) = \alpha(u)\alpha(t), \quad \forall t \in \mathbb{C}\}. \tag{3.1}$$

Since $\varphi(0) = 0$, we have $\Lambda_+ \neq \emptyset$. One can deduce easily the following lemma from the identification (3.1).

Lemma 3.1. *The discrete subset Λ_+ is a subgroup of the additive group $\mathbb{C}$. Moreover the restriction of α to Λ_+ gives a homomorphism of Λ_+ into the multiplicative group $\mathbb{C} \setminus \{0\}$. Especially $\alpha(u) \neq 0$ for any $u \in \Lambda_+$.*

As is known, any discrete subgroup Λ_+ of the additive group $\mathbb{C}$ is given by one of the following.

(1) There exist τ_1 and $\tau_2 \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Im} \tau_2/\tau_1 > 0$ and $\Lambda_+ = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$. In what follows, we call such a case *non-degenerate.*

(2) There exists $\lambda_0 \in \mathbb{C} \setminus \{0\}$ such that $\Lambda_+ = \mathbb{Z}\lambda_0$. We will consider this case in §6.

(3) $\Lambda_+ = 0$. We will consider this case in §5.

(See, for example, Pontryagin [14] ch.3, §19, example 33.) In short, we obtain

$$\Lambda_+ \cong \mathbb{Z}^2,\ \mathbb{Z} \text{ or } \{0\}\,.$$

In the succeeding sections, in the case where this subgroup is isomorphic to $\mathbb{Z}^2$, we prove that η and α are expressed by elliptic functions by a standard argument. In other two cases, we shall show that α is expressed by trigonometric functions or rational functions.

Next we study the discrete subsets Λ_- and Λ^α.

Lemma 3.2. *If $p \in \Lambda_-$, namely, p is a pole of φ and α, then*

$$\varphi(p-t)\varphi(t) = -\varphi'(0)\operatorname{Res}_p \varphi \neq 0\,,$$

and $\operatorname{ord}_p \varphi = \operatorname{ord}_p \alpha = -1$.

Proof. Choose a generic $t_0 \in \mathbb{C}$ such that $\varphi(t_0)$, $\varphi(p-t_0)$, $\alpha(t_0)$ and $\alpha(p-t_0) \neq 0, \infty$. From the equation (2.4) we have

$$\frac{\varphi'(0)}{\varphi(t_0)\varphi(s-t_0)}\left(1-\frac{\alpha(t_0)\alpha(s-t_0)}{\alpha(s)}\right) = \frac{1}{\varphi(s)}\left(\frac{\alpha'(s)}{\alpha(s)} - \alpha'(0)\right).$$

The left-hand side turns out to be $\varphi'(0)/(\varphi(t_0)\varphi(p-t_0)) \neq 0, \infty$ as $s \to p$. Hence

$$\lim_{s\to p}\frac{1}{\varphi(s)}\left(\frac{\alpha'(s)}{\alpha(s)} - \alpha'(0)\right) \neq 0, \infty,$$

so that $\operatorname{ord}_p \varphi = \operatorname{ord}_p(\alpha'(s)/\alpha(s) - \alpha'(0))$. Since $\operatorname{ord}_p \varphi < 0$, $s = p$ is a pole of $\alpha'(s)/\alpha(s) - \alpha'(0)$. Hence we have $\operatorname{ord}_p(\alpha'(s)/\alpha(s) - \alpha'(0)) = \operatorname{ord}_p(\alpha'(s)/\alpha(s)) = -1$. Thus we obtain $\operatorname{ord}_p \alpha = \operatorname{ord}_p \varphi = -1$.

Now $\varphi(s) = (s-p)^{-1}f(s)$ and $\alpha'(s)\alpha(s)^{-1} - \alpha'(0) = (s-p)^{-1}\beta(s)$ for some holomorphic functions f and β defined near p. Then $\beta(p) = -1$, $f(p) = \operatorname{Res}_p \varphi$, and so

$$\lim_{s\to p}\frac{1}{\varphi(s)}\left(\frac{\alpha'(s)}{\alpha(s)} - \alpha'(0)\right) = \lim_{s\to p}\frac{\beta(s)}{f(s)} = \frac{-1}{\operatorname{Res}_p \varphi}.$$

This means $\varphi'(0)/(\varphi(t_0)\varphi(p-t_0)) = -1/\operatorname{Res}_p \varphi$ for any generic t_0. Therefore we have

$$\varphi(t)\varphi(p-t) = -\varphi'(0)\operatorname{Res}_p \varphi$$

for any $t \in \mathbb{C}$. Since α is not obvious, we have $\varphi \neq 0$ and so $-\varphi'(0)\operatorname{Res}_p \varphi \neq 0$.

This completes the proof. $\square$

Corollary 3.3. *Λ_- is invariant under the translation by Λ_+ and $\sharp(\Lambda_-/\Lambda_+) \leq 1$.*

Proof. Let $p \in \Lambda_-$ and $u \in \Lambda_+$ be given. Recall $\alpha(u) \neq 0, \infty$. Then we have $\alpha(p+u) = \alpha(p)\alpha(u) = \infty$, so that $p + u \in \Lambda_-$. This means Λ_- is a Λ_+-invariant subset.

Now, for any p_1 and $p_2 \in \Lambda_-$, $\lim_{t\to p_2}\varphi(p_1 - t)\varphi(t) = -\varphi'(0)\operatorname{Res}_p \varphi \neq 0, \infty$. Hence $\lim_{t\to p_2}\varphi(p_1 - t) = 0$, and so $p_1 - p_2 \in \Lambda_+$. This means $\sharp(\Lambda_-/\Lambda_+) \leq 1$, as was to be shown. $\square$

Lemma 3.4. *Let $q \in \mathbb{C}$ be a zero of α, i.e., $q \in \Lambda^\alpha$. Then we have*

$$\alpha'(q) \neq 0 \quad \text{i.e.,} \quad \mathrm{ord}_q\, \alpha = 1. \tag{1}$$

$$\frac{\alpha(t)\alpha(q-t)}{\varphi(t)\varphi(q-t)} = -\frac{\alpha'(q)}{\varphi'(0)\varphi(q)} \neq 0. \tag{2}$$

Moreover Λ^α is invariant under the translation by Λ_+ and $\sharp(\Lambda^\alpha/\Lambda_+) \leq 1$.

Proof. Since $\Lambda_+ \cap \Lambda^\alpha = \emptyset$, we have $\varphi(q) \neq 0$. From the equation (2.4) we have

$$\frac{-\alpha(t)\alpha(q-t)}{\varphi(t)\varphi(q-t)} = \frac{\alpha'(q)}{\varphi'(0)\varphi(q)},$$

Since $\alpha/\varphi \neq 0$, we obtain (2) and $\alpha'(q) \neq 0$, so that $\mathrm{ord}_q\, \alpha = 1$.

Now let $q \in \Lambda^\alpha$ and $u \in \Lambda_+$ be given. Then we have $\alpha(q+u) = \alpha(q)\alpha(u) = 0$, so that $p + u \in \Lambda_-$. This means Λ^α is a Λ_+-invariant subset.

Finally, for any q_1 and $q_2 \in \Lambda_-$,

$$\frac{-\alpha(q_2)\alpha(q_1-q_2)}{\varphi(q_2)\varphi(q_1-q_2)} = \frac{\alpha'(q_1)}{\varphi'(0)\varphi(q_1)} \neq 0, \infty.$$

Since $\alpha(q_2)/\varphi(q_2) = 0$, we have $(\alpha/\varphi)(q_1 - q_2) = \infty$. From Lemma 2.1 (2) follows $q_1 - q_2 \notin \Lambda_-$. Thus we obtain $\varphi(q_1 - q_2) = 0$, i.e., $q_1 - q_2 \in \Lambda_+$. This means $\sharp(\Lambda^\alpha/\Lambda_+) \leq 1$. This completes the proof of Lemma 3.4. □

4. Non-degenerate case.

In this section we consider the case $\Lambda_+ \cong \mathbb{Z}^2$ i.e. the doubly-periodic case. The function η admits a periodicity with respect to the discrete subgroup Λ_+.

Lemma 4.1.

(1) *The set of all poles of the function η is equal to $\Lambda_+ \cup \Lambda_-$. All poles of η are simple, and* $\mathrm{Res}_p\, \eta = \mathrm{ord}_p\, \varphi$ *for $p \in \Lambda_+ \cup \Lambda_-$.*
(2) $\eta(z+p) = \eta(z), \quad p \in \Lambda_+, z \in \mathbb{C}$.
(3) $\mathrm{Res}_p\, \eta = 1, \quad p \in \Lambda_+$.
(4) $\mathrm{Res}_p\, \eta = -1, \quad p \in \Lambda_-$.

Proof. The assertions (1), (3) and (4) are clear due to $\eta = \varphi'/\varphi$. In fact, from the definition, $\Lambda_+ = \{\varphi = 0\}$ and $\Lambda_- = \{\varphi = \infty\}$.

Let $p \in \Lambda_+ = \{z \in \mathbb{C}; \varphi(z) = 0\}$. In order to prove (2), recall the function α is regular at $z = p$ because $\{\alpha = \infty\} = \{\varphi = \infty\}$ (Lemma 2.1 (1)). From Lemma 3.1, $\alpha(p) \neq 0$. By virtue of (3.1)

$$\begin{aligned}
\eta(t) - \eta(p-t) &= \frac{\partial}{\partial t} \log(\alpha(p) - \alpha(t)\alpha(p-t)) \\
&= \frac{\partial}{\partial t} \log(\alpha(p) - \alpha(t)\alpha(p)\alpha(-t)) \\
&= \frac{\partial}{\partial t} (\log \alpha(p) + \log(1 - \alpha(t)\alpha(-t))) \\
&= \frac{\partial}{\partial t} \log(1 - \alpha(t)\alpha(-t)) \\
&= \eta(t) - \eta(-t)
\end{aligned}$$

Hence $\eta(-t) = \eta(p-t)$ for all $t \in \mathbb{C}$ i.e. $\eta(z+p) = \eta(z)$ for all $z \in \mathbb{C}$. $\square$

From $\Lambda_+ \cong \mathbb{Z}^2$, there exist τ_1, τ_2 and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Im} \tau_2/\tau_1 > 0$ and $\Lambda_+ = \mathbb{Z}(\tau_1/\lambda) + \mathbb{Z}(\tau_2/\lambda)$.

Lemma 4.2. *There exists $\mu \in \mathbb{C}$ such that $\Lambda_- = (-\mu/\lambda) + \Lambda_+$.*

Proof. Suppose $\Lambda_- = \emptyset$. The set of the poles of η is Λ_+ from Lemma 4.1 (1). Thus, in a fundamental period-parallelogram for the elliptic function η, the number of poles of the function η is one and the pole is simple. There is no such an elliptic function [16 p.432], [6 p.157]. Then $\Lambda_- \neq \emptyset$.

Hence there exists $\mu \in \mathbb{C}$ such that $-\mu/\lambda \in \Lambda_-$. Because Λ_- is invariant under the translation by Λ_+, $(-\mu/\lambda) + \Lambda_+ \subset \Lambda_-$.

We take $p \in \Lambda_-$. By $\sharp(\Lambda_-/\Lambda_+) \leq 1$, $p - (-\mu/\lambda) \in \Lambda_+$. Then $p \in (-\mu/\lambda) + \Lambda_+$ i.e. $\Lambda_- \subset (-\mu/\lambda) + \Lambda_+$. Therefore $\Lambda_- = (-\mu/\lambda) + \Lambda_+$. $\square$

We can summarize the conclusion of the function $\eta(z)$ just obtained as follows:

(1) The function $\eta(z)$ is doubly-periodic with periods $\tau_1/\lambda, \tau_2/\lambda$.

(2) The set of poles of η is $\Lambda_+ \cup ((-\mu/\lambda) + \Lambda_+)$, and all poles are simple.

(3) $\operatorname{Res}_p \eta = \begin{cases} 1, & p \in \Lambda_+, \\ -1, & p \in (-\mu/\lambda) + \Lambda_+. \end{cases}$

By [16 p.449], [6 p.177], the function $\eta(z)$ is expressed by the function $\zeta(z)$:

Theorem 4.3.

$$\eta(z) = A + \lambda\zeta(\lambda z; \tau_1, \tau_2) - \lambda\zeta(\lambda z + \mu; \tau_1, \tau_2),$$

where $A \in \mathbb{C}$ and

$$\zeta(z; \tau_1, \tau_2) = \frac{1}{z} + \sum_{\substack{\omega = m_1\tau_1 + m_2\tau_2 \\ (m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

Proof. By [16 p.449], [6 p.177], we get

$$\eta(z) = A + \zeta(z; \frac{\tau_1}{\lambda}, \frac{\tau_2}{\lambda}) - \zeta(z + \frac{\mu}{\lambda}; \frac{\tau_1}{\lambda}, \frac{\tau_2}{\lambda})$$

for some constant A. From [6 p.184]

$$\zeta(\lambda z; \lambda\tau_1, \lambda\tau_2) = \frac{1}{\lambda}\zeta(z; \tau_1, \tau_2).$$

Hence

$$\eta(z) = A + \lambda\zeta(\lambda z; \tau_1, \tau_2) - \lambda\zeta(\lambda z + \mu; \tau_1, \tau_2).$$

We have thus proved the theorem. $\square$

Proposition 4.4.

$$\varphi(z) = \exp(Az+B)\frac{\sigma(\lambda z)}{\sigma(\lambda z+\mu)}\,,$$

where A is in Theorem 4.3, *and $B \in \mathbb{C}$, and*

$$\sigma(z;\tau_1,\tau_2) = z \prod_{\substack{\omega=m_1\tau_1+m_2\tau_2\\(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}}} \{(1-\frac{z}{\omega})\exp(\frac{z}{\omega}+\frac{1}{2}(\frac{z}{\omega})^2)\}\,.$$

Proof. By $\zeta(z) = (d/dz)\log\sigma(z)$,

$$\begin{aligned}\eta(z) &= A + \frac{d}{dz}\log\sigma(\lambda z) - \frac{d}{dz}\log\sigma(\lambda z+\mu)\\ &= \frac{d}{dz}\log(e^{Az}\frac{\sigma(\lambda z)}{\sigma(\lambda z+\mu)})\,.\end{aligned}$$

From $\eta(z) = (d/dz)\log\varphi(z)$,

$$\log\varphi(z) = \log(e^{Az}\frac{\sigma(\lambda z)}{\sigma(\lambda z+\mu)}) + B\,.$$

Thus

$$\varphi(z) = e^{Az+B}\frac{\sigma(\lambda z)}{\sigma(\lambda z+\mu)}\,.$$

We have completed the proof of Proposition 4.4. □

Next we shall determine the function α.

Lemma 4.5.

(1) *The set of poles of the function α'/α is $\Lambda^\alpha \cup \Lambda_-$. All poles of α'/α are simple, and* $\mathrm{Res}_p\,\alpha'/\alpha = \mathrm{ord}_p\,\alpha$ *for $p \in \Lambda^\alpha \cup \Lambda_-$.*
(2) $(\alpha'/\alpha)(z+p) = (\alpha'/\alpha)(z), \quad p \in \Lambda_+, z \in \mathbb{C}$.
(3) $\mathrm{Res}_p\,\alpha'/\alpha = 1, \quad p \in \Lambda^\alpha$.
(4) $\mathrm{Res}_p\,\alpha'/\alpha = -1, \quad p \in \Lambda_-$.

Proof.

(1) We note that Λ_- is the set of the poles of α by Lemma 2.1 (1). By means of this, we obtain the result easily.
(2) By Lemma 3.1, we get

$$\alpha(t+u) = \alpha(u)\alpha(t) \quad (\forall t \in \mathbb{C}, \forall u \in \Lambda_+)\,.$$

Then $\alpha'(t+u) = \alpha(u)\alpha'(t)$. From two equations above,

$$\frac{\alpha'(t+u)}{\alpha(t+u)} = \frac{\alpha'(t)}{\alpha(t)} \quad (\forall t \in \mathbb{C}, \forall u \in \Lambda_+)\,.$$

(3) By Lemma 3.4 (1), if $p \in \Lambda^\alpha$, then $\mathrm{ord}_p\,\alpha = 1$. Hence $\mathrm{Res}_p(\alpha'/\alpha) = 1$ for $p \in \Lambda^\alpha$.
(4) From Lemma 2.1 (2) and Lemma 3.2, we have $\mathrm{ord}_p\,\alpha = \mathrm{ord}_p\,\varphi = -1$ for $p \in \Lambda_-$. Thus $\mathrm{Res}_p(\alpha'/\alpha) = -1$ for $p \in \Lambda_-$. □

Lemma 4.6. *There exists $\nu \in \mathbb{C}$ such that $\Lambda^\alpha = (-\nu/\lambda) + \Lambda_+$.*

Proof. Suppose $\Lambda^\alpha = \emptyset$. The set of the poles of α'/α is Λ_- from Lemma 4.5 (1). We note that $\Lambda_- = (-\mu/\lambda) + \Lambda_+$. Thus, in a fundamental period-parallelogram for the elliptic function α'/α, the number of poles of the function α'/α is one and the pole is simple. There is no such an elliptic function [16 p.432], [6 p.157]. Then $\Lambda^\alpha \neq \emptyset$.

Hence there exists $\nu \in \mathbb{C}$ such that $-\nu/\lambda \in \Lambda^\alpha$. Because Λ^α is invariant under the translation by Λ_+ and $\sharp(\Lambda^\alpha/\Lambda_+) \leq 1$, we have $\Lambda^\alpha = (-\nu/\lambda) + \Lambda_+$. □

We can summarize the conclusion of the function $(\alpha'/\alpha)(z)$ just obtained as follows:

(1) The function $(\alpha'/\alpha)(z)$ is doubly-periodic with periods $\tau_1/\lambda, \tau_2/\lambda$.
(2) The set of poles of α'/α is $((-\nu/\lambda) + \Lambda_+) \cup ((-\mu/\lambda) + \Lambda_+)$, and all poles are simple.
(3) $\operatorname{Res}_p \alpha'/\alpha = \begin{cases} 1, & p \in (-\nu/\lambda) + \Lambda_+, \\ -1, & p \in (-\mu/\lambda) + \Lambda_+. \end{cases}$

Theorem 4.7.

$$\alpha(z) = \exp(\rho z)\frac{\sigma(\mu;\tau_1,\tau_2)\sigma(\lambda z+\nu;\tau_1,\tau_2)}{\sigma(\nu;\tau_1,\tau_2)\sigma(\lambda z+\mu;\tau_1,\tau_2)}$$

for some $\rho \in \mathbb{C}$.

Proof. By [16 p.449], [6 p.177],

$$\begin{aligned}\frac{\alpha'(z)}{\alpha(z)} &= \rho + \zeta(z+\frac{\nu}{\lambda};\frac{\tau_1}{\lambda},\frac{\tau_2}{\lambda}) - \zeta(z+\frac{\mu}{\lambda};\frac{\tau_1}{\lambda},\frac{\tau_2}{\lambda}) \\ &= \rho + \lambda\zeta(\lambda z+\nu;\tau_1,\tau_2) - \zeta(\lambda z+\mu;\tau_1,\tau_2) \\ &= \frac{d}{dz}\log e^{\rho z}\frac{\sigma(\lambda z+\nu)}{\sigma(\lambda z+\mu)}.\end{aligned}$$

Thus

$$\alpha(z) = Ce^{\rho z}\frac{\sigma(\lambda z+\nu)}{\sigma(\lambda z+\mu)}.$$

From $\alpha(0) = 1$, $C = \sigma(\mu)/\sigma(\nu)$. Therefore we obtain

$$\alpha(z) = e^{\rho z}\frac{\sigma(\mu)\sigma(\lambda z+\nu)}{\sigma(\nu)\sigma(\lambda z+\mu)}.$$

This completes the proof of Theorem 4.7. □

Lemma 4.8.

$$\psi(z) = \exp((\rho - A)z - 2B)\frac{\sigma(\mu)\sigma(\nu-\mu)\sigma(\lambda z+\mu+\nu)}{\sigma^2(\nu)\sigma(\lambda z+\mu)}.$$

Proof. By $\alpha(z) = e^{\rho z}(\sigma(\mu)\sigma(\lambda z+\nu))/(\sigma(\nu)\sigma(\lambda z+\mu))$,

$$\begin{aligned}\alpha(s)-\alpha(t)\alpha(s-t) &= e^{\rho s}\Big(\frac{\sigma(\mu)\sigma(\lambda s+\nu)}{\sigma(\nu)\sigma(\lambda s+\mu)} - \frac{\sigma(\mu)\sigma(\lambda t+\nu)\sigma(\mu)\sigma(\lambda(s-t)+\nu)}{\sigma(\nu)\sigma(\lambda t+\mu)\sigma(\nu)\sigma(\lambda(s-t)+\mu)}\Big)\\ &= e^{\rho s}\frac{\sigma(\mu)}{\sigma^2(\nu)\sigma(\lambda s+\mu)\sigma(\lambda t+\mu)\sigma(\lambda(s-t)+\mu)}\\ &\times\{\sigma(\nu)\sigma(\lambda t+\mu)\sigma(\lambda(s-t)+\mu)\sigma(\lambda s+\nu)\\ &\quad -\sigma(\lambda s+\mu)\sigma(\mu)\sigma(\lambda t+\nu)\sigma(\lambda(s-t)+\nu)\}\,.\end{aligned}$$

We use the three term equation of σ below.

$$\begin{aligned}&\sigma(x+y)\sigma(x-y)\sigma(z+w)\sigma(z-w)\\ &+\sigma(x+z)\sigma(x-z)\sigma(w+y)\sigma(w-y)\\ &+\sigma(x+w)\sigma(x-w)\sigma(y+z)\sigma(y-z)\\ &=0\,.\end{aligned}$$

By $\sigma(-x) = -\sigma(x)$,

$$\alpha(s)-\alpha(t)\alpha(s-t) = e^{\rho s}\frac{\sigma(\mu)\sigma(\lambda s+\mu+\nu)\sigma(\mu-\nu)\sigma(\lambda(t-s))\sigma(\lambda t)}{\sigma^2(\nu)\sigma(\lambda s+\mu)\sigma(\lambda t+\mu)\sigma(\lambda(s-t)+\mu)}\,.$$

On the other hand,

$$\varphi(t)\varphi(s-t) = e^{As+2B}\frac{\sigma(\lambda t)\sigma(\lambda(s-t))}{\sigma(\lambda t+\mu)\sigma(\lambda(s-t)+\mu)}\,.$$

From $\psi(s) = (\alpha(s)-\alpha(t)\alpha(s-t))/(\varphi(t)\varphi(s-t))$, we get the desired result. $\square$

5. Degenerate case, I.

Now we consider the most degenerate case $\Lambda_+ = \{0\}$. The main result of this section is

Theorem 5.1. *If $\Lambda_+ = 0$, then*

$$\alpha(x) = e^{\rho x}\frac{ax+b}{cx+b}$$

for some ρ, a, b and $c \in \mathbb{C}$ with $b(a-c) \neq 0$.

Conversely we deduce the following from some straightforward computation.

Proposition 5.2. *The function $\alpha(x) = e^{\rho x}(ax+b)(cx+b)^{-1}$ given in Theorem 5.1 satisfies the equation* (0.2)*:*

$$\alpha(x)\alpha'(y) - \alpha'(x)\alpha(y) = (\alpha(x+y) - \alpha(x)\alpha(y))(\eta(x) - \eta(y)).$$

Here $\eta(x)$ is given by

$$\eta(x) = \frac{b}{x(cx+b)} + A$$

for an arbitrary constant $A \in \mathbb{C}$.

Because of $\eta(x) = (\varphi'(x)/\varphi(x))$, we have

$$\varphi(x) = e^{Ax+B}\frac{x}{cx+b}$$

for an arbitrary constant $B \in \mathbb{C}$. Furthermore $\psi(x)$ is given by

$$\psi(x) = e^{(\rho-A)x-2B}\frac{(c-a)\{acx+b(a+c)\}}{cx+b}.$$

Now we will prove Theorem 5.1.

Proof of Theorem 5.1. Choose a generic point $s \in \mathbb{C}$ such that $\alpha(s) \neq \infty$ and $\psi(s) \neq 0, \infty$. Consider the meromorphic function of t

$$h_s(t) := \alpha(t)\alpha(s-t)$$

defined on the whole plane $\mathbb{C}$. From the equation (1.1)

$$\frac{\alpha(s) - h_s(t)}{\varphi(t)\varphi(s-t)} = \psi(s) \neq 0, \infty,$$

we have $h_s(t) = \alpha(s)$, if and only if $\varphi(t) = 0$ or $\varphi(s-t) = 0$. The latter condition is equivalent to $t = 0$ or s because of $\Lambda_+ = \{0\}$. Moreover the function $h_s(t) = \alpha(t)\alpha(s-t)$ has at most two zeroes and at most two poles on $\mathbb{C}$ from Corollary 3.3 and Lemma 3.4 (2). Therefore the meromorphic function h_s defined on the whole plane $\mathbb{C}$ has three exceptional values 0, ∞ and $\alpha(s)$ for such a generic s. In view of the great Picard theorem (see, for example, [5]), the function h_s extends itself to a meromorphic function defined on the whole Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Recall $\sharp\Lambda^\alpha \leq 1$ and $\sharp\Lambda_- \leq 1$. In other words, we have $\Lambda^\alpha = \emptyset$ or $\{\nu_0\}$ and $\Lambda_- = \emptyset$ or $\{\mu_0\}$ for some ν_0 and $\mu_0 \in \mathbb{C}$. So we have four possibilities:

$$(\sharp\Lambda^\alpha, \sharp\Lambda_-) = (0,0), (1,0), (0,1) \text{ and } (1,1).$$

Introduce a linear fraction $S(x)$ by

$$S(x) := 1,\ x - \nu_0,\ \frac{1}{x - \mu_0} \text{ and } \frac{x - \nu_0}{x - \mu_0},$$

respectively. Then we have $\alpha(x)/S(x) = e^{g(x)}$ for some entire function g. In fact, Lemma 3.4 and Lemma 3.2 imply $\operatorname{ord}_{\nu_0} \alpha = 1$ and $\operatorname{ord}_{\mu_0} \alpha = -1$, respectively. For a generic s, $e^{g(t)+g(s-t)} = h_s(t)/S(t)S(s-t)$ is also a meromorphic function of t on the whole $\mathbb{C} \cup \{\infty\}$, and furthermore it has no poles and no zeroes on $\mathbb{C}$. Therefore we have $e^{g(t)+g(s-t)} = c(s)$ for some constant $c(s)$ depending only on s. Differentiating it by the variable t, we obtain $g'(t) - g'(s-t) = 0$ for any $t \in \mathbb{C}$ and a generic s.

Consequently the derivative g' is constant, so that $\alpha(x) = Ce^{\rho x}S(x)$ for some C and $\rho \in \mathbb{C}$. Since α is not obvious, we have $S \neq 1$. Recall $\alpha(0) = 1$ from Lemma 2.2. It follows that $\alpha(x) = e^{\rho x}(ax + b)(cx + b)^{-1}$ for some a, b and $c \in \mathbb{C}$ with $(a - c)b \neq 0$. This completes the proof of Theorem 5.1. □

6. Degenerate case, II.

Finally we consider the singly-periodic case $\Lambda_+ \cong \mathbb{Z}$.

Theorem 6.1. *If $\Lambda_+ \cong \mathbb{Z}$, then*

$$\alpha(x) = e^{\rho x}\frac{a(e^{2x/\lambda} - 1) + b}{c(e^{2x/\lambda} - 1) + b}$$

for some ρ, λ, a, b and $c \in \mathbb{C}$ with $\lambda \neq 0$ and $b(a - c) \neq 0$.

Proof. Fix a generator $\lambda_0 \in \Lambda_+$. There exists some $\rho' \in \mathbb{C}$ such that $e^{\rho'\lambda_0}\alpha(\lambda_0) = 1$, since $\alpha(\lambda_0) \neq 0, \infty$. Remark that $e^{\rho' x}\alpha(x)$ is also a solution of the equation (0.2). So we may replace $\alpha(x)$ with $e^{\rho' x}\alpha(x)$. Then $\alpha(\lambda') = 1$ for all $\lambda' \in \Lambda_+$, so that

$$\alpha(x + \lambda') = \alpha(x) \tag{6.1}$$

for any $x \in \mathbb{C}$ and any $\lambda' \in \Lambda_+$.

Choose a generic point $s \in \mathbb{C}$ such that $\alpha(s) \neq \infty$ and $\psi(s) \neq 0, \infty$. Consider the meromorphic function of t

$$h_s(t) := \alpha(t)\alpha(s - t)$$

defined on the whole plane $\mathbb{C}$. From the equation (1.1)

$$\frac{\alpha(s) - h_s(t)}{\varphi(t)\varphi(s - t)} = \psi(s) \neq 0, \infty,$$

we have $h_s(t) = \alpha(s)$, if and only if $\varphi(t) = 0$ or $\varphi(s-t) = 0$. The latter condition is equivalent to $t \in \Lambda_+$ or $t \in s + \Lambda_+$. From (6.1) there exists a meromorphic function $k_s = k_s(\xi)$ defined on $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ such that

$$h_s(x) = k_s\left(e^{2\pi\sqrt{-1}x/\lambda_0}\right).$$

From what we have already shown, $k_s(\xi) = \alpha(s)$, if and only if $\xi = 1$ of $e^{2\pi\sqrt{-1}s/\lambda_0}$. Corollary 3.3 and Lemma 3.4 (2) imply that k_s has at most two zeroes and at most two poles on $\mathbb{C}^\times$. Therefore the meromorphic function k_s defined on $\mathbb{C}^\times$ has three exceptional values 0, ∞ and $\alpha(s)$ for such a generic s. In view of the great Picard theorem (see, for example, [5]), the function h_s extends itself to a meromorphic function defined on the whole Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Recall $\sharp\Lambda^\alpha/\Lambda_+ \leq 1$ and $\sharp\Lambda_-/\Lambda_+ \leq 1$. In other words, we have $\Lambda^\alpha = \emptyset$ or $\nu_0 + \Lambda_+$ and $\Lambda_- = \emptyset$ or $\mu_0 + \Lambda_+$ for some ν_0 and $\mu_0 \in \mathbb{C}$. So we have four possibilities:

$$(\sharp\Lambda_-/\Lambda_+, \sharp\Lambda^\alpha/\Lambda_+) = (0,0), (0,1), (1,0) \text{ and } (1,1).$$

Set $c' := e^{2\pi\sqrt{-1}\nu_0/\lambda_0}$ and $c'' := e^{2\pi\sqrt{-1}\mu_0/\lambda_0}$. We introduce a linear fraction $S(\xi)$ by

$$S(\xi) := 1,\ \xi - c',\ \frac{1}{\xi - c''},\ \text{and } \frac{\xi - c'}{\xi - c''},$$

respectively. Then there exists some holomorphic function g defined on the whole plane $\mathbb{C}$ such that

$$\alpha(x) = e^{g(x)} S\left(e^{2\pi\sqrt{-1}x/\lambda_0}\right)$$

from Lemma 3.4 and Lemma 3.2.

Set $\xi_s := e^{2\pi\sqrt{-1}s/\lambda_0}$ for $s \in \mathbb{C}$. Then, for a generic point s, the function $k_s(\xi)/S(\xi)S(\xi^{-1}\xi_s)$ is a meromorphic function defined on the whole $\mathbb{C} \cup \{\infty\}$, and has no poles and no zeroes on $\mathbb{C}^\times$. Therefore $k_s(\xi)/S(\xi)S(\xi^{-1}\xi_s) = b(s)\xi^n$ for some function $b(s)$ and some integer $n \in \mathbb{Z}$. From the definition of k_s,

$$e^{g(t)+g(s-t)} = \alpha(t)\alpha(s-t)/S\left(e^{2\pi\sqrt{-1}t/\lambda_0}\right) S\left(e^{2\pi\sqrt{-1}(s-t)/\lambda_0}\right) = b(s)e^{2\pi\sqrt{-1}nt/\lambda_0}.$$

Differentiating it by the variable t, we obtain

$$g'(t) - g'(s-t) - 2\pi\sqrt{-1}n/\lambda_0 = 0$$

for any s and $t \in \mathbb{C}$. Hence $n = 0$ and

$$\alpha(x) = e^{g(x)} S\left(e^{2\pi\sqrt{-1}x/\lambda_0}\right) = Ce^{\rho x} S\left(e^{2\pi\sqrt{-1}x/\lambda_0}\right)$$

for some C and $\rho \in \mathbb{C}$. Since α is not obvious, we have $S \neq 1$. Recall $\alpha(0) = 1$ from Lemma 2.2. Define $\lambda := \lambda_0/\pi\sqrt{-1}$. Then we obtain

$$\alpha(x) = e^{\rho x}\frac{a(e^{2x/\lambda} - 1) + b}{c(e^{2x/\lambda} - 1) + b}$$

for some a, b and $c \in \mathbb{C}$ with $(a-c)b \neq 0$. This completes the proof of Theorem 6.1. □

By some straightforward computation, we obtain

Lemma 6.2.

$$\alpha(x) = e^{\rho x} \frac{a(e^{2x/\lambda} - 1) + b}{c(e^{2x/\lambda} - 1) + b}$$

satisfies the equation (0.2)

$$\alpha(x)\alpha'(y) - \alpha'(x)\alpha(y) = (\alpha(x+y) - \alpha(x)\alpha(y))(\eta(x) - \eta(y)) ,$$

where

$$\eta(x) = A + \frac{2\lambda^{-1}e^{2x/\lambda}}{e^{2x/\lambda} - 1} - \frac{2\lambda^{-1}ce^{2x/\lambda}}{c(e^{2x/\lambda} - 1) + b}$$

for an arbitrary constant $A \in \mathbb{C}$.

Lemma 6.3.

$$\varphi(x) = e^{Ax+B} \frac{e^{2x/\lambda} - 1}{c(e^{2x/\lambda} - 1) + b} ,$$

where the constant A *is in the lemma above and* B *is an arbitrary constant.*

Proof. It is trivial because of $\eta(x) = (d/dx)\log\varphi(x)$. □

Lemma 6.4.

$$\psi(x) = e^{-Ax-2B} \frac{(a-c)\{-ac(e^{2x/\lambda} - 1) + b^2 - b(a+c)\}}{c(e^{2x/\lambda} - 1) + b} .$$

Proof. From $\psi(s) = (\alpha(s) - \alpha(t)\alpha(s-t))/(\varphi(t)\varphi(s-t))$, we get the desired result. □

Suppose $a \neq 0, b$ and $c \neq 0, b$. Then the solutions stated above are expressed in terms of the hyperbolic sine function. In fact, there exist μ and $\nu \in \mathbb{C}$ such that

$$\begin{cases} b = a(1 - e^{-2\nu}) , \\ b = c(1 - e^{-2\mu}) . \end{cases}$$

Then we get

$$\begin{aligned} \alpha(x) &= e^{\rho x} \frac{a(e^{2x/\lambda} - 1) + b}{c(e^{2x/\lambda} - 1) + b} \\ &= e^{\rho x} \frac{a(e^{2x/\lambda} - 1) + a(1 - e^{-2\nu})}{c(e^{2x/\lambda} - 1) + c(1 - e^{-2\mu})} \\ &= e^{\rho x} \frac{\sinh\mu \sinh(\lambda^{-1}x + \nu)}{\sinh\nu \sinh(\lambda^{-1}x + \mu)} . \end{aligned}$$

These solutions are obtained in [1].

On the other hand, let $a = 0$ and $c \neq b$, or let $a = b$ and $c \neq 0$. There exists $\mu \in \mathbb{C}$ such that $b = c(1 - e^{-2\mu})$. We note that $\alpha(0) = 1$, and, as a result,

$$\alpha(x) = e^{(\rho - \lambda^{-1})x} \frac{\sinh\mu}{\sinh(\lambda^{-1}x + \mu)} .$$

Next let $c = 0$ and $a \neq b$, or let $b = c$ and $a \neq 0$. There exist $\nu \in \mathbb{C}$ such that $b = a(1 - e^{-2\nu})$. We note that $\alpha(0) = 1$, and, as a result,

$$\alpha(x) = e^{(\rho + \lambda^{-1})x} \frac{\sinh(\lambda^{-1}x + \nu)}{\sinh\nu} .$$

The above two cases are not listed in [1], so that we obtain new Lax forms of the system in [1].

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