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# **Deductive hyperdigraphs**

**A method of describing diversity of coherences**

**Akira Higuchi, Kazumasa Matsuo & Toru Tsujishita**

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# Deductive hyperdigraphs

A method of describing diversity of coherences

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## Abstract

The main purpose of this paper is to introduce several graphical methods of describing deductive hyperdigraphs and explains its significance for description of complex systems.

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## Introduction

Although research of complex systems require description of various intricate interrelationship between their components, there seems to be few concepts which can handle the relationship among more than two components, such as

- the component  $a$  does not determine the behavior of the component  $c$  but  $a$  and  $b$  do determine  $c$ . On the other hand  $b$  is determined by the components  $a, c$ ,

or

- the component  $a$  determines both of the component  $b$  and  $c$ .

We say that there is a functional dependency of a component  $a$  on the components  $b_1, \dots, b_n$ , if the status of the latter determines that of the former. Note that the causality is a special functional dependency when temporal factor is taken into consideration. The functional dependencies in a system have rather complex relationship with each others and it is hopeless to try to express their totality without mathematical language.

In this paper, we show that the concept of deductive hyperdigraph is a mathematical concept which can precisely depict such totality. Moreover, this concept is helpful also for such diverse topics as metabolic systems, deduction systems, and concurrent systems.

A *hyperdigraph* consists of a set of nodes and a relation  $\vdash$  which relates finite subsets of nodes to nodes. We express the statement that the nodes  $a_1, \dots, a_n$  are related to  $b$  by

$$a_1, \dots, a_n \rightarrow b.$$

By this relation we intend to mean that the components  $a_1, \dots, a_n$  collectively determine the behavior of  $b$ , or, we can infer the behavior of the component  $b$  once we know the behavior of the components  $a_1, \dots, a_n$ .

A hyperdigraph is called *deductive* if

- it is *transitive* in the following sense: From

$$a_0, a_1, \dots, a_n \rightarrow b,$$

and

$$c_1, \dots, c_m \rightarrow a_0,$$

it follows

$$c_1, \dots, c_m, a_1, \dots, a_n \rightarrow b,$$

- it is *reflexive*, i.e.,  $a_1, \dots, a_n \rightarrow a_i$  for all  $i$ ,
- it is *monotone*, i.e.,  $a_1, \dots, a_n \rightarrow b$  implies  $a, a_1, \dots, a_n \rightarrow b$ .

In contrast to the digraphs which have the natural planar graph representation, there seems to be no such canonical representation for hyperdigraphs, although there are many representations of them, which are claimed to be minimal[AAS].

In this paper we also give various method of describing deductive hyperdigraphs. We illustrate these methods by describing 14 different coherence structures on systems composed of 3 elements.

One of the most concise method of description seems to be given by a set with lattice labeling. **A lattice labeling on a set  $V$**  is simply a map  $\lambda$  from  $V$  to a lattice  $L$ . This induces a deductive hyperdigraph structure on  $V$  by

$$a_1, \dots, a_n \rightarrow b \stackrel{def}{\iff} \lambda a_1 \vee \dots \vee \lambda a_n \geq \lambda b.$$

When we impose the condition that the image of  $\lambda$  generates  $L$  with respect to the join operation, then this description is essentially unique. Thus this allows us to enumerate all deductive hyperdigraph structures on a given set easily when the number of components is small. We can use it also to analyze interrelations between various deductive hyperdigraphs structures on a fixed set, which can be interpreted as the relation between coherences of complex systems.

**Notations**  $\mathcal{P} V$  will denote the powerset of a set  $V$ , which is regarded as a lattice by the usual inclusion orders. We usually write a singleton set  $\{v\}$  simply by  $v$  for brevity. We use the notation  $A \subseteq B$  when  $A$  is a subset of  $B$  and  $A \subset B$  when  $A \subseteq B$  but  $A \neq B$ .

Through this paper, *all sets are assumed to be finite.*

## 1 Deductive hyperdigraphs

### 1.1 Definition

A hyperdigraph is a pair  $\Gamma = (V, E)$ , with  $E \subseteq \mathcal{P} V \times V$ . We write  $W \rightarrow_{\Gamma} v$  when  $(W, v) \in E$ . If  $\{a_1, \dots, a_n\} = W$ , we use the notation  $a_1, \dots, a_n \rightarrow v$ . We often refer to a hyperdigraph simply by a pair  $(V, \rightarrow)$ .

Two hyperdigraphs  $(V_i, \rightarrow_i)$  ( $i = 1, 2$ ) are called *isomorphic* if there is a bijection  $f : V_1 \rightarrow V_2$  satisfying

$$W \rightarrow_1 v \iff fW \rightarrow_2 f v$$

for all  $W \subseteq V_1$  and  $v \in V_1$ .

A hyperdigraph  $\Gamma$  is called *transitive* if it satisfies the following condition:

$$(TR) \quad \frac{W \cup v \rightarrow v' \quad W' \rightarrow v}{W \cup W' \rightarrow v'},$$

i.e., if  $W \rightarrow v$  and  $W' \cup v \rightarrow v'$  holds, then  $W \cup W' \rightarrow v'$ .

It is easily shown that the transitivity is equivalent to the following apparently stronger condition:

$$(TR') \quad \frac{W \rightarrow v \quad \forall w \in W [U \rightarrow w]}{U \rightarrow v}$$

A hyperdigraph is called *reflexive* if  $W \rightarrow w$  for all  $v \in W$ , and *monotone* if

$$\frac{W \rightarrow v}{W \cup W' \rightarrow v}$$

for all  $W' \subseteq V$ .

A hyperdigraph is called *deductive* if it is reflexive, monotone and transitive. The set of all the deductive hyperdigraphs with the vertex set  $V$  is denoted by  $\text{DHGraph}(V)$ .

## 1.2 Examples

It is easily shown that the following  $(V, \rightarrow)$ 's are deductive hyperdigraphs.

### 1.2.1 Simple logic, empirical logic

Let  $V$  be a finite set of propositions. Define  $W \rightarrow v$  if the proposition  $v$  holds whenever every proposition in  $W$  is valid [C1]. The transitivity is nothing but the cut rule of deduction.

Such logic appears commonly in daily life as the empirical deduction based on memories of concrete events. In fact, when certain propositions  $p_1, \dots, p_n$  turn out to be true, we sometimes conclude that  $p$  is true when the proposition  $p$  was true under all the situations we can recall where  $p_1, \dots, p_n$  hold. Such empirical judgments are continuously changing according to the increase or decrease of the memory of empirical situations.

### 1.2.2 Mutual dependency of random variables

Let  $(X, \mathcal{B}, \mu)$  be a probability space. Let  $(x_v : X \rightarrow X_v)_{v \in V}$  be a finite family of random variables. This defines the following deductive hyperdigraph with base set  $V: a_1, \dots, a_n \rightarrow b$  when the support of the induced probability measure on  $X_{a_1} \times \dots \times X_{a_n} \times X_b$  is included in the graph of a function from  $X_{a_1} \times \dots \times X_{a_n}$  to  $X_b$ . This example can be considered as a special case of the next example.

### 1.2.3 Functional relations

Let  $\{X_v \mid v \in V\}$  be a family of nonempty sets and  $R \subseteq \prod_{v \in V} X_v$ . Define  $W \rightarrow_R v$  when

$$\frac{\forall w \in W [w(x) = w(y)]}{v(x) = v(y)}$$

for all  $x, y \in R$ . Here  $v(x)$  denotes the  $v$ -th component of  $x \in R$ .

**Remark.** It is known that every deductive hyperdigraph of finite degree can be realized in this way[A].

### 1.2.4 Algebraic structures

When a set  $V$  is given a family of operations, then every subset  $A \subseteq V$  defines a closed subset  $\bar{A} \subseteq V$  called its closure, which is the collection of the elements generated by  $A$  with the given operations. For example, if  $G$  is a group, then  $A \subseteq G$  generates a subgroup  $\bar{A} \subseteq G$ . We define  $a_1, \dots, a_n \rightarrow b$  if  $b$  is generated by  $a_1, \dots, a_n$ .

### 1.2.5 Metabolic systems

Let  $V$  be a finite sorts of chemical molecules. Define  $W \rightarrow v$  when molecules of sort  $v$  can be produced from the molecules of sorts in  $W$ . Under some situations, the monotonicity might be questionable, since some molecule can inhibit certain chemical reactions. However, if there are no such inhibiting molecules, it is obvious that  $W \rightarrow v$  defines a deductive hyperdigraph.

### 1.2.6 Events

Let  $V$  be a finite set of events of a system  $P$  of concurrent processes. Define  $W \rightarrow_P v$  if in every possible behavior of  $P$ , the event  $v$  must have occurred whenever every event in  $W$  has occurred. In other words, the totality of events in  $W$  cannot occur without accompanying the event  $v$ . The monotonicity is again questionable when certain set of events might conflict and cannot occur.

Consider the hyperdigraph  $W \rightarrow v$  defined by the condition that the events in  $W$  must have occurred whenever  $v$  is occurred. Then this is transitive and reflexive. However, this obviously not generally monotone, since otherwise  $V \rightarrow v$  for any  $v \in V$  and hence every event



must have occurred when some event has occurred, which is a very degenerate situation.

## 2 Equivalent concepts

The notion of deductive hyperdigraphs have three equivalent concepts, closure operator, Moore family and lattice labeled set.

### 2.1 Closure operators

A map  $C : \mathcal{P} V \rightarrow \mathcal{P} V$  is called a *closure operator on  $V$*  if it satisfies, for  $A, B \in \mathcal{P} V$ ,

$$(C1) \quad A \subseteq CA,$$

$$(C2) \quad A \subseteq B \implies CA \subseteq CB,$$

$$(C3) \quad CCA = CA.$$

We denote by  $\text{Con}(V)$  the set of all the closure operators on  $V$ . Note that we do not impose the condition

$$C\emptyset = \emptyset$$

which is sometimes included in the definition of closure operators.

### 2.2 Moore family

A family  $\mathcal{A}$  of subsets of  $V$  is called an *intersection structure on  $V$*  if  $A \cap B \in \mathcal{A}$  whenever  $A, B \in \mathcal{A}$ . An intersection structure  $\mathcal{A}$  is called a *Moore family* if contains  $V$ . We denote by  $\text{Moore}(V)$  the set of all the Moore family on  $V$ . We call a Moore family  $\mathcal{A}$  **totally alive** when it contains the empty set. When  $\mathcal{A}$  is not totally alive, then there are elements in the intersection of  $\mathcal{A}$  which may be regarded as constant components able to be ignored in many aspects.

### 2.3 Lattice labeling

A map  $\lambda$  from  $V$  to a lattice  $L$  is called a *lattice labeling of  $V$* . It is called *reduced* when  $\lambda$  generates  $L$  with respect to the join operation, i.e., each  $\ell \in L$  is the join of the image of a subset of  $V$ .

Two lattice labelings  $\lambda_i : V \rightarrow L_i$  ( $i = 1, 2$ ) are called isomorphic if there is a lattice isomorphism  $g : L_1 \rightarrow L_2$  satisfying  $\lambda_2 = g \circ \lambda_1$ .

A lattice labeled set is a pair  $(V, \lambda)$  of a set and a lattice labeling  $\lambda$ . If  $\lambda$  is reduced, we call  $(V, \lambda)$  a reduced lattice labeled set.

A lattice labeling  $\lambda : V \rightarrow L$  induces a map  $\lambda^* : L \rightarrow \mathcal{P} V$  defined by

$$\lambda^* \ell := \{ v \in V \mid \lambda v \leq \ell \}.$$

**Proposition 2.1** (i) *The map  $\lambda^*$  is a meet semilattice map, i.e.,*

$$\lambda^*(\ell_1 \wedge \ell_2) = \lambda^* \ell_1 \cap \lambda^* \ell_2.$$

*In particular, the image of  $\lambda^*$  is a Moore family. Furthermore*

$$\lambda^*(\ell_1 \vee \ell_2) = \overline{\lambda^* \ell_1 \cup \lambda^* \ell_2},$$

*where the closure is taken with respect to the image of  $\lambda^*$  which is meet closed family of subsets (cf. A.1).*

(ii) *The following conditions are equivalent to each others:*

(a) *The lattice labeling  $\lambda$  is reduced.*

(b) *For every  $\ell \in L$ ,*

$$\ell = \bigvee_{v \in \lambda^* \ell} \lambda v.$$

(c)  *$\lambda^*$  is injective.*

**Proof.** Let  $\ell_i \in L$  ( $i = 1, 2$ ). Then

$$\begin{aligned} \lambda^*(\ell_1 \wedge \ell_2) &= \{ v \mid \lambda v \leq \ell_1 \wedge \lambda v \leq \ell_2 \} \\ &= \{ v \mid \lambda v \leq \ell_1 \} \cap \{ v \mid \lambda v \leq \ell_2 \} \\ &= \lambda^* \ell_1 \cap \lambda^* \ell_2. \end{aligned}$$

Hence the assertion (i).

Suppose  $\lambda$  is reduced. Then, for  $\ell \in L$ ,  $\ell = \bigvee_{v \in I} \lambda v$  for some  $I \subseteq V$ . Since  $I \subseteq \lambda^* \ell$ , we have

$$\ell = \bigvee_{v \in I} \lambda v \leq \bigvee_{v \in \lambda^* \ell} \lambda v \leq \ell.$$

It is obvious that (b) implies (c).

Suppose now (c) holds. For  $\ell \in L$ , put

$$\ell' := \bigvee_{\lambda v \leq \ell} \lambda v.$$

Then  $\lambda^* \ell \leq \lambda^* \ell'$  since

$$v \in \lambda^* \ell \Rightarrow \lambda v \leq \ell \iff \lambda v \leq \ell' \iff v \in \lambda^* \ell'.$$

On the other hand,  $\lambda^* \ell' \leq \lambda^* \ell$  since  $\ell' \leq \ell$ . Hence  $\lambda^* \ell = \lambda^* \ell'$ , which implies  $\ell = \ell'$  by (c). **q.e.d.**

**Remarks** (i) Note that the Moore families coincide when they are induced by isomorphic lattice labelings. (ii) Note that there are researches with results representing a lattice concretely as a sublattice of the powerset lattice of a set [BF]. For example, given a lattice  $L$ , we define  $V$  to be the set of join-irreducible elements of  $L$  and consider the inclusion  $\iota : V \rightarrow L$ . Then this defines a reduced lattice labeling of  $V$  [Co]. However, for our purpose, the role of lattices are secondary compared to the base set  $V$  to be labeled. For example, when an element  $v \in V$  is labeled by a join-reducible element, this means that it is *redundant* since it is produced by  $\{ w \mid \lambda w < \lambda v \}$  because

$$\lambda v = \bigvee_{\lambda w < \lambda v} \lambda w.$$

## 2.4 Bijective correspondences

We show that the structures introduced above correspond bijectively to each others naturally.

### 2.4.1 Bijection between deductive hyperdigraphs and closure operators

Let  $\Gamma = (V, \rightarrow)$  be a deductive hyperdigraph. Define, for  $A \in \mathcal{P} V$ ,

$$C_\Gamma(A) := \{ v \in V \mid A \rightarrow v \}.$$

**Proposition 2.2**  $C_\Gamma$  is a closure operator.

**Proof.** The conditions (C1) and (C2) obviously hold. We have only to show that  $CCA \subseteq CA$ . Suppose  $v \in CCA$ , i.e.  $W \rightarrow v$  with  $W \subseteq CA$ . Then, for each  $w \in W$ , there is a  $U_w \subseteq A$  with  $U_w \rightarrow w$ . By (TR'), we have  $U := \cup_{w \in W} U_w \rightarrow v$ . Since  $U \subseteq A$ , we obtain  $v \in CA$ . **q.e.d.**

On the other hand, for a closure operator  $C$  on  $V$ , define

$$W \rightarrow_C v \stackrel{\text{def}}{\iff} v \in CW.$$

**Proposition 2.3** The pair  $(V, \rightarrow_C)$  is a deductive hyperdigraphs.

**Proof.** Obviously  $\rightarrow_C$  is reflexive and monotone. Suppose  $W \rightarrow v$  and  $W' \cup v \rightarrow v'$  hold. From  $W' \subseteq CW'$  and  $v \in CW$ , we have  $W' \cup v \subseteq CW \cup CW' \subseteq C(W \cup W')$ . Hence

$$v' \in C(W' \cup v) \subseteq CC(W \cup W') = C(W \cup W'),$$

which means  $W \cup W' \rightarrow v'$ .

**q.e.d.**

It is obvious that the above correspondences are inverse to each other.

**Proposition 2.4** *The correspondence which associates  $\Gamma$  with  $C_\Gamma$  is a bijection.*

### 2.4.2 Bijection between closure operators and Moore families

Let  $C$  be a closure operator on  $V$ . A subset  $A$  of  $V$  is called *closed with respect to  $C$*  if it is a fixed point of  $C$ , i.e.,  $CA = A$ . The set of all the closed sets will be denoted by  $\mathcal{A}_C$ , which is a Moore family by Proposition A.1 since it obviously contains  $V$ .

Conversely, suppose  $\mathcal{A}$  is a Moore family on  $V$ . Define

$$C_{\mathcal{A}}X := \bigcap_{X \subseteq A \in \mathcal{A}} X.$$

Then this is a closure operator again by A.1.

By A.1 we have

**Proposition 2.5** *The correspondence which associates a closure operator  $C$  to the Moore family  $\mathcal{A}_C$  is bijective.*

### 2.4.3 Bijection between Moore families and lattice labelings

Let  $\mathcal{A}$  be a Moore family on  $V$ . Define  $\lambda_{\mathcal{A}}v := C_{\mathcal{A}}\{v\}$ .

**Proposition 2.6** *The lattice labeling  $\lambda_{\mathcal{A}} : V \rightarrow \mathcal{A}$  is reduced.*

**Proof.** Suppose  $A \in \mathcal{A}$ . Since  $\lambda_{\mathcal{A}}a = C_{\mathcal{A}}a \subseteq A$  for all  $a \in A$ , we have

$$A \subseteq \bigcup_{a \in A} \lambda_{\mathcal{A}}a \subseteq A,$$

whence  $A = \bigvee_{a \in A} \lambda_{\mathcal{A}}a$ .

**q.e.d.**

Suppose now  $\lambda : V \rightarrow L$  is a lattice labeling on  $V$ . Define  $\mathcal{A}_\lambda := \{ \lambda * \ell \mid \ell \in L \}$ . By Proposition 2.1, we have obviously the following.

**Proposition 2.7** *If  $\lambda$  is a lattice labeling, then the family  $\mathcal{A}_\lambda$  is a Moore family.*

We have the following proposition.

**Proposition 2.8** *The correspondence is bijective which associates a Moore family  $\mathcal{A}$  to the isomorphic class, represented by  $\lambda_{\mathcal{A}}$ , of reduced lattice labelings.*

**Proof.** Since the lattice for the labeling  $\lambda_{\mathcal{A}}$  is  $\mathcal{A}$  itself, the correspondence is injective. It is surjective, since for a reduced lattice labeled set  $(V, \lambda : V \rightarrow L)$ , the Moore family  $\mathcal{A}_\lambda$  is isomorphic as a lattice to  $L$  and induces a lattice labeling obviously isomorphic to  $\lambda$ . **q.e.d.**

#### 2.4.4 Bijection between lattice labeled sets to deductive hyperdigraphs

The composition of the above bijections gives one between isomorphic classes of reduced lattice labelings to deductive hyperdigraphs. Its direct description is as follows. A lattice labeling  $\lambda : V \rightarrow L$  induces a deductive hyperdigraph  $\rightarrow$  defined by

$$W \rightarrow v \stackrel{def}{\iff} \lambda W \geq \lambda v,$$

where  $\lambda W := \bigvee_{w \in W} \lambda w$ .

**Proposition 2.9** *The pair  $(V, \rightarrow)$  is a deductive hyperdigraph.*

Although it is easy to trace the correspondences constructed above to show this, we prove this directly because of the importance of the representation of deductive hyperdigraphs by lattice labelings.

**Proof.** Obviously  $\rightarrow_C$  is reflexive and monotone. Suppose  $W \rightarrow v$  and  $W' \cup v \rightarrow v'$  hold.

$$\begin{aligned} \lambda(W \cup W') &= \lambda W \vee \lambda W' \\ &\geq \lambda v \vee \lambda W' \\ &= \lambda(v \cup W') \\ &\geq \lambda v'. \end{aligned}$$

Hence  $W \cup W' \rightarrow v'$ .

**q.e.d.**

### 2.4.5 Summary

We have proved the following.

**Theorem 2.10** *The correspondences*

$$\Gamma \leftrightarrow C \leftrightarrow \mathcal{A} \leftrightarrow (V, \lambda)$$

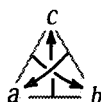
give bijections between the following sets

- deductive hyperdigraphs on  $V$ ,
- closure operators on  $V$ ,
- Moore families on  $V$ ,
- lattice labelings on  $V$ .

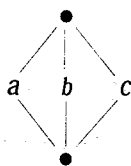
## 2.5 Examples

### 2.5.1 Example 1

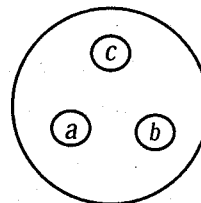
In this example, besides the emptyset and the total set  $abc$  there are only three nontrivial closed sets  $a, b$  and  $c$ . Hence the closure of any subset with two elements is the total space, which means that any two components dominates the other one. Since singleton sets are closed, no single component determine the others.



Deductive hyperdigraph



Labeled lattice



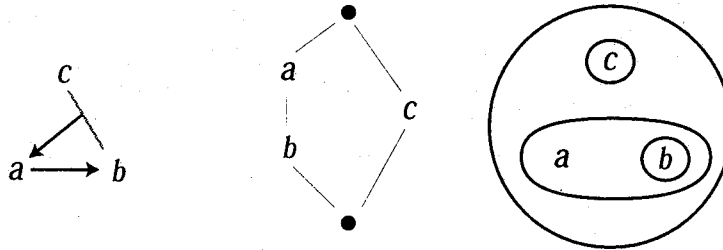
Moore family

### 2.5.2 Example 2

In this case the nontrivial closed sets are  $b, c$  and  $ab$ . Hence the closure of  $a$  is  $ab$ , which means  $a \rightarrow b$ . In other words, the component  $a$  dominates  $b$ . Furthermore the closure of  $bc$  is  $abc$ , which means  $b, c \rightarrow a$ , i.e., the components  $b, c$  together dominate  $a$  although neither  $b$  nor  $c$  can do it since  $b, c$  are closed sets. We may say that once the component  $c$  is frozen the components  $a$  and  $b$  dominates each other.

$A$	closure of $A$
$\emptyset$	$\emptyset$
$a$	$a$
$b$	$b$
$c$	$c$
$ab$	$abc$
$ac$	$abc$
$bc$	$abc$
$abc$	$abc$

Table 1: Closure operator



Deductive hyperdigraph

Labeled lattice

Moore family

$A$	closure of $A$
$\emptyset$	$\emptyset$
$a$	$ab$
$b$	$b$
$c$	$c$
$ab$	$ab$
$ac$	$abc$
$bc$	$abc$
$abc$	$abc$

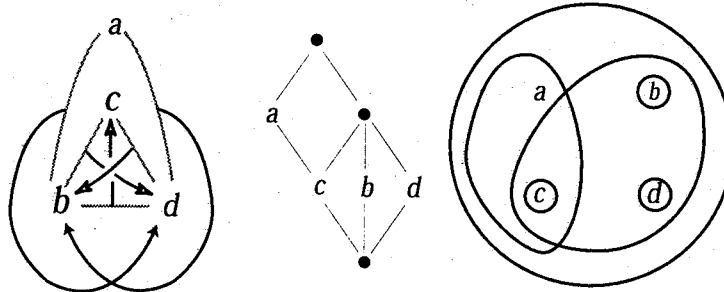
Table 2: Closure operator

### 2.5.3 Example 3

There are only five nontrivial closed sets  $b, c, d, ab, bcd$ . Hence as in Example 1, any two of  $bcd$  determines the other. The closure of  $ab$  is  $abcd$ , whence  $ab$  dominates  $c$  and  $d$ :

$$a, b \rightarrow c \quad a, b \rightarrow d.$$

Note that either of these hyperarcs induce the other. For example,  $a, b \rightarrow c$  and  $b, c \rightarrow d$  implies  $a, b, c \rightarrow d$  by the transitivity.



Deductive hyperdigraph

Labeled lattice

Moore family

$A$	closure of $A$
$\emptyset$	$\emptyset$
$b$	$b$
$c$	$c$
$d$	$d$
$a, ac$	$ac$
$bc, bd, cd, bcd$	$bcd$
$ab, ad, abc, abd, acd, abcd$	$abcd$

Table 3: Closure operator



### 3 The lattice of deductive hyperdigraphs

#### 3.1 Poset of deductive hyperdigraphs

The set of deductive hyperdigraph structures on a set  $V$  is a Moore family on  $\mathcal{P} V \times V$ , where  $\rightarrow := \rightarrow_1 \cap \rightarrow_2$  is defined by

$$W \rightarrow v \stackrel{def}{\iff} W \rightarrow_1 v \wedge W \rightarrow_2 v.$$

We call the closure of a subset  $U$  of  $\mathcal{P} V \times V$  with respect to the closure operator corresponding to this Moore family, as **the deductive hyperdigraph generated by  $U$** .

The natural partial order of  $\mathcal{P}(\mathcal{P} V \times V)$  induces one on  $\text{DHGraph}(V)$ , which can be described as

$$\rightarrow_1 \leq \rightarrow_2 \stackrel{def}{\iff} \frac{W \rightarrow_1 v}{W \rightarrow_2 v}.$$

This partial order has the following description in each cryptomorphic expression.

##### 3.1.1 Closure operator

Let  $C_i$  ( $i = 1, 2$ ) be closure operators on  $V$ . Define

$$C_1 \leq C_2 \stackrel{def}{\iff} C_1 W \subseteq C_2 W \quad \text{for all } W \subseteq V.$$

##### 3.1.2 Moore families

For Moore families  $\mathcal{A}_i$  ( $i = 1, 2$ ), define

$$\mathcal{A}_1 \leq \mathcal{A}_2 \stackrel{def}{\iff} \mathcal{A}_1 \supseteq \mathcal{A}_2.$$

A stronger intersection structure as fewer closed sets.

##### 3.1.3 Lattice labelings

Let  $\lambda_i : V \rightarrow L_i$  ( $i = 1, 2$ ) be lattice labelings. Define

$$\lambda_1 \leq \lambda_2$$

when there is an injective meet lattice map  $i : L_2 \rightarrow L_1$  such that the closure operator  $C : L_1 \rightarrow L_1$ , corresponding to the Moore family  $\text{Im}(i)$ , satisfies

$$C(\lambda_2(v)) = i(\lambda_2(v))$$

for all  $v \in V$ . Recall here that the meet semilattice map preserves the top element which is the meet of the empty set.

### 3.1.4 Equivalence

**Proposition 3.1** *Let  $\rightarrow_i, C_i, \mathcal{A}_i, \lambda_i$  ( $i = 1, 2$ ) correspond to each other under the cryptomorphisms, then*

$$\begin{aligned} \rightarrow_1 \leq \rightarrow_2 & \iff C_1 \leq C_2 \\ & \iff \mathcal{A}_1 \leq \mathcal{A}_2 \\ & \iff \lambda_1 \leq \lambda_2 \end{aligned}$$

We prove this in the Appendix.

## 3.2 Existence of graded poset structure

The poset of Moore families on  $V$  has a graded poset structure [H, BDK]. The following is the key lemma behind this fact.

**Lemma 3.2** ([H] Lemma3, Lemma 6) *Let  $\mathcal{A}$  be a Moore family on  $V$ . Then*

$$\mathcal{A} \setminus \{X\} \leftrightarrow X \in \mathcal{A}$$

*is a one-to-one correspondence between the Moore families which cover  $\mathcal{A}$  with respect to the order " $\leq$ " and those elements  $X$  of  $\mathcal{A}$  satisfying*

$$X \neq \bigcap \{A \in \mathcal{A} \mid A \supset X\}.$$

Let  $\mathcal{A}$  be a Moore family. We call an element  $x \in \mathcal{A}$  **removable** if

$$x \neq \bigcap_{y \in \mathcal{A}, y \neq x} y,$$

i.e. it is meet irreducible. The above lemma says that if  $x$  is removable, then  $\mathcal{A} \setminus \{x\}$  is a Moore family again.

An element  $y \notin \mathcal{A}$  is called **addable to  $\mathcal{A}$**  if

$$x \cap y \in \mathcal{A} \text{ for all } x \in \mathcal{A}.$$

It is obvious that if  $y$  is addable then  $\mathcal{A} \cup \{y\}$  is a Moore family.

### 3.3 Deformation of deductive hyperdigraphs

When a multicomponent system varies, the mutual dependencies of components change. The above lemma 3.2 describes how the change takes place successively. Changes can be described as the succession of atomic changes, which either remove removable closed sets or add addable sets.

## 4 Description of deductive hyperdigraphs of degree up to 4

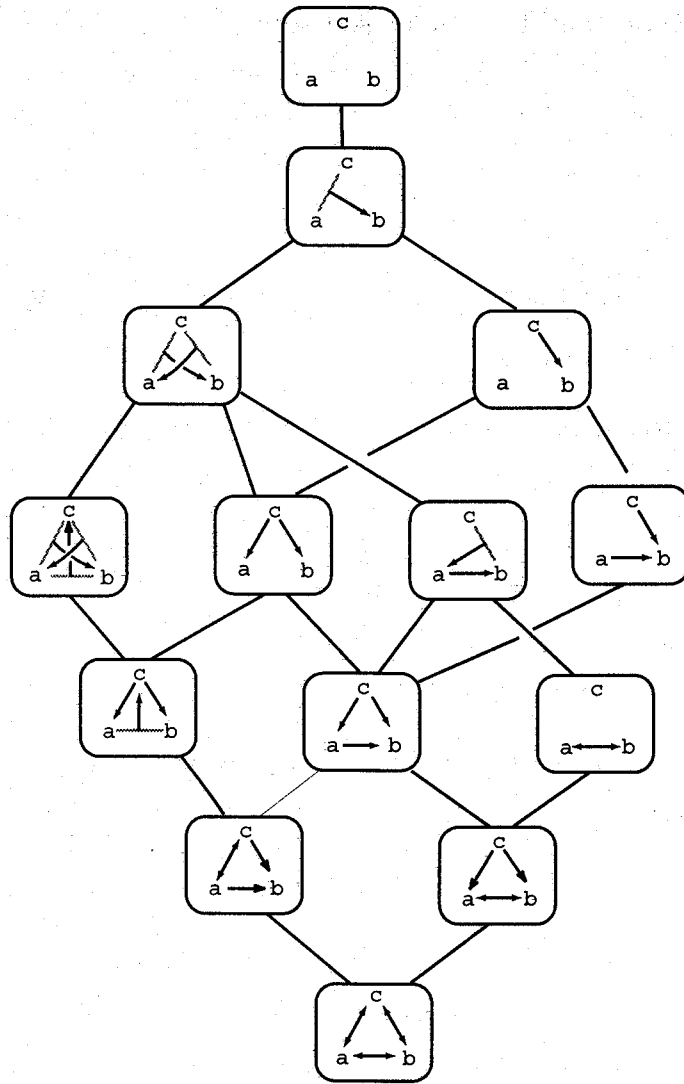
### 4.1 Deductive hyperdigraphs of degree three

There are 14 isomorphic types of deductive hyperdigraphs on a set of three elements, which will be graphically described in the following tree ways:

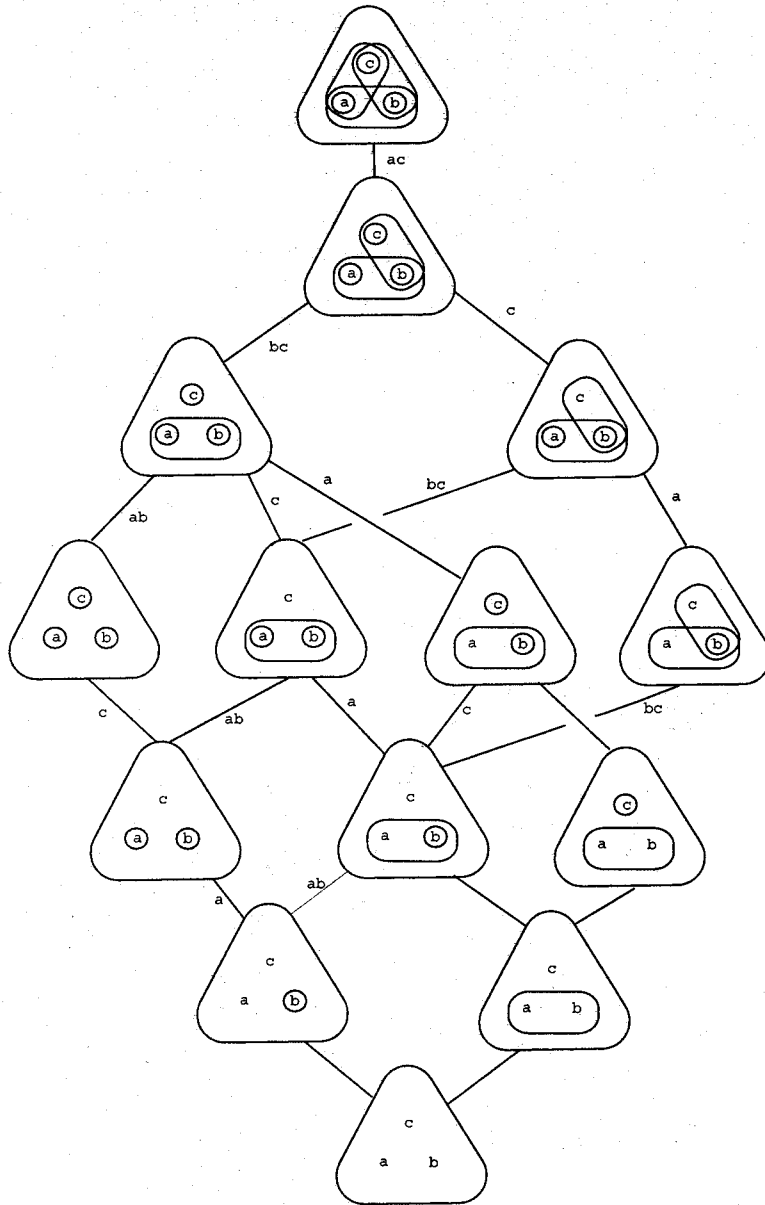
- representation by generating arcs,
- representation by closed sets,
- representation by labeled lattice.

#### 4.1.1 Representation by generating arcs

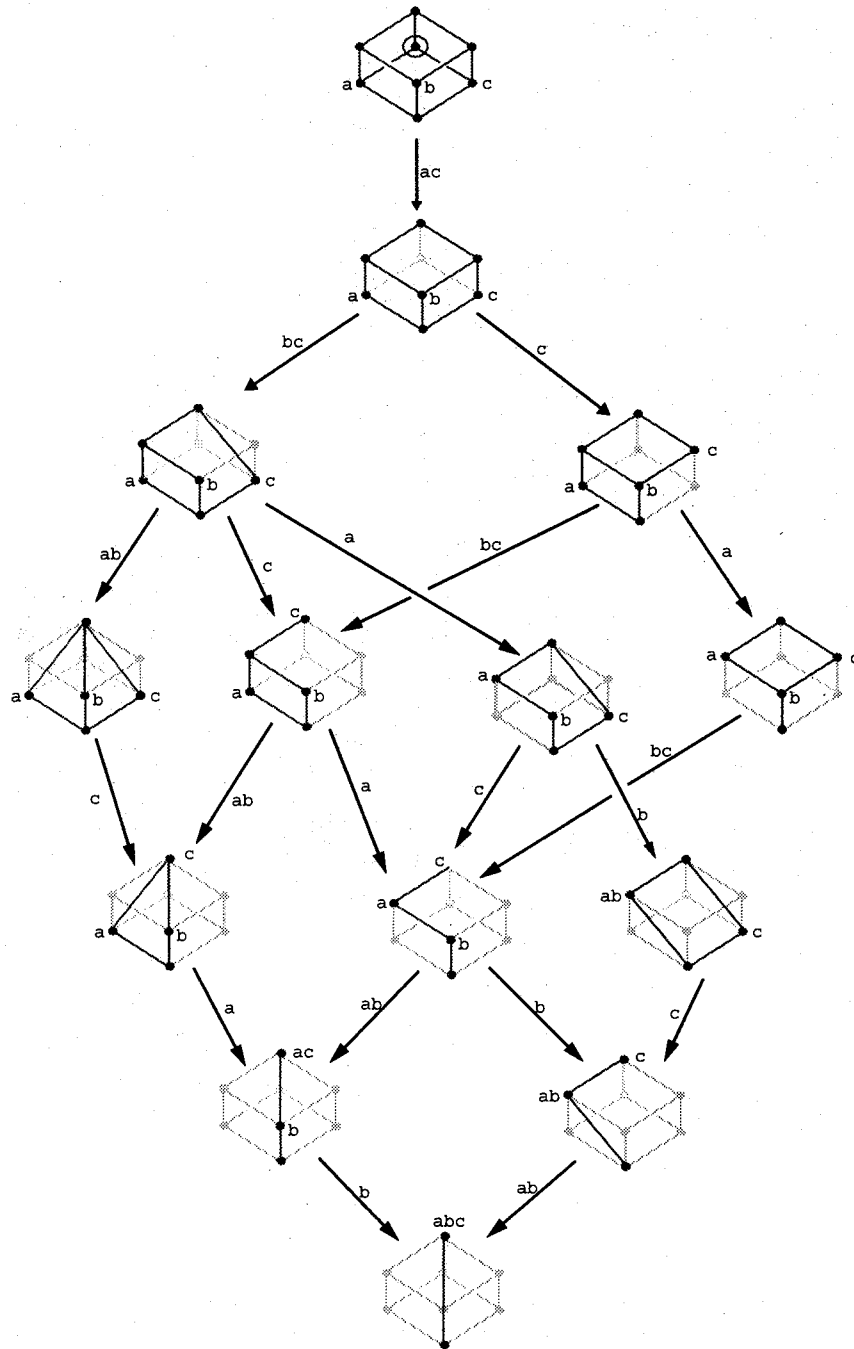
We say that  $W \rightarrow a$  is minimal, if there is no  $W' \subset W$  with  $W' \rightarrow a$ . When  $ab \rightarrow c$  is minimal, we write an arc from middle of the edge  $ab$  to the vertex  $c$ .



### 4.1.2 Representation by closed sets



### 4.1.3 Representation by labeled lattice



We note that when the emptyset is join irreducible, then we can remove it. Hence we have in fact five more diagrams.

## 4.2 deductive hyperdigraphs of degree four

### 4.2.1 graded lattice structure

There are 184 deductive hyperdigraphs of degree four. They form a graded lattice whose shape is as follows:

### 4.2.2 An example of generation of coherence

Every path in Figure 1 from the bottom to the top describes a shortest path of change coherence, from the weakest to the strongest. We pick up arbitrary such path and draw the coherence of each step by hyperdigraph and by lattice labeled set in Figures 2,3.

## 5 Application to description of complex systems

### 5.1 Aspects of interrelationship among components

We can read various aspects of interrelationship among components from labeled lattices.

1. When a node is labeled by a single label  $x$ , then the  $x$ -component by itself dominates no other ones.
2. When a node is labeled by several labels  $x_1, \dots, x_m$ , then any of these dominates the others. Namely these components are completely coherent and any of them dominates all the others. When this node is the bottom, then these components are frozen, whereas the empty set is closed, these components combined together have only *one degree of freedom*.
3. For the deductive hypergraph given by the following graph, the components  $a$  and  $b$  even combined together dominate no other ones.

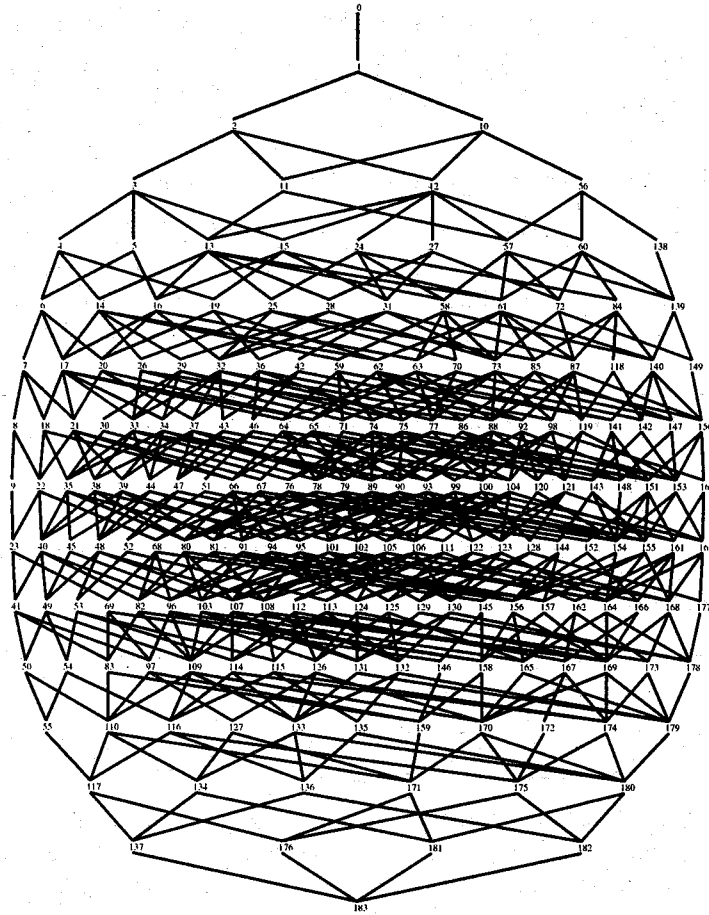


Figure 1: Hasse diagram of the lattice of closure operators of degree 4



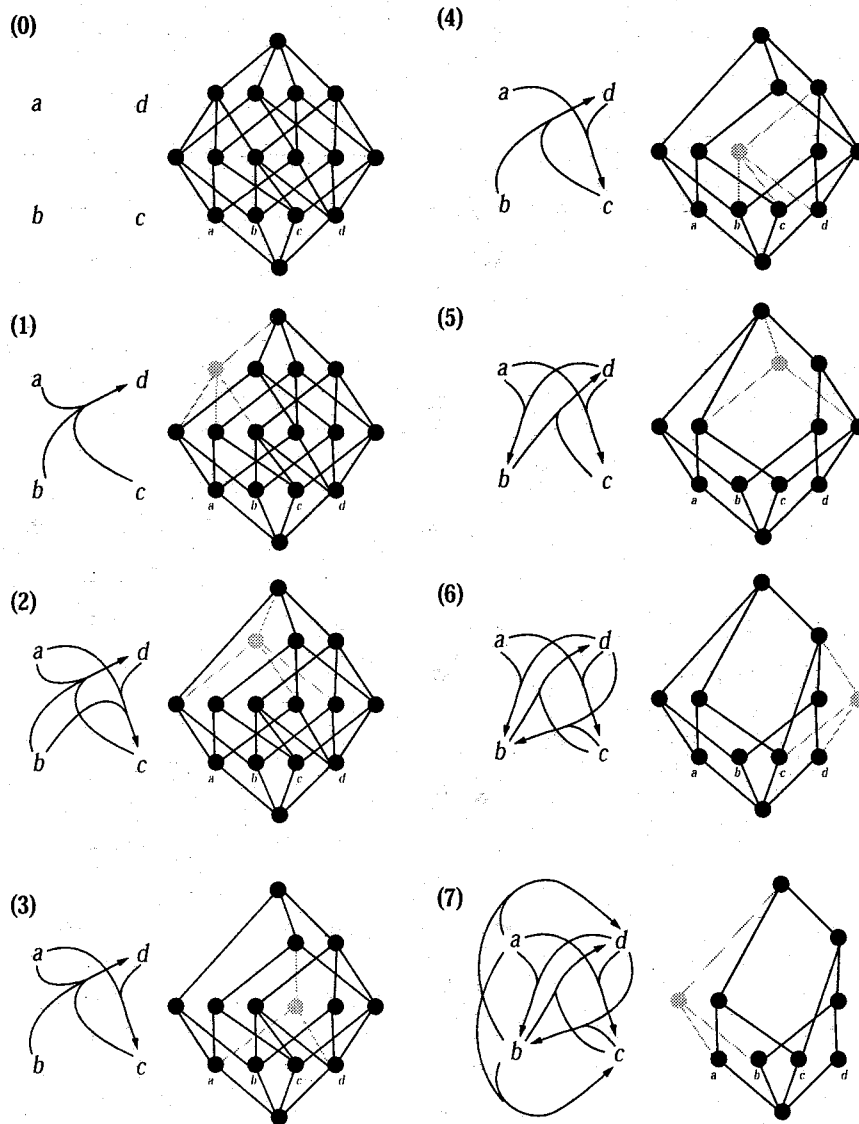


Figure 2: Change of coherence (i). (0) draws the weakest coherence where every component is independent of other ones. At the stage (3) the components *a, d* suffice to dominate *c*. At the stage (5) the components except *a* are dependent on some of the other components. any of the pairs  $\langle a, d \rangle, \langle a, d \rangle, \langle a, d \rangle, \langle a, d \rangle$ , two components dominate the other ones.

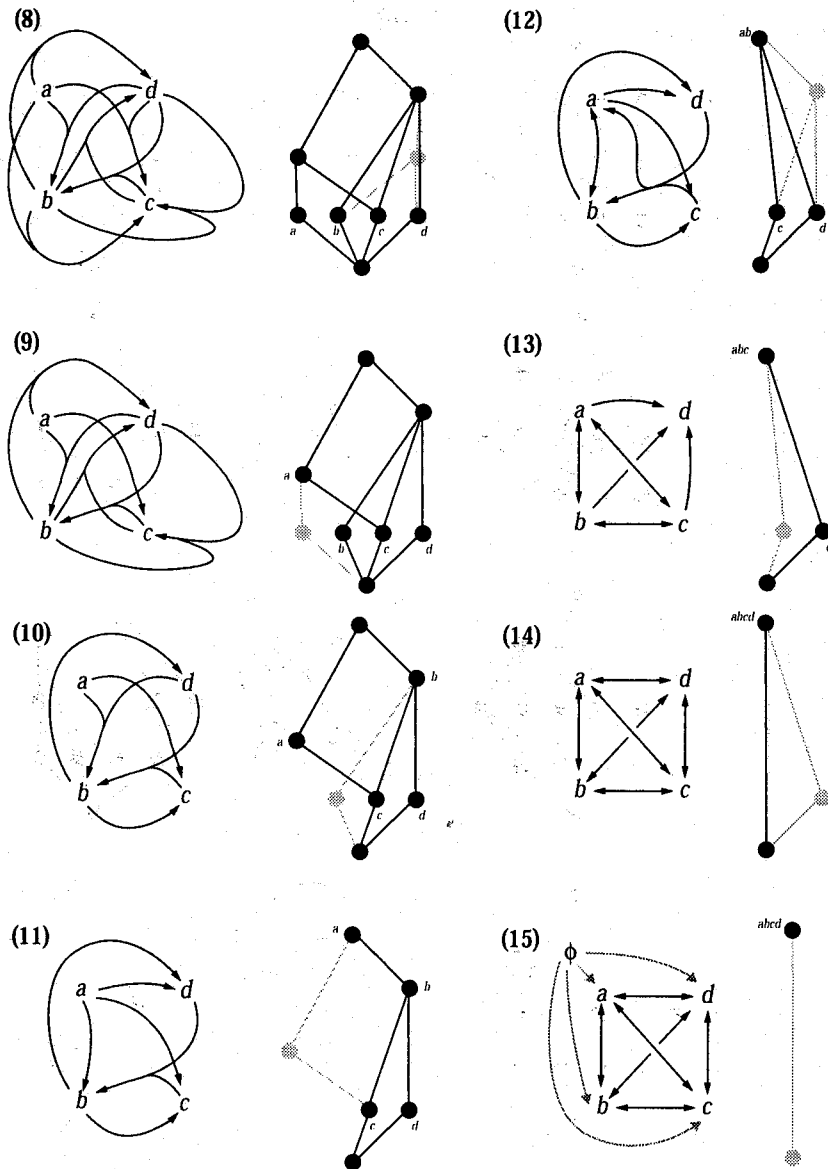
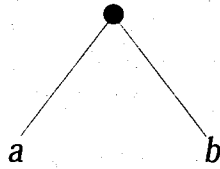
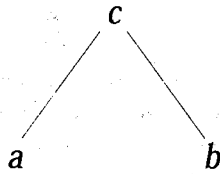


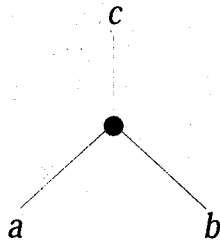
Figure 3: Change of coherence (ii). At the stage (9), the component  $a$  alone begins to dominate  $c$  and at the stage (11) it begins to dominate the whole system. At the stage (14) every component is interlocked to each others and at final stage (15) every component gets frozen.



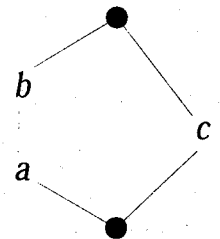
4. The following graph means  $c \rightarrow a$ ,  $c \rightarrow b$  and  $a, b \rightarrow c$ . Namely, the component  $c$  dominates both  $a$  and  $b$ , but is dominated by  $a$  and  $b$  combined together. However neither  $a$  nor  $b$  dominates  $c$  in isolation.



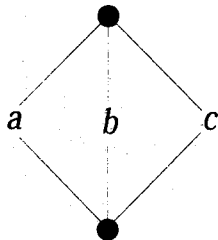
5. The following graph means  $c \rightarrow a$  and  $c \rightarrow b$  but  $a, b \rightarrow c$  does not hold. Namely, the component  $c$  dominates  $a$  and  $b$  but  $a$  and  $b$  even combined together cannot dominate  $c$ .



6. The following graph describes the deductive hyperdigraph of Example 1. The component  $a$  dominates  $b$  and  $b$  with the help of  $c$  in turn dominates  $a$ . Recall that this is the lattice called pentagon which prevents the whole lattice being modular.



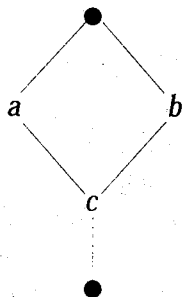
7. The following graph means  $a, b \rightarrow c, b, c \rightarrow a, c, a \rightarrow b$ , namely, any two of the three dominate the other one. Recall that this is the lattice called diamond which prevents a modular lattice being distributive.



8. For the deductive hyperdigraph given by the following graph, each of  $a$  and  $b$  dominates  $c$ . But this might give rise to conflicts between  $a$  and  $b$  when they want to control  $c$  independently. In other words, this coherence implies that there is certain relationship between  $a$  and  $b$ , which might be very strong. In the context of functional dependencies, this case is described by the equations

$$x_c = f(x_a) = g(x_b),$$

which obviously induces implicit functional relationship  $f(x_a) = g(x_b)$ . Although generally this induces almost functional relationship  $a \leftrightarrow b$  when the relation is differentiable, it is not necessary the case with general case.



## 5.2 Change of coherence caused by freezing components

When some of the components of a system is frozen being clamped from outside, the mode of mutual coherence changes in various ways.

The lattice labelings gives us concise description of such changes.

Let  $\lambda : V \rightarrow L$  be the current mode of coherence, where  $V$  is the set of components of the system. Suppose the components in  $W \subset V$  is frozen. Then the set of possibly active components of the system turns to  $V \setminus W$  and the resulting coherence is stronger or equal to the one given by the lattice labeling

$$\lambda_W : V \setminus W \rightarrow L_W,$$

where

$$L_W := \{ \ell \in L \mid \ell \geq \ell_W := \bigvee_{w \in W} \lambda_w \},$$

and

$$\lambda_W v := \lambda v \vee \ell_W.$$

The justification that why this construction describes the lower bound of the effect of freezing components can be seen by first analyzing the deductive hyperdigraph  $\rightarrow'$  of the resulting coherence. Since the components in  $W$  are fixed, every component actually dominates  $W$  and its closure. Hence  $U \rightarrow' v$  if  $U \cup W \rightarrow v$ . So, if we define

$$U \rightarrow_W v \stackrel{\text{def}}{\iff} U \cup W \rightarrow v,$$

then  $\rightarrow' \geq \rightarrow_W$ .

The lattice labeling of  $\rightarrow_W$  is given by extracting those closed sets of  $\rightarrow$  which includes  $W$ , whence the image of the labeling  $\lambda_W$  of  $\rightarrow_W$  is  $L_W$ . Moreover,

$$\lambda_W(v) = \bigcap_{v \in U, W \subseteq U} U = \overline{v \cup W} = \overline{\{v\}} \vee \overline{W} = \lambda v \vee \ell_W.$$

We illustrate this operation by the following example. When the component 1 is frozen, then the component 3 begins to dominate 5 since  $1 \vee 3 \geq 5$ . Similarly 2 begins to dominate 6 since  $2 \vee 1 \leq 6$ . When 6 is frozen, then 3 and 7 begin to dominate the entire system.

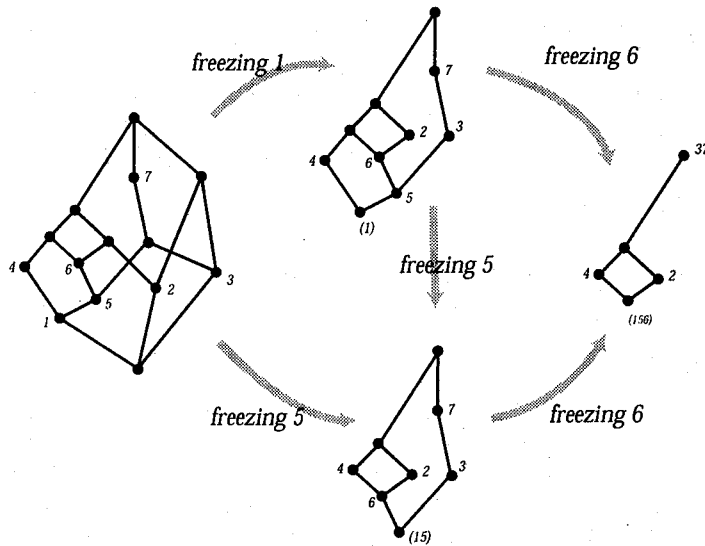


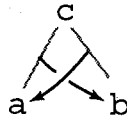
Figure 6: Change of coherence by freezing components

### 5.3 Relation with the environment

The deductive hypergraphs offers a rich language to talk about interrelationship between a system and its environment. Especially, it can express the subtleties of the influence of the environment on the systems, which seems not possible for the natural language.

We illustrate only a few of such examples.

#### 5.3.1 Example 1



Suppose  $a, b$  belongs to the system and  $c$  is outside of the system. Then this coherence means that with the factor  $c$  one can control the mode of mutual determination of the components  $a, b$  of the system. If  $c$  is frozen, then the component  $a, b$  acts in a fixed interlocked mode.

Suppose  $b, c$  belongs to the system and  $a$  is outside of the system. Then the system dominates the part  $a$  of the environment. However by  $a$  one can effect the mutual relationship between  $b$  and  $c$ . In fact,

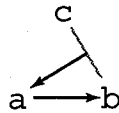
by getting hold of  $a$ , one can control the behavior of the system in the component  $b$  by closely observing the behavior of  $c$ . However, in this case, there might occur inconsistency or conflict in the control since the controller  $c$  itself can be controlled by the system.

By this example it should be realized that

$$a, c \rightarrow b$$

under a coherence does not imply that  $a, c$  controls  $b$ . For example, if  $b$  is rigid, then presence of this coherence implies certain constraint between  $a, c$ .

### 5.3.2 Example 2



Suppose  $a, b$  belongs to the system and  $c$  is outside of the system. Then the component  $b$  is determined by the component  $a$  internally, whereas  $a$  is determined by  $b$  when  $c$  is controlled from outside. In this case the control can effect only the inverse determination from  $b$  to  $a$ .

Suppose now  $b, c$  belongs to the system and  $a$  is outside of the system. Then one can control  $b$  of the system directly by  $a$  but, the system can get hold of  $a$  when  $c$  is used jointly with  $b$ . So here again there is conflict when this coherence is utilized both by the environment and by the system.

## 6 Concluding remarks

Deductive hyperdigraphs are closely related to various topics.

**Multicategory** If we give names to hyperarrows as

$$\varphi : a_1, \dots, a_n \rightarrow b,$$

then we come roughly to the notion of symmetric multicategory of Lambek[L].

**Infinite base set** When the base set  $V$  is infinite, we must use the concept of compactness. The closure operators should be restricted to satisfy the compactness condition

$$CY = \bigcup_{A \subset Y, A \text{ is finite}} CA.$$

We must also impose a Moore family  $\mathcal{A}$  the compactness condition

$$Y = \bigcup_{A \subset Y, A \in \mathcal{A}} A.$$

The lattice used in labeling should be algebraic lattice in the sense that every elements can be expressed as the join of compact elements.

**Domain** When the monotonicity condition is dropped to a certain degree, we obtain a structure studied in the domain theory. We will discuss such structure in future [HMT].

**Matroid** If a deductive hyperdigraph satisfies the following condition, then it is a structure usually called *matroid*. If  $W \rightarrow b$  but  $W' \not\rightarrow b$  for all proper subset  $W' \subset W$ , then  $W \cup b \setminus w \rightarrow w$  for all  $w \in W$ . The family of closed sets forms a semi-modular lattice. Only a few of deductive hyperdigraphs are matroids. For example, among the 14 deductive hyperdigraph of degree 3, there only three such ones. The matroid has yet another definition by *submodular* function  $\rho : \mathcal{P}(V) \rightarrow \mathbf{N}$ , where submodularity means that

(R1)  $\rho \emptyset = 0$ ,

(R2)  $\rho X$  is less than the number of elements of  $X$ ,

(R3)  $\rho$  is submodular, namely,

$$\rho(A \cup B) \leq \rho(A) + \rho(B) - \rho(A \cap B).$$

When the condition R2 is dropped, we obtain *polymatroid*. When  $\rho$  is a polymatroid, we can define a deductive hyperdigraph  $(V, \rightarrow)$  by

$$W \rightarrow v \stackrel{def}{\iff} \rho W = \rho(W \cup v).$$

We can show conversely that every deductive hyperdigraph can be obtained in this way using the Dilworth embedding of lattices in to matroids [DW, Theorem 14.1], although non isomorphic polymatroids can give a same deductive hyperdigraph.



**Chu spaces** A relation  $R \subseteq X \times Y$  is sometimes called a *Chu space* [P] or *contexts* in the formal concept analysis [DP, Ch 11], plays various roles in many branch of mathematics. It is well-known that this defines a pair of closure operators which have anti-isomorphic lattice of closed sets [B], called *Galois connection* since it is a general mechanism behind the formal aspects of the Galois theory. In this case, two sets are labeled by one lattice, although it is used with the opposite order for one set.

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## Proofs of propositions

### A Closure operator of lattices

Let  $L$  be a lattice. A map  $C : L \rightarrow L$  is called a closure operator if for all  $l, l_i \in L$  ( $i = 1, 2$ ),

$$\mathbf{C1}' \quad Cl \geq l,$$

$$\mathbf{C2}' \quad l_1 \geq l_2 \Rightarrow Cl_1 \geq Cl_2,$$

$$\mathbf{C3}' \quad CC = C.$$

**Proposition A.1** *There is a bijective correspondence between*

- a closure operator  $C : L \rightarrow L$ ,
- a meet sublattice  $M$  of  $L$  containing the top of  $L$ ,

given by

$$C_M(l) := \bigcap_{m \geq l, m \in M} m,$$

and

$$M_C := \{ l \mid Cl = l \}.$$

**Proof.** First we show that  $M_C$  is meet closed. Suppose  $l_i$  ( $i = 1, 2$ )  $\in M_C$ . Then  $C(l_1 \wedge l_2) \leq Cl_i$  ( $i = 1, 2$ ) imply  $C(l_1 \wedge l_2) \leq Cl_1 \wedge Cl_2 = l_1 \wedge l_2$ . Since  $C$  is monotone, it follows  $l_1 \wedge l_2$  is closed.

Next we show that  $C_M$  is a closure operator. Obviously  $C := C_M$  satisfies (C1') and (C2'). Since  $Cl \in M$  for all  $l \in M$  and  $Cl' = l'$  for all  $l' \in M$ , we have  $CCl = Cl$ , whence (C3').

Now we show that the above correspondences are inverses to each other.

Let  $M = M_C$ . Since  $Cl \in M$ , we have

$$C_M l \leq C_M Cl = Cl.$$

On the other hand, since  $C_M l$  is closed with respect  $C$ ,  $C$  is monotone and  $C_M l \geq l$ , we have

$$C_M l = CC_M l \geq Cl.$$

Hence  $C_M l = Cl$ .

Put now  $C = C_M$ . Let  $m \in M$ . Then  $Cm = C$ , whence  $m \in M_C$ . Suppose conversely  $l \in M_C$ . Then

$$l = Cl = \bigcap_{m \geq l, m \in M} m \in M,$$

since  $M$  is meet closed. Hence  $M_C = M$ .

q.e.d.

## B Proof of Proposition 3.1.4

**Proof.** Suppose  $\rightarrow_1 \leq \rightarrow_2$ . For  $A \in \mathcal{P}V$ , let  $v \in C_1A$ . Then  $A \rightarrow_1 v$ , whence  $A \rightarrow_2$ . Thus we have  $v \in C_2A$ . This implies  $C_1A \subseteq C_2A$ .

Suppose now  $C_1 \leq C_2$  and  $A \in \mathcal{A}_2$ .

$$A \subseteq C_1A \subseteq C_2A = A,$$

whence  $A \in \mathcal{A}_1$  and we have  $\mathcal{A}_2 \subseteq \mathcal{A}_1$ .

Suppose  $\mathcal{A}_1 \leq \mathcal{A}_2$ . Let  $\iota : \mathcal{A}_2 \rightarrow \mathcal{A}_1$  be the inclusion map. Then the closure operator corresponding to  $\iota$  (cf. the appendix) is  $C_2$  and we have obviously

$$C_2\lambda_1v = C_2C_1v = {}^*C_2v = \lambda_2v,$$

where  $*$  holds because  $C_2v \subseteq C_2C_1v \subseteq C_2C_2v = C_2v$ .

Suppose now that  $(V, \lambda_1) \leq (V, \lambda_2)$ . There is by definition an injective meet semilattice map  $\iota : L_2 \rightarrow L_1$  with

$$C\lambda_2v = \iota\lambda_2v,$$

where  $C : L_1 \rightarrow L_1$  is the closure operator corresponding to the meet sublattice  $\text{Im}(\iota) \subseteq L_1$ . Let  $W \rightarrow_1 v$ , i.e.,

$$\lambda_1 \leq \bigvee_{w \in W} \lambda_1w.$$

Put  $m = \iota \bigvee_{w \in W} \lambda_2w$ . Then

$$m \geq \iota\lambda_2w = C\lambda_1w \geq \lambda_1w$$

for all  $w \in W$ , whence

$$m \geq \bigvee_{w \in W} \lambda_1w \geq \lambda_1v$$

. Since  $m$  is  $C$ -closed, we obtain

$$m \geq C\lambda_1 v = \iota\lambda_2 v.$$

Since  $\iota$  is injective, we obtain

$$\bigvee_{w \in W} \lambda_2 w \geq \lambda_2 v,$$

i.e.  $W \rightarrow_2 v$ .

**q.e.d.**