Title	Norm inequalities for some singular integral operators
Author(s)	Nakazi, T.
Citation	Hokkaido University Preprint Series in Mathematics, 474, 1-13
Issue Date	1999-11-1
DOI	10.14943/83620
Doc URL	http://hdl.handle.net/2115/69224
Type	bulletin (article)
File Information	pre474.pdf

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Series #474. November 1999

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Norm Inequalities for Some Singular Integral Operators

by

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AMS Subject Classification ; Primary 45 E 10, 47 B 35, Secondary 46 J 15, 46 K 50

Keywords : Singular integral operator, norm, analytic operator algebra, lifting theorem

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

Abstract. Let $\mathcal{B}$ be a von Neumann algebra and P a selfdjoint projection. For A and B in $\mathcal{B}$, set $S_{A,B} = AP + BQ$ where $Q = I - P$. The operator $S_{A,B}$ will be called a singular integral operator. When $\mathcal{B} = L^\infty(T)$ where $L^\infty(T)$ is the usual Lebesgue space on the unit circle and P is an analytic projection, in [6] we established formulae for norms of $S_{A,B}$ and $(S_{A,B})^{-1}$. In this paper, if $\mathcal{A} = \{D \in \mathcal{B} : PDP = D\}$ and $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property, then we will establish formulae of norms of $S_{A,B}$ and $(S_{A,B})^{-1}$. These formulae are operator theoretic and different from the previous ones. There are several examples such that $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. As result, we give several interesting inequalities.

§1. Introduction

Let m denote the normalized Lebesgue measure on the unit circle T. For $1 \leq p \leq \infty$, $L^p(T)$ denotes the usual Lebesgue space on T and $H^p(T)$ denotes the usual Hardy space on T. The canonical example of a singular integral operator is the operator defined by

$$(S_{a,b}F)(\zeta) = \frac{a(\zeta)+b(\zeta)}{2}F(\zeta) + \frac{a(\zeta)-b(\zeta)}{2}(SF)(\zeta)$$

on $L^2(T)$; here, $a(\zeta)$ and $b(\zeta)$ denote functions in $L^\infty(T)$ and

$$(SF)(\zeta) = \frac{1}{\pi i}\int_T \frac{f(\eta)}{\eta - \zeta} d\eta \quad (a.e. \zeta \in T),$$

the integral being a Cauchy principal value (cf.[4]). Then $P = (I+S)/2$ is a selfadjoint projection from $L^2(T)$ to $H^2(T)$, $Q = (I-S)/2$ is a selfadjoint projection from $L^2(T)$ to $e^{-i\theta}\overline{H^2(T)}$, and $P+Q = I$ where I denotes the identity operator. Hence

$$S_{a,b} = aP + bQ.$$

The following inequalities are well known and not difficult to establish.

$$\max\{\|a\|_\infty, \|b\|_\infty\} \leq \|S_{a,b}\| \leq \|\sqrt{|a|^2+|b|^2}\|_\infty.$$

and

$$\inf_{\|F\|_2=1} \|S_{a,b}F\|_2^2 \leq ess \inf_T (\min\{|a|^2, |b|^2\}).$$

In the previous paper, the author and T.Yamamoto [6] showed the following theorems.

Theorem A. *Let $a, b \in L^\infty(T)$. Then*

$$\|S_{a,b}\|^2 = \inf_{k \in H^\infty(T)} \left\| \frac{|a|^2+|b|^2}{2} + \sqrt{|a\bar{b}+k|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2} \right\|_\infty .$$

Theorem B. *Let $a, b \in L^\infty(T)$. Then*

$$\inf_{F \in L^2(T), \|F\|_2=1} \|S_{a,b}F\|_2^2$$

$$= \sup_{k \in H^\infty(T)} \left(ess \inf_T \left(\frac{|a|^2+|b|^2}{2} - \sqrt{|a\bar{b}+k|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2} \right) \right).$$

In this paper, we give formulae which are similar to those of Theorems A and B. In fact, we prove them for more general situations.

Let K be a complex Hilbert space and H the closed subspace of K. Let P be a selfadjoint projection from K to H and $Q = I - P$ where I denotes the identity operator on K. $\mathcal{B}$ denotes a von Neumann algebra on K which contains I and $\mathcal{A}$ denotes a (perhaps nonselfadjoint) weakly closed subalgebra of $\mathcal{B}$ which has H as an invariant subspace. For A and B in $\mathcal{B}$, set

$$S_{A,B} = AP + BQ.$$

The operator $S_{A,B}$ is called a singular integral operator. In this paper, we give formulae of norms of $S_{A,B}$ and $(S_{A,B})^{-1}$. The formulae are little bit complicated. However, as result we give the following simple inequalities.

$$\begin{aligned}
&\inf_{D\in\mathcal{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} + |\langle (B^*A+D)F, G\rangle| \right\} \\
\leq\ & \|S_{A,B}\|^2 \\
\leq\ & \inf_{D\in\mathcal{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \{\max(\|AF\|^2, \|BG\|^2) + |\langle (B^*A+D)F, G\rangle|\}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{D\in\mathcal{A}} \inf_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} - |\langle (B^*A+D)F, G\rangle| \right\} \\
\geq\ & \|S_{A,B}^{-1}\|^2 \\
\geq\ & \sup_{D\in\mathcal{A}} \inf_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \{\min(\|AF\|^2, \|BG\|^2) - |\langle (B^*A+D)F, G\rangle|\}.
\end{aligned}$$

In §2, we give a definition of a lifting property of $(\mathcal{B}, \mathcal{A}, P)$ and several examples which have such a property. In §3, we study the norm of $S_{A,B}$ when $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. In §4, we study the norm of $S_{A,B}^{-1}$ when $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. In §5, we give several remarks.

§2. Lifting theorem

In this section, we recall a special case of a lifting theorem which was proved in [5]. The classical case was proved by Cotlar and Sadosky [1].

Suppose $\mathcal{T} = (T_{ij})$ is a 2×2 operator matrix on $K \oplus K$ where $T_{ij} \in \mathcal{B}$ $(i,j = 1,2)$, $T_{11} \geq 0$, $T_{22} \geq 0$ and $T_{21}^* = T_{12}$. $[\mathcal{B}]$ denotes the set of such operator matrices $\mathcal{T}$. $[\mathcal{A}]_0$ denotes the subset of $[\mathcal{B}]$ such that $T_{12} \in \mathcal{A}$ and $T_{11} = T_{22} = 0$. Let us denote

$$\mathcal{T}[f_1, f_2] = \sum_{i,j=1}^{2} \langle T_{ij} f_i, f_j \rangle.$$

If $\mathcal{T}$ satisfies $\mathcal{T}[f_1, f_2] \geq 0$ for all f_1 in H $(resp. K)$ and f_2 in $H^\perp$ $(resp. K)$, then $\mathcal{T}$ is said to be positive on $H \oplus H^\perp$ $(resp. K \oplus K)$ where $H^\perp$ is the orthogonal complement of H in K. When $\mathcal{T}$ in $[\mathcal{B}]$ and $\mathcal{T}$ is positive on $H \oplus H^\perp$, if we can find $\tilde{\mathcal{T}}$ in $\mathcal{T} + [\mathcal{A}]_0$ which is positive on $K \oplus K$ then we say that $\mathcal{T}$ has a lifting $\tilde{\mathcal{T}}$. If any positive $\mathcal{T}$ in $[\mathcal{B}]$ on $H \oplus H^\perp$ has a lifting $\tilde{\mathcal{T}}$, we say that $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property where P is a selfadjoint projection of K onto H.

Example. We give several examples of $(\mathcal{B}, \mathcal{A}, P)$ which have a lifting property (cf.[5]).

(1) Let $\mathcal{B} = \mathcal{L}(K)$ be the set of all bounded linear operators on K, P a selfadjoint projection from K to H and

$$\mathcal{A} = \{A \in \mathcal{L}(K) : PAP = AP\}.$$

(2) Let U be a bilateral shift operator on K with $UP = PUP$, where P is a selfadjoint projection from K to H. Suppose $\bigcap_{n=0}^{\infty} U^n(H) = \{0\}$ and $\bigcup_{n=0}^{\infty} U^{*n}(H)$ is dense in K. Let $\mathcal{B} = \{T \in \mathcal{L}(K)\ ;\ UT = TU\}$ and $\mathcal{A} = \{A \in \mathcal{B}\ ;\ PAP = AP\}$.

(3) Let $\mathcal{B}_1$ be a factor with faithful semifinite normal trace τ and let $\mathcal{E}$ be a complete nest of selfadjoint projections in $\mathcal{B}_1$. Let $L^p = L^p(\mathcal{B}, \tau)$ $(1 \leq p \leq \infty)$, be the usual noncommutative Lebesgue spaces and define the noncommutative Hardy space $H^p = H^p(\mathcal{B}_1, \mathcal{E}, \tau)$ to be the closed subspace of L^p of elements A for which $(1 - P_1)AP_1 = 0$ for all $P_1 \in \mathcal{E}$. Suppose that $\mathcal{B} = L^\infty, \mathcal{A} = H^\infty, K = L^2$ and $H = H^2$.

(4) Let $\mathcal{A}_1$ be a weak-*Dirichlet algebra of $L^\infty(\mu)$ where μ is a probability measure. The abstract Hardy space $H^p(\mu)$, $1 \leq p \leq \infty$, associated with $\mathcal{A}_1$ are defined as follows. For $1 \leq p \leq \infty$, $H^p(\mu)$ is the $L^p(\mu)$-closure of $\mathcal{A}_1$, while $H^\infty(\mu)$ is defined to be the weak-*closure of $\mathcal{A}_1$ in $L^\infty(\mu)$. Suppose $\mathcal{B} = L^\infty(\mu)$, $\mathcal{A} = H^\infty(\mu)$, $K = L^2(\mu)$ and $H = H^2(\mu)$.

In the latter section, when $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property we study the norms of the sigular integral operator and its inverse.

§3. Inequality for norm of $S_{A,B}$

In this section, we assume that $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. Assuming that $\|AP\| = \|A\|$ and $\|BQ\| = \|B\|$, we give a formula and inequalities for the norm of $S_{A,B}$. In general,

$$S_{A,B}^* S_{A,B} = (PA^* + QB^*)(AP + BQ) = PA^*AP + QB^*BQ + QB^*AP + PA^*BQ.$$

If B^*A is in $\mathcal{A}$, then $\|S_{A,B}\| = \max(\|A\|, \|B\|)$. Not assuming that B^*A is in $\mathcal{A}$, Proposition 1 gives a formula similar to Theorem A.

Proposition 1. *Let A and B be in $\mathcal{B}$. If* $\max(\|A\|, \|B\|) \le \|S_{A,B}\|$*, then*

$$\|S_{A,B}\|^2$$

$$= \inf_{D \in \mathcal{A}} \sup_{\substack{F,G \in K \\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2 + \|BG\|^2}{2} + \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2} \right\}.$$

There exists at least a D in $\mathcal{A}$ which gives the infimum.

Proof. Put $\gamma = \|S_{A,B}\|$, then

$$\|Af + Bg\|^2 \le \gamma^2 \|f + g\|^2$$

where $f \in H$ and $g \in H^\perp$. Hence

$$\langle (\gamma^2 - A^*A)f, f\rangle + \langle (\gamma^2 - B^*B)g, g\rangle - 2Re\langle (B^*A - \gamma^2)f, g\rangle \ge 0$$

for all $f \in H$ and $g \in H^\perp$. Suppose $\mathcal{T} = [T_{ij}]$, where $T_{11} = \gamma^2 - A^*A$, $T_{22} = \gamma^2 - B^*B$ and $T_{12} = T_{21}^* = B^*A - \gamma^2$. By hypothesis on A and $B, T_{11} \ge 0$ and $T_{22} \ge 0$. Hence $\mathcal{T}$ is positive on $H \oplus H^\perp$. Since $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property, there exists D in $\mathcal{A}$ such that $\tilde{\mathcal{T}} = [\tilde{\mathcal{T}}_{ij}]$ is positive on $K \oplus K$ where $\tilde{\mathcal{T}}_{11} = T_{11}$, $\tilde{\mathcal{T}}_{22} = T_{22}$ and $\tilde{\mathcal{T}}_{12} = \tilde{\mathcal{T}}_{21}^* = T_{12} + D$. Hence

$$\langle (\gamma^2 - A^*A)F, F\rangle \langle (\gamma^2 - B^*B)G, G\rangle \ge |\langle (B^*A + D)F, G\rangle|^2$$

for any $F \in K$ and $G \in K$, and so

$$\begin{aligned} &\gamma^4 \|F\|^2 \|G\|^2 - \gamma^2 (\|AF\|^2 \|G\|^2 + \|F\|^2 \|BG\|^2) \\ + \quad &\|AF\|^2 \|BG\|^2 - |\langle (B^*A + D)F, G\rangle|^2 \ge 0 \end{aligned}$$

for any $F \in K$ and $G \in K$. If $\|F\| = \|G\| = 1$, then

$$\gamma^4 - \gamma^2(\|AF\|^2 + \|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A + D)F, G\rangle|^2 \ge 0.$$

Therefore

$$\gamma^2 \le \frac{\|AF\|^2 + \|BG\|^2}{2} - \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2}$$

or

$$\gamma^2 \geq \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}.$$

The first inequality above is not valid because $\gamma^2 \geq \max\{\|AF\|^2,\ \|BG\|^2\}$ for any $F, G \in K$ with $\|F\| = \|G\| = 1$. Thus

$$\gamma^2 \geq \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2} \right\}.$$

Conversely if

$$\gamma_0^2 = \inf_{D\in\mathcal{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2} \right\},$$

for any $\varepsilon > 0$ there exists D in $\mathcal{A}$ such that

$$(\gamma_0+\varepsilon)^4 - (\gamma_0+\varepsilon)^2(\|AF\|^2+\|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A+D)F,G\rangle|^2 \geq 0$$

for any $F, G \in K$ with $\|F\| = \|G\| = 1$. Hence

$$(\gamma_0+\varepsilon)^4 - (\gamma_0+\varepsilon)^2(\|Af\|^2+\|Bg\|^2) + \|Af\|^2\|Bg\|^2 - |\langle B^*Af,g\rangle| \geq 0$$

for any $f \in H$ and $g \in H^\perp$ with $\|f\| = \|g\| = 1$ because $(1-P)DP = 0$. This implies $\gamma_0 + \varepsilon \geq \|S_{A,B}\|$ and so $\gamma_0 \geq \|S_{A,B}\|$ because ε is arbitrary.

Corollary 1. *Let A and B be in $\mathcal{B}$. If $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ where a and b are complex numbers, then*

$$\|S_{A,B}\|^2 = \frac{|a|^2+|b|^2}{2} + \sqrt{\|B^*A+\mathcal{A}\|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2}.$$

Proof. Since $(AP)^*(AP) = PA^*AP = |a|^2P$, $\|AP\| = \|A\|$, Similary $\|BQ\| = \|B\|$.

Corollary 2.*Let A be in $\mathcal{B}$ and $B = I$. Then*

$$\|S_{A,I}\|^2 = \inf_{D\in\mathcal{A}} \sup_{\substack{F\in K\\ \|F\|=1}} \left\{ \frac{\|AF\|^2+1}{2} + \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2} \right\}$$

Proof. It is a corollary of the proof of Proposition 1. In fact, note that

$$(\gamma^2 - 1)\langle(\gamma^2 - A^*A)F, F\rangle \geq |\langle(A + D)F, G\rangle|^2$$

for any $G \in K$ with $\|G\| = 1$ if and only if

$$(\gamma^2 - 1)\langle(\gamma^2 - A^*A)F, F\rangle \geq \|(A + D)F\|^2.$$

Theorem 2. *Let A and B be in $\mathcal{B}$. If* $\max(\|A\|, \|B\|) \leq \|S_{A,B}\|$, *then*

$$\begin{aligned}
&\inf_{D\in\mathcal{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{\frac{\|AF\|^2 + \|BG\|^2}{2} + |\langle(B^*A + D)F, G\rangle|\right\} \\
&\leq \|S_{A,B}\|^2 \\
&\leq \inf_{D\in\mathcal{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{\max(\|AF\|^2, \|BG\|^2) + |\langle(B^*A + D)F, G\rangle|\right\}
\end{aligned}$$

Proof. This is an immediate consequence of Proposition 1. In fact, it can be shown by the following elementary inequality :

$$\begin{aligned}
&\frac{\|AF\|^2 + \|BG\|^2}{2} + \sqrt{|\langle(B^*A + D)F, G\rangle|^2} \\
\leq\ & \frac{\|AF\|^2 + \|BG\|^2}{2} + \sqrt{|\langle(B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2} \\
\leq\ & \frac{\|AF\|^2 + \|BG\|^2}{2} + \sqrt{|\langle(B^*A + D)F, G\rangle|^2} + \sqrt{\left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2}
\end{aligned}$$

§4. Inequality for norm of $(S_{A,B})^{-1}$

In this section, we assume that $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. Under some conditions on A and B, we give a formula and inequalities of $\inf\{\|S_{A,B}F\| \ ; \ F \in K, \|F\| = 1\}$. As in a formula of $\|S_{A,B}\|$, even if $(\mathcal{B}, \mathcal{A}, P)$ does not have a lifting property, in general it is easy to see that

$$\inf\{\|S_{A,B}F\| \ ; \ F \in K, \|F\| = 1\} = \min(\inf \|AF\|, \inf \|BF\|).$$

when A and B^* are in $\mathcal{A}$. Proposition 3 implies that if B^*A is in $\mathcal{A}$, then the same thing is true when $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property.

Proposition 3. *Let A and B be in $\mathcal{B}$. If* $\inf\{\|Af\|\ ;\ f \in H, \|f\| = 1\} = \inf\{\|AF\|\ ;\ F \in K, \|F\| = 1\}$ *and* $\inf\{\|Bg\|\ ;\ g \in H^\perp,\ \|g\| = 1\} = \inf\{\|BG\|\ ;\ G \in K, \|G\| = 1\}$, *then*

$$\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 =$$

$$\sup_{D\in\mathcal{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2 + \|BG\|^2}{2} - \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2} \right\}.$$

There exists at least D in $\mathcal{A}$ which gives the supremum.

Proof, Put $\varepsilon = \inf\{\|S_{A,B}F\|\ ;\ F \in K, \|F\| = 1\}$, then

$$\|Af + Bg\|^2 \geq \varepsilon^2 \|f + g\|^2$$

where $f \in H$ and $g \in H^\perp$. Hence

$$\langle (A^*A - \varepsilon^2)f, f\rangle + \langle (B^*B - \varepsilon^2)g, g\rangle - 2Re\langle (B^*A - \varepsilon^2)f, g\rangle \geq 0$$

for all $f \in H$ and $g \in H^\perp$. Suppose $\mathcal{T} = [T_{ij}]$ where $T_{11} = A^*A - \varepsilon^2$, $T_{22} = B^*B - \varepsilon^2$ and $T_{12} = T_{21}^* = B^*A - \varepsilon^2$. By hypothesis on A and B, $T_{11} \geq 0$ and $T_{22} \geq 0$. Hence T is positive on $H \oplus H^\perp$. As in the proof of Proposition 1, since $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property, there exists D in $\mathcal{A}$ such that

$$\langle (A^*A - \varepsilon^2)F, F\rangle \langle (B^*B - \varepsilon^2)G, G\rangle \geq |\langle (B^*A + D)F, G\rangle|^2$$

for any $F \in K$ and $G \in K$, and so

$$\varepsilon^4 - \varepsilon^2(\|AF\|^2 + \|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A + D)F, G\rangle|^2 \geq 0$$

for any $F \in K$ and $G \in K$ with $\|F\| = \|G\| = 1$. Therefore

$$\varepsilon^2 \leq \frac{\|AF\|^2 + \|BG\|^2}{2} - \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2}$$

or

$$\varepsilon^2 \geq \frac{\|AF\|^2 + \|BG\|^2}{2} + \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2}$$

By hypotheses on A and $B, A^*A \geq \varepsilon^2$ and $B^*B \geq \varepsilon^2$, and so

$$\varepsilon^2 \leq (\|AF\|^2 + \|BG\|^2)/2$$

for any $F, G \in K$ with $\|F\| = \|G\| = 1$. This implies that the second inequality is not valid. Thus

$$\varepsilon^2 \leq$$

$$\inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2 + \|BG\|^2}{2} - \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2} \right\}.$$

The converse can be shown as in the proof of Proposition 1.

Corollary 3. *Let A and B be in $\mathcal{B}$. If A and B satisfy the condition in Proposition 3, then*

$$\begin{aligned}
&\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\quad \geq \sup_{D\in\mathcal{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \{\min(\|AF\|^2, \|BG\|^2) - |\langle (B^*A+D)F, G\rangle|\} \\
&\quad \geq \min\Big(\inf_{\substack{F\in K \\ \|F\|=1}} \|AF\|^2, \inf_{G\in K} \|BG\|^2\Big) - \|B^*A+\mathcal{A}\|.
\end{aligned}$$

Corollary 4. *Let A and B be in $\mathcal{B}$. If $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ where a and b are complex numbers, then*

$$\begin{aligned}
&\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\quad = \frac{|a|^2+|b|^2}{2} - \sqrt{\|B^*A+\mathcal{A}\|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2}.
\end{aligned}$$

Corollary 5. *Let A be in $\mathcal{B}$ and $B = I$. Then*

$$\begin{aligned}
&\inf \|S_{A,I}F\|^2 \\
&\quad = \sup_{D\in\mathcal{A}} \inf_{\substack{F\in K \\ \|F\|=1}} \left\{\frac{\|AF\|^2+1}{2} - \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2}\right\}.
\end{aligned}$$

Proof. It is a corollary of the proof of Proposition 3 similarly to the proof of Corollary 2.

Theorem 4. *Let A and B be in $\mathcal{B}$. If* $\inf\{\|Af\|\ ;\ f\in H, \|f\|=1\} = \inf\{\|AF\|\ ;\ F\in K, \|F\|=1\}$ *and* $\inf\{\|Bg\|\ ;\ g\in H^\perp, \|g\|=1\} = \inf\{\|BG\|\ ;\ G\in K, \|G\|=1\}$, *then*

$$\begin{aligned}
&\sup_{D\in\mathcal{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \left\{\frac{\|AF\|^2+\|BG\|^2}{2} - |\langle (B^*A+D)F, G\rangle|\right\} \\
&\geq \inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\geq \sup_{D\in\mathcal{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \{\min(\|AF\|^2,\ \|BG\|^2) - |\langle (B^*A+D)F, G\rangle|\}.
\end{aligned}$$

Proof. This is an immediate result of Proposition 3 and the proof is similar to that of Theorem 2.

§5. Applications and remarks

(I) Suppose $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. For A in $\mathcal{B}$, the Toeplitz operator T_A is defined by $T_A f = P(Af)$ for f in H. Then we can give an operator version of Theorem 2 in [6]. That is, if $\inf\{\|Af\| \ ; \ f \in H, \ \|f\| = 1\} = \inf\{\|AF\| \ ; \ F \in K, \ \|K\| = 1\}$, then

$$\begin{aligned} &\inf_{f \in H \ \|f\|=1} \|T_A f\|^2 \\ &= \sup_{D \in \mathcal{A}} \{ \inf_{F \in K \ \|F\|=1} (\|AF\|^2 - \|(A+D)F\|^2) \}. \end{aligned}$$

This has an application. That is, T_A is left invertible if and only if there exist D in $\mathcal{A}$ and $\varepsilon > 0$ such that

$$\|AF\|^2 \geq \varepsilon + \|(A+D)F\|^2$$

for any F in K with $\|F\| = 1$. This is equivalent to

$$A^* A \geq \varepsilon + (A+D)^*(A+D).$$

It may be interesting that we did not use the factorization theorem in the proof of the result above. When A is a unitary operator, T_A is left invertible if and only if there exists a D in $\mathcal{A}$ such that $\|A + D\| < 1$. This implies Theorem 2 in [7] where $\mathcal{B} = \mathcal{L}(K)$, that is, Example (1) in Section 2, and Theorem 7.30 in [3] where $\mathcal{B} = L^\infty(T)$, that is , the classical case.

(II) Suppose $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property, We give an application of Corollary 5. Let A be a nonzero operator in $\mathcal{B}$ with $\inf\{\|AF\| \ ; \ F \in K, \ \|F\| = 1\} = \inf\{\|Af\| \ ; \ f \in H, \ \|f\| = 1\}$. $S_{A,I}$ is left invertible if and only if there exist D in $\mathcal{A}$ and $\varepsilon > 0$ such that

$$\|AF\|^2 \geq \varepsilon + \|(A+D)F\|^2$$

for any F in K with $\|F\| = 1$.

We give a proof. If $S_{A,I}$ is left invertible, by Corollary 5 there exist $D \in \mathcal{A}$ and $1 > \delta > 0$ such that

$$\frac{\|AF\|^2 + 1}{2} - \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2 - 1}{2}\right)^2} \geq \delta$$

for any $F \in K$ with $\|F\| = 1$. This implies that

$$(1 - \delta)\|AF\|^2 \geq (1 - \delta)\delta + \|(A+D)F\|^2$$

for any $F \in K$ with $\|F\| = 1$. Setting $\varepsilon = (1-\delta)\delta$ the necessity follows. Conversely if there exist D in $\mathcal{A}$ and $\varepsilon > 0$ such that

$$\|AF\|^2 \geq \varepsilon + \|(A+D)F\|^2$$

for any F in K with $\|F\| = 1$,

$$\begin{aligned}
&\frac{\|AF\|^2+1}{2} - \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2} \\
\geq\ & \frac{\|AF\|^2+1}{2} - \sqrt{\|AF\|^2 - \varepsilon + \left(\frac{\|AF\|^2-1}{2}\right)^2} \\
=\ & \frac{\|AF\|^2+1}{2} - \sqrt{\left(\frac{\|AF\|^2+1}{2}\right)^2 - \varepsilon^2} \\
=\ & \varepsilon^2 \Big/ \left\{\frac{\|AF\|^2+1}{2} + \sqrt{\left(\frac{\|AF\|^2+1}{2}\right)^2 - \varepsilon^2}\right\}
\end{aligned}$$

for any $F \in K$ with $\|F\| = 1$. Now Corollary 5 implies that $\inf_{F\in K\ \|F\|=1} \|S_{A,I}F\| > 0$.

In an abstract situation (see Example (1) in Section 2), Shinbrot [7] studied singular integral operators.

(III) We don't assume that $(\mathcal{B}, \mathcal{A}, P)$ has a lifting property. We can prove the following :

(1) Let A and B be in $\mathcal{B}$. Then

$$\begin{aligned}
&\|S_{A,B}\|^2 \\
=\ & \sup_{\substack{f\in H,\ g\in H^\perp \\ \|f\|=\|g\|=1}} \left\{\frac{\|Af\|^2 + \|Bg\|^2}{2} + \sqrt{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2 - \|Bg\|^2}{2}\right)^2}\right\}.
\end{aligned}$$

Hence if $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ then

$$\|S_{A,B}\|^2 = \frac{|a|^2 + |b|^2}{2} + \sqrt{\|QB^*AP\|^2 + \left(\frac{|a|^2 - |b|^2}{2}\right)^2}.$$

(2) Let A and B be in $\mathcal{B}$. Then

$$\begin{aligned}
&\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
=\ & \inf_{\substack{f\in H\ g\in H^\perp \\ \|f\|=\|g\|=1}} \left\{\frac{\|Af\|^2 + \|Bg\|^2}{2} - \sqrt{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2 - \|Bg\|^2}{2}\right)^2}\right\}.
\end{aligned}$$

Hence if $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ then

$$\inf_{F \in K \ \|F\|=1} \|S_{A,B}F\|^2 = \frac{|a|^2 + |b|^2}{2} - \sqrt{\|QB^*A\|^2 + \left(\frac{|a|^2 - |b|^2}{2}\right)^2}$$

(IV) Let $\mathcal{S}$ be a set of unitary operators on K and suppose the set $\{Vf \ ; \ V \in \mathcal{S}, f \in H\}$ is dense in K. If A is in the commutant of $\mathcal{S}$, then

$$\|A\| = \|AP\|$$

and

$$\inf_{F \in K \ \|F\|=1} \|AF\| = \inf_{F \in H \ \|f\|=1} \|Af\|.$$

This can be proved as in the proof of $(viii)$ of Theorem 4 in [2].

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