



Title	Asymptotics for prediction errors of stationary processes with reflection positivity
Author(s)	Inoue, A.; Kasahara, Y.
Citation	Hokkaido University Preprint Series in Mathematics, 475, 1-15
Issue Date	1999-12-1
DOI	10.14943/83621
Doc URL	<a href="http://hdl.handle.net/2115/69225">http://hdl.handle.net/2115/69225</a>
Type	bulletin (article)
File Information	pre475.pdf



[Instructions for use](#)

**ASYMPTOTICS FOR PREDICTION ERRORS  
OF STATIONARY PROCESSES  
WITH REFLECTION POSITIVITY**

**A. INOUE AND Y. KASAHARA**

**Series #475. December 1999**

HOKKAIDO UNIVERSITY  
PREPRINT SERIES IN MATHEMATICS

- #449 G. Ishikawa, Developable hypersurfaces and homogeneous spaces in a real projective space, 15 pages. 1999.
- #450 A. Asao, Infinite dimensional analysis and analytic number theory, 35 pages. 1999.
- #451 T. Nakazi, Riesz's functions and Carleson inequalities, 9 pages. 1999.
- #452 A. Inoue, Asymptotics for the partial autocorrelation function of a stationary process, 37 pages. 1999.
- #453 J. Adachi, Generic knots in contact 3-manifolds, 14 pages. 1999.
- #454 H. Kubo and K. Kubota, Asymptotic behavior of classical solutions to a system of semilinear wave equations in low space dimensions, 25 pages. 1999.
- #455 F. Oda and T. Yoshida, Crossed Burnside rings [I]: The fundamental theorem, 50 pages. 1999.
- #456 D. Pei, T. Sano and E. Torii, Generic differential geometry of hyperbolic plane curves, 13 pages. 1999.
- #457 A. Arai and M. Hirokawa, Ground states of a general class of quantum field Hamiltonians, 55 pages. 1999.
- #458 S. Izumiya and N. Takeuchi, Singularities of ruled surfaces in  $\mathbf{R}^3$ , 11 pages. 1999.
- #459 Y. Giga and M.-H. Sato, On semicontinuous solutions for general Hamilton-Jacobi equations, 8 pages, 1999.
- #460 H. Furuhashi and T. Kurose, Self-dual centroaffine surfaces of codimension two with constant affine mean curvature, 14 pages. 1999.
- #461 M.-H. Giga, Y. Giga and R. Kobayashi, Very singular diffusion equations, 37 pages, 1999.
- #462 T. Nakazi, Essential norms of some singular integral operators, 6 pages, 1999.
- #463 R. Agemi, Global existence of nonlinear elastic waves, 57 pages, 1999.
- #464 T. Mikami, Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric, 52 pages. 1999.
- #465 N. Kawazumi and Y. Shibukawa, The meromorphic solutions of the Bruschi-Calogero equation, 20 pages. 1999.
- #466 S. Izumiya, G. Kossioris and G. Makrakis, Multivalued solutions to the eikonal equation in stratified media, 24 pages. 1999.
- #467 K. Kubota and K. Yokoyama, Global existence of classical solutions to systems of nonlinear wave equations with different speed of propagation, 105 pages. 1999.
- #468 A. Higuchi, K. Matsue and T. Tsujishita, Deductive hyperdigraphs – a method of describing diversity of coherences, 35 pages. 1999.
- #469 M. Nakamura and T. Ozawa, Small solutions to nonlinear Schrödinger equations in the Sobolev spaces, 26 pages. 1999.
- #470 M. Nakamura and T. Ozawa, Small solutions to nonlinear wave equations in the Sobolev spaces, 27 pages. 1999.
- #471 K. Ito and Y. Kohsaka, Stability of a stationary solution for evolving boundaries of symmetric three-phases driven by surface diffusion, 36 pages. 1999.
- #472 Y. Giga and M.-H. Sato, A level set approach to semicontinuous viscosity solutions for Cauchy problems, 32 pages. 1999.
- #473 R. Yoneda, Characterization of Bloch space and Besov spaces by oscillations, 31 pages. 1999.

# ASYMPTOTICS FOR PREDICTION ERRORS OF STATIONARY PROCESSES WITH REFLECTION POSITIVITY

A. INOUE AND Y. KASAHARA

ABSTRACT. We consider the stationary processes that have completely monotone autocovariance functions  $R(\cdot)$ . We prove that regular variation of  $R(\cdot)$  implies an asymptotic formula for the prediction error.

## 1. INTRODUCTION

Let  $(X_t) = (X_t : t \in \mathbb{R})$  be a real, centered, mean-continuous, weakly stationary process defined on a probability space  $(\Omega, \mathcal{F}, P)$ , which we shall simply call a *stationary process*. We write  $R(\cdot)$  for the autocovariance function of  $(X_t)$ :

$$R(t) := E[X_t X_0] \quad (t \in \mathbb{R}).$$

This paper is concerned with the asymptotic behavior of the prediction error of  $(X_t)$ . We are especially interested in the case in which  $R(\cdot)$  is regularly varying with negative index, that is,

$$\forall \lambda > 0, \quad \lim_{t \rightarrow \infty} \frac{R(\lambda t)}{R(t)} = \lambda^{-p}$$

for some  $p > 0$  (see Bingham *et al.* [1]). This implies that  $R(t)$  is 'close' to the power function  $t^{-p}$  for large  $t$ .

Let  $H$  be the closed real linear hull of  $\{X_s : s \in \mathbb{R}\}$  in  $L^2(\Omega, \mathcal{F}, P)$ . Clearly  $H$  is a real Hilbert space with inner product  $(Y_1, Y_2) := E[Y_1 Y_2]$  and norm  $\|Y\| := (Y, Y)^{1/2}$ . For  $I \subset \mathbb{R}$ , put  $H_I$  for the closed subspace of  $H$  spanned by  $\{X_s : s \in I\}$ , and  $P_I$  for the orthogonal projection operator of  $H$  onto  $H_I$ . We write  $P_I^\perp$  for the projection operator onto the orthogonal complement  $H_I^\perp$  of  $H_I$ , i.e.,  $P_I^\perp Y = Y - P_I Y$  for  $Y \in H$ . Then, for  $T > 0$  and  $t > 0$ , the projection  $P_{[-t, 0]} X_T$  may be regarded as the best linear predictor of  $X_T$  on the observations  $\{X_s : -t \leq s \leq 0\}$ , hence  $P_{[-t, 0]}^\perp X_T = X_T - P_{[-t, 0]} X_T$  as its prediction error. Similarly,  $P_{(-\infty, 0]}^\perp X_T$  may be regarded as the prediction error for the prediction of  $X_T$  on the observations  $\{X_s : -\infty < s \leq 0\}$ .

We define

$$V_T(t) := \|P_{[-t, 0]}^\perp X_T\|^2 - \|P_{(-\infty, 0]}^\perp X_T\|^2 \quad (T > 0, t > 0).$$

After preliminary investigations, we reached the working hypothesis that, for a wide class of  $(X_t)$ , regular variation of  $R(\cdot)$  would imply the following asymptotic

---

*Date:* 25 November 1999.

*1991 Mathematics Subject Classification.* Primary 60G25; Secondary 42A08.

*Key words and phrases.* prediction error, stationary process, regular variation, reflection positivity.

formula:

$$(1.1) \quad V_T(t) \sim \int_t^\infty \left\{ \frac{R(s)}{\int_{-s}^s R(u) du} \right\}^2 ds \cdot \left\{ \int_0^T C(s) ds \right\}^2 \quad (t \rightarrow \infty),$$

where  $C(\cdot)$  is the canonical representation kernel of  $(X_t)$  (see (2.1) below).

In [7], (1.1) is proved for a special stationary process that has a spectral density proportional to  $|\xi|^{-q}/(1+\xi^2)$  ( $0 < q < 1$ ). The proof is by an explicit calculation of  $\|P_{[-t,0]}^\perp X_T\|$  using special functions, and so is not applicable to general stationary processes. Recently, one of the authors proved an analogue of (1.1) for discrete-time stationary processes in [6], under some assumptions on the AR( $\infty$ )-coefficients and MA( $\infty$ )-coefficients. The present paper arose from an attempt to prove (1.1) by following a continuous-time analogue of the method of [6]. Unfortunately, because of difficulties which are proper to continuous-time processes, it is still difficult to fully achieve this hope. However we have at least one nice class of stationary processes to which we can apply the method. It is the class of stationary processes with *reflection positivity*, that is, those with completely monotone autocovariance functions.

A stationary process with autocovariance function  $R(\cdot)$  is said to have reflection positivity, or simply (RP), if there exists a finite Borel measure  $\sigma$  on  $(0, \infty)$  such that

$$(1.2) \quad R(t) = \int_0^\infty e^{-|t|\lambda} \sigma(d\lambda) \quad (t \in \mathbb{R}).$$

For example, the stationary processes with  $R(t) = (1 + |t|)^{-p}$ , for  $0 < p < \infty$ , have (RP) (see §4). We refer to, e.g., Okabe [9], [11] and [3, 4, 5] for earlier work on stationary processes with (RP). We may say that the class of stationary processes with (RP) is especially suited for asymptotic analysis. In this paper, we use the class to check the validity of (1.1).

To state the results, we recall some notation from regular variation theory. We write  $\mathcal{R}_0$  for the class of slowly varying functions at infinity: the class of positive, measurable  $\ell$ , defined on some neighborhood  $[A, \infty)$  of infinity, such that

$$\forall \lambda > 0, \quad \lim_{x \rightarrow \infty} \ell(\lambda x)/\ell(x) = 1.$$

Let  $\ell \in \mathcal{R}_0$  and choose  $B$  so that  $\ell(\cdot)$  is locally bounded on  $[B, \infty)$  (cf. [1, Corollary 1.4.2]). When we say  $\int^\infty \ell(s) ds/s = \infty$ , it means that  $\int_B^\infty \ell(s) ds/s = \infty$ . If so, then we define another slowly varying function  $\tilde{\ell}$  by

$$\tilde{\ell}(x) := \int_B^x \frac{\ell(s)}{s} ds \quad (x \geq B)$$

(see [1, §1.5.6]). By the assumption  $\int^\infty \ell(s) ds/s = \infty$ , the asymptotic behavior of  $\tilde{\ell}(x)$  as  $x \rightarrow \infty$  does not depend on the choice of  $B$ .

Here is the main theorem.

**Theorem 1.1.** *Let  $T > 0$ ,  $p > 0$  and  $\ell \in \mathcal{R}_0$ . Let  $(X_t)$  be a stationary process with (RP). We assume*

$$(1.3) \quad R(t) \sim t^{-p} \ell(t) \quad (t \rightarrow \infty).$$

(i) If  $0 < p < 1$ , then

$$V_T(t) \sim t^{-1} \frac{(1-p)^2}{4} \cdot \left\{ \int_0^T C(s) ds \right\}^2 \quad (t \rightarrow \infty).$$

(ii) If  $p = 1$  and  $\int_0^\infty \ell(s) ds/s = \infty$ , then

$$V_T(t) \sim t^{-1} \left\{ \frac{\ell(t)}{2\bar{\ell}(t)} \right\}^2 \cdot \left\{ \int_0^T C(s) ds \right\}^2 \quad (t \rightarrow \infty).$$

(iii) If  $1 < p < \infty$  or if  $p = 1$  and  $\int_0^\infty \ell(s) ds/s < \infty$ , then

$$V_T(t) \sim \frac{t^{-(2p-1)} \{\ell(t)\}^2}{(2p-1) \{\int_{-\infty}^\infty R(s) ds\}^2} \cdot \left\{ \int_0^T C(s) ds \right\}^2 \quad (t \rightarrow \infty).$$

We remark that (i)–(iii) above correspond to the following cases, respectively: (i)'  $\int_0^\infty R(t) dt = \infty$  and  $0 < p < 1$ , (ii)'  $\int_0^\infty R(t) dt = \infty$  and  $p = 1$ , (iii)'  $\int_0^\infty R(t) dt < \infty$ . Stationary processes that satisfy (1.3) with  $0 < p < 1$  are called *long-memory processes*. We see that the result (i) for long-memory processes is especially simple. For, in this case, the index over  $t$  is  $-1$ , and so does not depend on  $p$ , and the slowly varying function  $\ell(\cdot)$  does not appear.

We can state the results (i)–(iii) above more simply as follows:

**Theorem 1.1'.** *Let  $p > 0$ ,  $T > 0$  and  $\ell \in \mathcal{R}_0$ , and let  $(X_t)$  be a stationary process with (RP). Then (1.3) implies (1.1).*

We briefly explain the method of the proof of Theorem 1.1. We define, for  $t > 0$  and  $n = 1, 2, \dots$ ,

$$P_t^n := P_{(-\infty, 0]} \quad \text{if } n \text{ is odd,} \quad := P_{[-t, \infty)} \quad \text{if } n \text{ is even.}$$

Then, for  $t > 0$ ,  $T > 0$  and  $n \geq 2$ , repeated use of the orthogonal decompositions

$$(1.4) \quad P_{[-t, 0]}^\perp = P_{(-\infty, 0]}^\perp + P_{[-t, 0]}^\perp P_{(-\infty, 0]} = P_{[-t, \infty)}^\perp + P_{[-t, 0]}^\perp P_{[-t, \infty)},$$

yields

$$(1.5) \quad \begin{aligned} \|P_{[-t, 0]}^\perp X_T\|^2 &= \|P_{(-\infty, 0]}^\perp X_T\|^2 + \sum_{k=1}^{n-1} \|(P_t^{k+1})^\perp P_t^k \cdots P_t^1 X_T\|^2 \\ &\quad + \|P_{[-t, 0]}^\perp P_t^n \cdots P_t^1 X_T\|^2 \end{aligned}$$

(compare [6, (4.2)]). If we set

$$U_T^n(t) := \|(P_t^{n+1})^\perp P_t^n \cdots P_t^1 X_T\|^2 \quad (n = 1, 2, \dots),$$

$$Z_T^n(t) := \|P_{[-t, 0]}^\perp P_t^n \cdots P_t^1 X_T\|^2 \quad (n = 1, 2, \dots),$$

then (1.5) may be written as

$$(1.6) \quad V_T(t) = \sum_{k=1}^{n-1} U_T^k(t) + Z_T^n(t)$$

Eq. (1.6) suggests that we could reduce our problem to the asymptotics of  $U_T^k(t)$  by showing that  $Z_T^n(t)$  is small enough (for  $n$  large if necessary). The point is that  $U_T^n(t)$  ( $n = 1, 2, \dots$ ) are easier to handle than  $V_T(t)$ . This is because  $U_T^n(t)$  are

defined only in terms of the projection operators  $P_{(-\infty, 0]}$  and  $P_{[-t, \infty)}$ . Analytically, these operators are much easier to handle than  $P_{[-t, 0]}$ .

To prove (ii) and (iii) of Theorem 1.1, it is enough to use (1.6) with only  $n = 2$ . In fact, in this case, we can show that  $V_T(t) \sim U_T^1(t)$  as  $t \rightarrow \infty$ . The proof of (i) is much harder. For, in this case,  $U_T^n(t)$  for  $n \geq 2$  are not negligible since they turn out to have the same order of asymptotics as that of  $U_T^1(t)$ . In order to prove (i), we must determine the asymptotics of  $U_T^n(t)$  for every  $n \geq 1$ .

To follow the line above, we need the representation of  $P_{(-\infty, 0]}X_t$  in terms of  $\{X_s : -\infty < X_s \leq 0\}$ . Formally, this would be of the form  $\int_0^\infty k_t(s)X_{-s}ds$  with some kernel  $k_t(\cdot)$ . Wiener tackled this problem in [13] and formally gave a formula identifying  $k_t(\cdot)$  in terms of the outer function of the stationary process (see [2, §4.4]). However the derivation is purely formal ([2, p. 92]). In §2, we prove such a representation theorem for stationary processes that satisfy (RP). It is at this point where the assumption (RP) is used essentially.

## 2. PROJECTION

In what follows, throughout this paper, we assume that the autocovariance function  $R(\cdot)$  of  $(X_t)$  is of the form (1.2) with a finite Borel measure  $\sigma$  on  $(0, \infty)$ . Then we know that  $(X_t)$  is purely nondeterministic:  $\cap_t H_{(-\infty, t]} = \{0\}$ . Let  $\Delta(\cdot)$  be the spectral density of  $(X_t)$ :  $R(t) = \int_{-\infty}^\infty e^{-it\xi} \Delta(\xi) d\xi$  for  $t \in \mathbb{R}$ . It follows that  $\{\log \Delta(\xi)\}/(1 + \xi^2) \in L^1(\mathbb{R})$ . We write  $h(\cdot)$  for the outer function of  $(X_t)$ :

$$h(z) := \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1 + \xi z}{\xi - z} \cdot \frac{\log \Delta(\xi)}{1 + \xi^2} d\xi \right\} \quad (\Im z > 0).$$

The *canonical representation kernel*  $C(\cdot)$  of  $(X_t)$  is defined by

$$(2.1) \quad C(t) := \frac{1}{2\pi} \int_{-\infty}^\infty e^{-it\xi} h(\xi) d\xi$$

in  $L^2(\mathbb{R})$ -sense, where  $h(\cdot) := \text{l.i.m.}_{\eta \downarrow 0} h(\cdot + i\eta) \in L^2(\mathbb{R})$ . The kernel  $C(\cdot)$  vanishes on  $(-\infty, 0)$  and satisfies the following equalities:

$$(2.2) \quad h(z) = \int_0^\infty e^{izt} C(t) dt \quad (\Im z > 0),$$

$$(2.3) \quad R(t) = \int_0^\infty C(t+s)C(s) ds \quad (t \geq 0).$$

See, e.g., Rozanov [12], Dym-McKean [2] and [9, §2] for details.

By [10, Theorem A] and [4, Theorems 2.4 and 2.5], there exists a unique Borel measure  $\nu$  on  $(0, \infty)$  such that

$$(2.4) \quad h(z) = \int_0^\infty \frac{1}{\lambda - iz} \nu(d\lambda) \quad (\Im z > 0), \quad \int_0^\infty \int_0^\infty \frac{1}{\lambda + \lambda'} \nu(d\lambda) \nu(d\lambda') < \infty.$$

It follows that the canonical representation kernel  $C(\cdot)$  of  $(X_t)$  admits the following representation:

$$C(t) = I_{(0, \infty)}(t) \int_0^\infty e^{-t\lambda} \nu(d\lambda) \quad (t \in \mathbb{R}).$$

By (2.4) as well as Theorems S1.5.1, S1.5.2 and Lemma S1.5.2 of Kac and Krein [8],  $1/h(z)$  admits the representation

$$(2.5) \quad \frac{1}{h(z)} = b -iaz - iz \int_0^\infty \frac{1}{(\lambda - iz)\lambda} \tau(d\lambda) \quad (\Im z > 0),$$

where  $a \geq 0$ ,  $b \geq 0$  and  $\tau$  is a Borel measure on  $(0, \infty)$  such that

$$\int_0^\infty \frac{1}{\lambda(1+\lambda)} \tau(d\lambda) < \infty.$$

We note that the triple  $(a, b, \tau)$  is determined uniquely by  $h(\cdot)$  hence by  $(X_t)$ . We define

$$G(t) := \int_0^\infty e^{-t\lambda} \lambda^{-1} \tau(d\lambda) \quad (t > 0).$$

Given a measurable function  $f : (0, \infty) \rightarrow \mathbb{R}$ , we define its Laplace transform

$$\hat{f}(x) := \int_0^\infty e^{-xt} f(t) dt$$

for  $x > 0$  such that the integral converges absolutely. It follows from (2.2) and (2.5) that

$$(2.6) \quad \hat{C}(x) \{b + ax + x\hat{G}(x)\} = 1 \quad (x > 0).$$

This implies

$$(2.7) \quad aC(t) + b \int_0^t C(s) ds + \int_0^t G(s)C(t-s) ds = 1 \quad (t > 0)$$

since the Laplace transforms of both sides coincide by (2.6).

We set

$$A(t) := \int_0^\infty e^{-t\lambda} \tau(d\lambda) \quad (t > 0).$$

Then  $A(t) = -\dot{G}(t)$  for every  $t > 0$ . For  $t > 0$ , we define

$$K_t(s) := \int_0^t A(s+u)C(t-u) du \quad (s > 0).$$

As (2.7) implies  $\int_0^\infty K_t(s) ds = \int_0^t G(s)C(t-s) ds \leq 1$ ,  $K_t(\cdot)$  is integrable on  $(0, \infty)$  for every  $t > 0$ .

Here is the representation of  $P_{(-\infty, 0]} X_t$  in terms of  $\{X_s : -\infty < s \leq 0\}$ .

**Theorem 2.1.** For  $t > 0$ , we have

$$(2.8) \quad P_{(-\infty, 0]} X_t = aC(t)X_0 + \int_0^\infty K_t(s)X_{-s} ds,$$

where the integral converges absolutely in  $H$ .

*Proof.* Let

$$(2.9) \quad X_u = \int_{-\infty}^\infty C(u-s)\xi(ds) = \int_{-\infty}^u C(u-s)\xi(ds) \quad (u \in \mathbb{R})$$



be the canonical representation of  $(X_u)$  (see, e.g., [12, Ch. III, §3]). Define the linear map  $T : L^2(\mathbb{R}) \rightarrow H$  by

$$(2.10) \quad Tf := \int_{-\infty}^{\infty} f(-s)\xi(ds) \quad (f \in L^2(\mathbb{R})).$$

Then  $T$  is a Hilbert space isomorphism of  $L^2(\mathbb{R})$  onto  $H$ ; for example, the representation (2.9) implies that  $T$  is onto. From (2.9) we see that  $(T^{-1}X_u)(\cdot) = C(u + \cdot)$ . Moreover, since  $P_{(-\infty, 0]}Tf = \int_{-\infty}^0 f(-s)\xi(ds)$ , we have

$$(2.11) \quad (T^{-1}P_{(-\infty, 0]}Tf)(u) = I_{(0, \infty)}(u)f(u) \quad (f \in L^2(\mathbb{R})),$$

in particular,  $(T^{-1}P_{(-\infty, 0]}X_t)(\cdot) = I_{(0, \infty)}(u)C(t + \cdot)$ . Thus, (2.8) follows if we prove

$$I_{(0, \infty)}(u)C(t + u) = aC(t)C(u) + \int_0^{\infty} K_t(s)C(u - s)ds \quad (u \in \mathbb{R}).$$

However, since both sides of the equality above vanish for  $u \leq 0$ , it is enough to prove

$$(2.12) \quad C(t + u) = aC(t)C(u) + \int_0^u K_t(s)C(u - s)ds \quad (u > 0).$$

Let  $x > 0$ . Then by (2.6) and the equalities

$$\begin{aligned} \int_0^{\infty} e^{-xu}C(t + u)du &= x \int_0^{\infty} e^{-xs} \left\{ \int_t^{t+s} C(u)du \right\} ds, \\ \hat{G}(x) \int_0^{\infty} e^{-xu}C(t + u)du &= \int_0^{\infty} e^{-xs} \left\{ \int_0^s G(u)C(t + s - u)du \right\} ds, \end{aligned}$$

the Laplace transform  $\int_0^{\infty} e^{-xu}C(t + u)du$  is equal to

$$\hat{C}(x) \left\{ b + ax + x\hat{G}(x) \right\} \int_0^{\infty} e^{-xu}C(t + u)du = x\hat{C}(x)\hat{F}_t(x)$$

where

$$F_t(s) := bC(t + s) + a \int_t^{t+s} C(u)du + \int_0^s G(u)C(t + s - u)du.$$

On the other hand, it follows from integration by parts that

$$\begin{aligned} \hat{K}_t(x) &= \int_0^t G(u)C(t - u)du - x \int_0^{\infty} e^{-xs} \left\{ \int_0^t G(s + u)C(t - u)du \right\} ds \\ &= x \int_0^{\infty} e^{-xs} \left\{ \int_0^t G(u)C(t - u)du - \int_s^{t+s} G(u)C(t + s - u)du \right\} ds, \end{aligned}$$

so that the Laplace transform of the right-hand side of (2.12) is equal to  $x\hat{C}(x)\hat{H}_t(x)$ , where

$$H_t(s) := bC(t) + \int_0^t G(u)C(t - u)du - \int_s^{t+s} G(u)C(t + s - u)du.$$

However, we have from (2.7) that  $F_t(s) = H_t(s)$  for every  $s > 0$ , and so the uniqueness of Laplace transforms implies (2.12).  $\square$

**Corollary 2.2.** *Let  $t > 0$ ,  $T > 0$  and  $n \in \mathbb{N}$ .*

(i) If  $n$  is odd, then we have

$$P_t^n \cdots P_t^1 X_T = \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) X_{-t-s_n} ds_n \pmod{H_{[-t,0]}}.$$

(ii) If  $n$  is even, then we have

$$P_t^n \cdots P_t^1 X_T = \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) X_{s_n} ds_n \pmod{H_{[-t,0]}}.$$

*Proof.* It follows from Theorem 2.1 that, for  $u > 0$ ,

$$(2.13) \quad P_{(-\infty,0]} X_u = \int_0^\infty K_u(t+s) X_{-t-s} ds \pmod{H_{[-t,0]}}.$$

Let  $S_v$  ( $v \in \mathbb{R}$ ) and  $\theta$  be the Hilbert space automorphisms of  $H$  such that  $S_v(X_s) = X_{v+s}$ ,  $\theta(X_s) = X_{-s}$  for every  $s \in \mathbb{R}$ . Then we have

$$(2.14) \quad S_v^{-1} = S_{-v}, \quad \theta^{-1} = \theta, \quad P_{[-t,\infty)} = (\theta S_t)^{-1} P_{(-\infty,0]} (\theta S_t).$$

Therefore, we have from (2.13) that, for  $u > 0$ ,

$$(2.15) \quad P_{[-t,\infty)} X_{-t-u} = S_{-t} \theta P_{(-\infty,0]} X_u = \int_0^\infty K_u(t+s) X_s ds \pmod{H_{[-t,0]}}.$$

The absolute convergence of the integrals in (2.13) and (2.15) allow us to use them repeatedly, and so we obtain the corollary.  $\square$

For  $T > 0$ ,  $t > 0$ ,  $u > 0$  and  $k \in \mathbb{N}$ , we define  $D_T^k(t, u)$  by

$$\begin{aligned} D_T^1(t, u) &:= \int_0^\infty C(v_1) K_T(t+v_1+u) dv_1, \\ D_T^2(t, u) &:= \int_0^\infty dv_2 C(v_2) \int_0^\infty dv_1 C(v_1) \\ &\quad \int_0^\infty A(t+s_1+v_2+u) K_T(t+s_1+v_1) ds_1, \\ D_T^k(t, u) &:= \int_0^\infty dv_k C(v_k) \cdots \int_0^\infty dv_1 C(v_1) \int_0^\infty ds_{k-1} A(t+s_{k-1}+v_k+u) \\ &\quad \int_0^\infty ds_{k-2} A(t+s_{k-1}+s_{k-2}+v_{k-1}) \cdots \int_0^\infty ds_2 A(t+s_3+s_2+v_3) \\ &\quad \int_0^\infty A(t+s_2+s_1+v_2) K_T(t+s_1+v_1) ds_1 \quad (k \geq 3). \end{aligned}$$

We note that the integrals above are well-defined since all the integrands are non-negative.

Recall  $U_T^n(t)$  from §1. Here is the representation of  $U_T^n(t)$  in terms of  $A(\cdot)$  and  $C(\cdot)$ .

**Proposition 2.3.** For  $T > 0$ ,  $t > 0$  and  $n \in \mathbb{N}$ , we have  $U_T^n(t) = \int_0^\infty D_T^n(t, u)^2 du$ .

*Proof.* By Corollary 2.2 and (2.14),  $U_T^n(t)$  is equal to

$$\left\| P_{(-\infty, 0]}^\perp \int_0^\infty ds_1 K_T(t + s_1) \int_0^\infty ds_2 K_{s_1}(t + s_2) \cdots \int_0^\infty K_{s_{n-1}}(t + s_n) X_{s_n} ds_n \right\|^2$$

C (use the maps  $S_t$  and  $\theta$  in the proof of Corollary 2.2 if  $n$  is even). Recall the map  $T : L^2(\mathbb{R}) \rightarrow H$  from (2.10). Since (2.11) implies

$$(T^{-1} P_{(-\infty, 0]}^\perp T f)(u) = I_{(-\infty, 0)}(u) f(u) \quad (f \in L^2(\mathbb{R})),$$

we see that  $U_T^n(t) = \int_0^\infty d_T^n(t, u)^2 du$ , where we write  $d_T^n(t, u)$  for the integral

$$\int_0^\infty ds_1 K_T(t + s_1) \int_0^\infty ds_2 K_{s_1}(t + s_2) \cdots \int_u^\infty K_{s_{n-1}}(t + s_n) C(s_n - u) ds_n.$$

It is enough to prove  $d_T^n(t, u) = D_T^n(t, u)$ . By the Fubini-Tonelli theorem,  $d_T^n(t, u)$  is equal to

$$\int_0^\infty dv_n C(v_n) \int_0^\infty ds_{n-1} K_{s_{n-1}}(t + s_n + u) \int_0^\infty ds_{n-2} K_{s_{n-2}}(t + s_{n-1}) \cdots \int_0^\infty ds_2 K_{s_2}(t + s_3) \int_0^\infty K_{s_1}(t + s_2) K_T(t + s_1) ds_1.$$

First,  $\int_0^\infty K_{s_1}(t + s_2) K_T(t + s_1) ds_1$  is equal to

$$\begin{aligned} & \int_0^\infty \left\{ \int_0^{s_1} C(v_1) A(t + s_2 + s_1 - v_1) dv_1 \right\} K_T(t + s_1) ds_1 \\ &= \int_0^\infty dv_1 C(v_1) \int_0^\infty A(t + s_2 + s_1) K_T(t + s_1 + v_1) ds_1. \end{aligned}$$

Second,  $\int_0^\infty K_{s_2}(t + s_3) A(t + s_2 + s_1) ds_2$  is equal to

$$\begin{aligned} & \int_0^\infty \left\{ \int_0^{s_2} C(v_2) A(t + s_3 + s_2 - v_2) dv_2 \right\} A(t + s_2 + s_1) ds_2 \\ &= \int_0^\infty dv_2 C(v_2) \int_0^\infty A(t + s_3 + s_2) A(t + s_2 + s_1 + v_2) ds_2. \end{aligned}$$

Repeating this argument, we finally obtain  $d_T^n(t, u) = D_T^n(t, u)$ , as desired.  $\square$

### 3. PROOF OF THEOREM 1.1

For simplicity, we set

$$J(t) := \int_0^t C(s) ds \quad (t > 0).$$

*Proof of Theorem 1.1(iii).* It follows from (2.3) that

$$(3.1) \quad \left\{ \int_0^\infty C(s) ds \right\}^2 = 2 \int_0^\infty R(t) dt.$$

Since the assumptions on  $p$  and  $\ell(\cdot)$  imply  $\int_0^\infty R(t) dt < \infty$ , we have  $\int_0^\infty C(s) ds < \infty$ . Therefore, applying the monotone convergence theorem to (2.6), we obtain

$$b = \left\{ \int_0^\infty C(s) ds \right\}^{-1}$$

(cf. [9, Lemma 2.7]). Hence we have from [3, Lemma 3.8] that

$$C(t) \sim bt^{-p}\ell(t) \quad (t \rightarrow \infty).$$

If  $1 < p < \infty$ , then it follows from [3, Theorem 4.1] that

$$G(t) \sim t^{1-p}\ell(t) \cdot \frac{b^3}{p-1} \quad (t \rightarrow \infty).$$

Since  $G(t) = \int_t^\infty A(s)ds$  and  $A(\cdot)$  is decreasing, this implies

$$(3.2) \quad A(t) \sim t^{-p}\ell(t)b^3 \quad (t \rightarrow \infty)$$

(cf. [1, §1.7.3]). On the other hand, if  $p = 1$  and  $\int^\infty \ell(s)ds/s < \infty$ , then as in the proof of [5, Theorem 1.2] we see that  $G(\cdot) \in \Pi_\ell$  with  $\ell$ -index  $-b^3$ . However, by de Haan's monotone convergence theorem (cf. [1, Theorem 3.6.8]), this also implies (3.2).

By (1.6) with  $n = 2$ , we have the lower bound for  $V_T(t)$ :  $V_T(t) \geq U_T^1(t)$ . On the other hand, by (1.4) and Theorem 2.1, we have the upper bound for  $V_T(t)$ :

$$(3.3) \quad \begin{aligned} V_T(t) &= \left\| P_{[-t,0]}^\perp P_{(-\infty,0]} X_T \right\|^2 = \left\| P_{[-t,0]}^\perp \int_t^\infty K_T(s) X_{-s} ds \right\|^2 \\ &\leq \left\| \int_t^\infty K_T(s) X_{-s} ds \right\|^2 = 2 \int_t^\infty ds K_T(s) \int_0^\infty K_T(s+u) R(u) du. \end{aligned}$$

First, we consider the lower bound. Since

$$(3.4) \quad A(t+T)J(T) \leq K_T(t) \leq A(t)J(T),$$

it follows from (3.2) that

$$(3.5) \quad K_T(t) \sim t^{-p}\ell(t)b^3J(T) \quad (t \rightarrow \infty).$$

As in [3, Lemma 3.8], this implies

$$\int_0^\infty C(v)K_T(t+v)dv \sim t^{-p}\ell(t)b^2J(T) \quad (t \rightarrow \infty),$$

which by (3.1) yields

$$U_T^1(t) = \int_t^\infty \left\{ \int_0^\infty C(v)K_T(u+v)dv \right\}^2 du \sim \frac{t^{-(2p-1)} \{\ell(t)\}^2 J(T)^2}{(2p-1) \{\int_{-\infty}^\infty R(s)ds\}^2} \quad (t \rightarrow \infty).$$

Next we consider the upper bound. From (3.4), it follows, as above, that

$$\int_0^\infty K_T(t+u)R(u)du \sim K_T(t) \int_0^\infty R(u)du \quad (t \rightarrow \infty),$$

hence

$$2 \int_t^\infty ds K_T(s) \int_0^\infty K_T(s+u)R(u)du \sim \frac{t^{-(2p-1)} \{\ell(t)\}^2 J(T)^2}{(2p-1) \{\int_{-\infty}^\infty R(s)ds\}^2} \quad (t \rightarrow \infty).$$

Since the asymptotics for the upper and lower bounds coincide, (iii) follows.  $\square$

*Proof of Theorem 1.1(ii).* We have from [5, Theorem 5.2] that

$$C(t) \sim t^{-1} \frac{\ell(t)}{\{2\tilde{\ell}(t)\}^{1/2}} \quad (t \rightarrow \infty),$$

so that

$$(3.6) \quad J(t) \sim \{2\tilde{\ell}(t)\}^{1/2} \quad (t \rightarrow \infty)$$

(cf. [6, (5.29)]).

Since  $\int_0^\infty C(t)dt = \infty$ , we see that  $b = 0$ , by letting  $x \downarrow 0$  in (2.6). Hence, comparing (2.6) with [5, (6.1)], we find, as in [5, Theorem 1.1(3)], that  $G(\cdot) \in \Pi_{\ell_1}$  with  $\ell_1$ -index  $-1$ , where  $\ell_1(t) := \ell(t)\{2\tilde{\ell}(t)\}^{-3/2}$ . Therefore, by de Haan's monotone density theorem, we have  $A(t) \sim t^{-1}\ell_1(t)$  as  $t \rightarrow \infty$ , and so

$$(3.7) \quad K_T(t) \sim t^{-1}\ell_1(t)J(T) \quad (t \rightarrow \infty).$$

Since  $-\dot{K}_T(t) = -\int_0^T \dot{A}(t+v)C(T-v)dv$  is decreasing in  $t$ , this implies

$$(3.8) \quad -\dot{K}_T(t) \sim t^{-2}\ell_1(t)J(T) \quad (t \rightarrow \infty).$$

By the Fubini-Tonelli theorem, we have

$$\int_0^\infty K_T(t+v)C(v)dv = -\int_0^\infty \dot{K}_T(t+s)J(s)ds.$$

Using (3.6) and (3.8), we apply [5, Proposition 4.3] to the above to obtain

$$(3.9) \quad \int_0^\infty K_T(t+v)C(v)dv \sim t^{-1} \frac{\ell(t)}{2\tilde{\ell}(t)} J(T) \quad (t \rightarrow \infty),$$

and so

$$U_T^{\frac{1}{2}}(t) = \int_t^\infty \left\{ \int_0^\infty C(v)K_T(u+v)dv \right\}^2 du \sim t^{-1} \left\{ \frac{\ell(t)}{2\tilde{\ell}(t)} \right\}^2 J(T)^2 \quad (t \rightarrow \infty).$$

On the other hand, in the same way as above, it follows that

$$\int_0^\infty K_T(t+u)R(u)du \sim t^{-1} \frac{\ell(t)}{2^{3/2}\tilde{\ell}(t)^{1/2}} J(T) \quad (t \rightarrow \infty),$$

which with (3.7) implies

$$2 \int_t^\infty ds K_T(s) \int_0^\infty K_T(s+u)R(u)du \sim t^{-1} \left\{ \frac{\ell(t)}{2\tilde{\ell}(t)} \right\}^2 J(T)^2 \quad (t \rightarrow \infty).$$

Thus the asymptotic behavior of the upper bound in (3.3) is equal to that of the lower bound  $U_T^{\frac{1}{2}}(t)$ , and so (ii) follows.  $\square$

*Proof of Theorem 1.1(i).* Step 1. For simplicity, we use the following number  $d \in (0, 1/2)$  rather than  $p$ :

$$d := \frac{1-p}{2}.$$

Then it follows from [5, Theorem 4.1] that

$$(3.10) \quad C(t) \sim t^{-(1-d)} \left\{ \frac{\ell(t)}{B(d, 1-2d)} \right\}^{1/2} \quad (t \rightarrow \infty),$$

where  $B(\cdot, \cdot)$  is the beta function ( we note that the index  $-1/2$  in [5, (4.3)] is mistaken; it should be  $1/2$ ). As in the previous proof, we have  $b = 0$ . Therefore, comparing [5, (3.5)] with (2.6), we obtain, as in [5, Theorem 1.1(i)],

$$G(t) \sim t^{-d} \left\{ \frac{\ell(t)}{B(d, 1-2d)} \right\}^{-1/2} \frac{\sin(\pi d)}{\pi} \quad (t \rightarrow \infty).$$

By the monotone convergence theorem, this implies

$$(3.11) \quad A(t) \sim t^{-(d+1)} \left\{ \frac{\ell(t)}{B(d, 1-2d)} \right\}^{-1/2} \frac{d \sin(\pi d)}{\pi} \quad (t \rightarrow \infty).$$

Step 2. We prove, for  $T > 0$  and  $k \in \mathbb{N}$ ,

$$(3.12) \quad U_T^k(t) \sim t^{-1} a_k \sin^{2k}(\pi d) J(T)^2 \quad (t \rightarrow \infty),$$

where

$$a_k := \frac{(2k-2)!!}{\pi^2 (2k-1)!! k} \quad (k \in \mathbb{N}).$$

Choose  $\delta$  from the interval  $(0, \min(d, 1/(8k-2)))$ . By (3.10) and (3.11), there exists  $M > 0$  such that

$$(3.13) \quad 0 \leq \frac{A(t)}{A(s)} \leq 2(t/s)^{-(1+d-\delta)} \quad (t \geq s \geq M),$$

$$(3.14) \quad 0 \leq \frac{C(t)}{C(s)} \leq 2 \max \left\{ (t/s)^{-(1-d+\delta)}, (t/s)^{-(1-d-\delta)} \right\} \quad (t \geq M, s \geq M)$$

(see [1, Theorem 1.5.6]). We set, for  $t > 0$ ,

$$(3.15) \quad C_0(t) := I_{(0,M)}(t)C(t), \quad C_1(t) := I_{[M,\infty)}(t)C(t).$$

For  $k \geq 3$  and  $(i_1, \dots, i_k) \in \{0, 1\}^k$ , we write  $M_T(t, u; i_1, \dots, i_k)$  for the integral

$$\begin{aligned} & \int_0^\infty dv_k \frac{C_{i_k}(tv_k)}{C(t)} \dots \int_0^\infty dv_1 \frac{C_{i_1}(tv_1)}{C(t)} \int_0^\infty ds_{k-1} \frac{A(t(1+s_{k-1}+v_k+u))}{A(t)} \\ & \int_0^\infty ds_{k-2} \frac{A(t(1+s_{k-1}+s_{k-2}+v_{k-1}))}{A(t)} \dots \int_0^\infty ds_2 \frac{A(t(1+s_3+s_2+v_3))}{A(t)} \\ & \int_0^\infty \frac{A(t(1+s_2+s_1+v_2))}{A(t)} \cdot \frac{K_T(t(1+s_1+v_1))}{A(t)} ds_1. \end{aligned}$$

We also write

$$\begin{aligned} M_T(t, u; i_1) &:= \int_0^\infty \frac{C_{i_1}(tv_1)}{C(t)} \cdot \frac{K_T(t(1+v_1+u))}{A(t)} dv_1, \\ M_T(t, u; i_1, i_2) &:= \int_0^\infty dv_2 \frac{C_{i_2}(tv_2)}{C(t)} \int_0^\infty dv_1 \frac{C_{i_1}(tv_1)}{C(t)} \\ & \int_0^\infty \frac{A(t(1+s_1+v_2+u))}{A(t)} \cdot \frac{K_T(t(1+s_1+v_1))}{A(t)} ds_1. \end{aligned}$$

Then, from Proposition 2.3, we have

$$U_T^k(t) = t^{4k-1} \{A(t)C(t)\}^{2k} \int_0^\infty \left\{ \sum_i M_T(t, u; i_1, \dots, i_k) \right\}^2 du,$$

where  $\sum_i = \sum_{(i_1, \dots, i_k) \in \{0,1\}^k}$ . As in the proof of [6, Proposition 6.1], it follows that

$$\lim_{t \rightarrow \infty} \int_0^\infty M_T(t, u; i_1, \dots, i_k)^2 du = \begin{cases} \frac{\pi^{2k} a_k J(T)^2}{d^{2k}} & (i_1 = 1, \dots, i_k = 1), \\ 0 & ((i_1, \dots, i_k) \neq (1, \dots, 1)) \end{cases}$$

(use (3.4) and [6, (6.15)]), hence

$$\lim_{t \rightarrow \infty} \int_0^\infty \left\{ \sum_i M_T(t, u; i_1, \dots, i_k) \right\}^2 du = \frac{\pi^{2k} a_k J(T)^2}{d^{2k}}.$$

Since (3.10) and (3.11) imply

$$(3.16) \quad t^{4k-1} \{A(t)C(t)\}^{2k} \sim t^{-1} \left\{ \frac{d \sin(d\pi)}{\pi} \right\}^{2k} \quad (t \rightarrow \infty),$$

(3.12) follows.

Step 3. Recall  $Z_T^n(t)$  from §1. By Corollary 2.2, we see that  $Z_T^n(t)$  is at most

$$\begin{aligned} & \left\| \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) X_{s_n} ds_n \right\|^2 \\ &= \int_{-\infty}^\infty \left\{ \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \right. \\ & \quad \left. \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) C(s_n+u) ds_n \right\}^2 du \\ &= U_T^n(t) + W_T^n(t), \end{aligned}$$

where

$$W_T^n(t) := \int_0^\infty \left\{ \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \right. \\ \left. \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) C(s_n+u) ds_n \right\}^2 du.$$

Therefore, we have from (1.6) that

$$(3.17) \quad \sum_{k=1}^{n-1} U_T^k(t) \leq V_T(t) \leq \sum_{k=1}^n U_T^k(t) + W_T^n(t) \quad (n = 2, 3, \dots).$$

As in the proof of Proposition 2.3, we see that  $W_T^n(t)$  is equal to

$$\begin{aligned} & \int_0^\infty \left\{ \int_0^\infty dv_n C(v_n+u) \cdots \int_0^\infty dv_1 C(v_1) \int_0^\infty ds_{n-1} A(t+s_{n-1}+v_n) \right. \\ & \quad \left. \int_0^\infty ds_{n-2} A(t+s_{n-1}+s_{n-2}+v_{n-1}) \cdots \int_0^\infty ds_2 A(t+s_3+s_2+v_3) \right. \\ & \quad \left. \int_0^\infty A(t+s_2+s_1+v_2) K_T(t+s_1+v_1) ds_1 \right\}^2 du. \end{aligned}$$

We prove, for  $T > 0$  and  $n \geq 3$ ,

$$(3.18) \quad W_T^n(t) \sim t^{-1} b_n(d) \sin^{2n}(\pi d) J(T)^2 \quad (t \rightarrow \infty),$$

where we write  $b_n(d)$  for the integral

$$\left(\frac{d}{\pi}\right)^{2n} \int_0^\infty du \left\{ \int_0^\infty \frac{dv_n}{(v_n+u)^{1-d}} \int_0^\infty \frac{dv_{n-1}}{(v_{n-1})^{1-d}} \cdots \int_0^\infty \frac{dv_1}{(v_1)^{1-d}} \right. \\ \left. \int_0^\infty \frac{ds_{n-1}}{(v_n+s_{n-1}+1)^{1+d}} \int_0^\infty \frac{ds_{n-2}}{(v_{n-1}+s_{n-1}+s_{n-2}+1)^{1+d}} \right. \\ \left. \cdots \int_0^\infty \frac{ds_2}{(v_3+s_3+s_2+1)^{1+d}} \int_0^\infty \frac{ds_1}{(v_2+s_2+s_1+1)^{1+d}(v_1+s_1+1)^{1+d}} \right\}^2.$$

Choose  $\delta$  from the interval  $(0, \min\{(1-2d)/2, d/(4n-3)\})$ , and take  $M > 0$  so large that both (3.13) and (3.14) hold. We define  $C_0(\cdot)$  and  $C_1(\cdot)$  by (3.15), and  $\varphi_0(\cdot)$  and  $\varphi_1(\cdot)$  by

$$\varphi_0(t) := I_{(0,M)}(t), \quad \varphi_1(t) := I_{[M,\infty)}(t) \quad (t > 0).$$

For  $(i_1, \dots, i_n) \in \{0, 1\}^n$ , we define  $N_T(t, u; i_1, \dots, i_n)$  by

$$N_T(t, u; i_1, \dots, i_n) := \int_0^\infty dv_n \varphi_{i_n}(tv_n) \frac{C(tv_n+u)}{C(t)} \int_0^\infty dv_{n-1} \frac{C_{i_{n-1}}(tv_{n-1})}{C(t)} \\ \cdots \int_0^\infty dv_1 \frac{C_{i_1}(tv_1)}{C(t)} \int_0^\infty ds_{k-1} \frac{A(t(1+s_{k-1}+v_k))}{A(t)} \\ \int_0^\infty ds_{k-2} \frac{A(t(1+s_{k-1}+s_{k-2}+v_{k-1}))}{A(t)} \cdots \int_0^\infty ds_2 \frac{A(t(1+s_3+s_2+v_3))}{A(t)} \\ \int_0^\infty \frac{A(t(1+s_2+s_1+v_2))}{A(t)} \cdot \frac{K_T(t(1+s_1+v_1))}{A(t)} ds_1.$$

Then, it follows that

$$W_T^n(t) = t^{4n-1} \{A(t)C(t)\}^{2n} \int_0^\infty \left\{ \sum_i N_T(t, u; i_1, \dots, i_n) \right\}^2 du,$$

where  $\sum_i = \sum_{(i_1, \dots, i_n) \in \{0, 1\}^n}$  as above. As in the proof of [6, Proposition 6.2], we have

$$\lim_{t \rightarrow \infty} \int_0^\infty N_T(t, u; i_1, \dots, i_n)^2 du = \begin{cases} \frac{\pi^{2n} b_n(d) J(T)^2}{d^{2n}} & (i_1 = 1, \dots, i_n = 1), \\ 0 & ((i_1, \dots, i_n) \neq (1, \dots, 1)) \end{cases}$$

(use (3.4)), hence

$$\lim_{t \rightarrow \infty} \int_0^\infty \left\{ \sum_i N_T(t, u; i_1, \dots, i_n) \right\}^2 du = \frac{\pi^{2n} b_n(d) J(T)^2}{d^{2n}}.$$

This and (3.16) yield (3.18).



Step 4. It follows from (3.12), (3.17) and (3.18) that

$$\begin{aligned} \sum_{k=1}^{n-1} a_k \sin^{2k}(\pi d) &\leq J(T)^{-2} \liminf_{t \rightarrow \infty} V_T(t)t \leq J(T)^{-2} \limsup_{n \rightarrow \infty} V_T(t)t \\ &\leq \sum_{k=1}^n a_k \sin^{2k}(\pi d) + b_n(d) \sin^{2n}(\pi d). \end{aligned}$$

However, we have  $b_n(d) \leq \pi^{-2} \tan^2(\pi d)$  for  $n \geq 3$  (see the proof of [6, Theorem 6.4]), hence, letting  $n \rightarrow \infty$ , we obtain

$$\lim_{t \rightarrow \infty} V_T(t)t = J(T)^2 \sum_{k=1}^{\infty} a_k \sin^{2k}(\pi d).$$

Since  $\sum_1^{\infty} a_k x^{2k} = \pi^{-2} \arcsin^2 x$  for  $|x| < 1$ , this yields the desired result.  $\square$

#### 4. EXAMPLES

Let  $(X_t)$  be a stationary process with autocovariance function  $R(\cdot)$  and canonical representation kernel  $C(\cdot)$ . Let  $T > 0$ .

*Example 1.* If  $R(t) = (1 + |t|)^{-p}$  for  $0 < p < \infty$ , then  $(X_t)$  has (RP). For,

$$\frac{1}{(1 + |t|)^p} = \int_0^{\infty} e^{-|t|\lambda} \frac{e^{-\lambda} \lambda^{p-1}}{\Gamma(p)} d\lambda \quad (t \in \mathbb{R}).$$

In this case,  $C(\cdot)$  seems to have no simple representation. By Theorem 1.1, we have the following results (we omit the case (i)  $0 < p < 1$ ):

(ii) if  $p = 1$ , then

$$V_T(t) \sim \frac{1}{4t(\log t)^2} \cdot \left\{ \int_0^T C(s) ds \right\}^2 \quad (t \rightarrow \infty);$$

(iii) if  $1 < p < \infty$ , then

$$V_T(t) \sim t^{-(2p-1)} \cdot \frac{4}{(2p-1)(p-1)^2} \left\{ \int_0^T C(s) ds \right\}^2 \quad (t \rightarrow \infty).$$

*Example 2.* Let  $1/2 < q < \infty$ . If  $C(t) = (1 + t)^{-q}$  ( $t > 0$ ), then  $(X_t)$  has (RP) (see [4, Theorem 2.6]).

(i) If  $1/2 < q < 1$ , then

$$R(t) \sim t^{-(2q-1)} \int_0^{\infty} \frac{1}{s^q(1+s)^q} ds \quad (t \rightarrow \infty)$$

(cf. [5, Theorem 4.1]). Hence, by Theorem 1.1(i), we have

$$V_T(t) \sim t^{-1} \cdot \{(1 + T)^{1-q} - 1\}^2 \quad (t \rightarrow \infty).$$

(ii) If  $q = 1$ , then  $R(t) \sim t^{-1} \log t$  as  $t \rightarrow \infty$  by [5, Theorem 5.2]. Hence, by Theorem 1.1(ii), we have

$$V_T(t) \sim \frac{\{\log(1 + T)\}^2}{t(\log t)^2} \quad (t \rightarrow \infty).$$

(iii) If  $q > 1$ , then  $R(t) \sim t^{-q} \int_0^\infty C(s) ds$  as  $t \rightarrow \infty$  (cf. [3, Lemma 3.8]). Therefore, by Theorem 1.1(iii) and (3.1), we have

$$V_T(t) \sim t^{-(2q-1)} \cdot \frac{\{1 - (1+T)^{1-q}\}^2}{2q-1} \quad (t \rightarrow \infty).$$

#### REFERENCES

- [1] N. H. Bingham, C. M. Goldie and J. L. Teugels, "Regular Variation," 2nd ed., Cambridge University Press, Cambridge, UK, 1989.
- [2] H. Dym and H. P. McKean, "Gaussian Processes, Function Theory, and the Inverse Spectral Problem," Academic Press, New York, 1976.
- [3] A. Inoue, The Alder-Wainwright effect for stationary process with reflection positivity, *J. Math. Soc. Japan* **43** (1991), 515-526.
- [4] A. Inoue, On the equations of stationary process with divergent diffusion coefficients, *J. Fac. Sci. Univ. Tokyo Sect. IA* **40** (1993), 307-336.
- [5] A. Inoue, Regularly varying correlation functions and KMO-Langevin equations, *Hokkaido Math. J.* **26** (1997), 1-26.
- [6] A. Inoue, Asymptotics for the partial autocorrelation function of a stationary process, *J. Analyse Math.*, to appear.
- [7] A. Inoue and Y. Kasahara, On the asymptotic behavior of the prediction error of a stationary process, in "Trends in Probability and Related Analysis" (N. Kono and N. R. Shieh, Ed.), pp. 207-218, World Scientific, Singapore, 1999.
- [8] I. S. Kac and M. G. Krein, R-functions - analytic functions mapping the upper halfplane into itself, in "American Mathematical Society Translations, Series 2, Vol. 103: Nine Papers in Analysis", pp. 1-18, American Mathematical Society, Providence, R.I., 1974.
- [9] Y. Okabe, On KMO-Langevin equations for stationary Gaussian processes with  $\bar{T}$ -positivity, *J. Fac. Sci. Univ. Tokyo Sec. IA* **33** (1986), 1-56.
- [10] Y. Okabe, On the theory of discrete KMO-Langevin equations with reflection positivity (II), *Hokkaido Math. J.* **17** (1988), 1-44.
- [11] Y. Okabe and A. Inoue, On the exponential decay of the correlation functions for KMO-Langevin equations, *Japan. J. Math.* **18** (1992), 13-24.
- [12] Y. A. Rozanov, "Stationary Random Processes," Holden-Day, San Francisco, CA, 1967.
- [13] N. Wiener, "Extrapolation, Interpolation, and Smoothing of Stationary Time Series," Wiley, New York, 1949.

(A. Inoue) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN

*E-mail address:* inoue@math.sci.hokudai.ac.jp

(Y. Kasahara) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN

*E-mail address:* y-kasa@math.sci.hokudai.ac.jp