ON AFFINE PARALLELS OF GENERIC PLANE CURVES

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ON AFFINE PARALLELS OF GENERIC PLANE CURVES

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

ABSTRACT. We introduce the notion of affine parallels to convex plane curves and study these bifurcations.

1. INTRODUCTION

The parallel to a curve in Euclidean plane has been classically known and studied in many articles (cf., [4,5]). The main tools in these articles are distance-squared functions. Applying ordinary techniques of the singularity theory to these functions, it has been studied how singularities of parallels bifurcate if we alter the distance from the original curve. In this paper we define the notion of affine parallels to convex curves in Affine plane under the framework of equi-affine geometry. We adopt affine distance-cubed functions of convex curves which have been introduced in [10] instead of distance-squared functions.

On the other hand, it is well known that the notions of the Voronoi diagram is applied to various kinds of fields in mathematical science. One of the applications of the Voronoi diagram is given in the theory of the computer vision. We understand that the Voronoi diagram is deeply related to the symmetry sets of curves in the singularity theory. It is known that the symmetry set is the self intersection locus of parallels of plane curves. So one of the start lines for the study of the computer vision is to consider the symmetry sets. However, there are no definition of the Voronoi diagram in Affine plane. When we make the new definitions of the affine Voronoi diagram, it is natural to start the study of affine parallels by using the similar argument as in the case for Euclidean plane curves.

In [7,8] P. J. Giblin and G. Sapiro studied the affine symmetry sets of plane curves. We remark that the affine symmetry set of a plane curve is the self intersection locus of affine parallels of the plane curve.

This paper is divided into four sections. The main result is Theorem 2.2 which is formulated in Section 2. The proof of Theorem 2.2 is given in Section 3. We give some examples with pictures which illustrate the results of Theorem 2.2 in Section 4.

The basic techniques we use in this paper depend heavily on those in the paper of J. W. Bruce [4] and the book of J. W. Bruce and P. J. Giblin [5].

All curves and maps considered here are of class C^∞ unless otherwise stated.

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2. Basic notions

We now present basic concepts on affine differential geometry of plane curves. For more details on classical results, see [2,11,13].

Let $\mathbb{R}^2$ be an affine plane which adopts the coordinate such that the area of the parallelogram spanned by two vectors $a = (a_1, a_2)$, $b = (b_1, b_2)$ is given by the determinant of a and b, that is $|\ a\ \ b\ | = a_1b_2 - a_2b_1$. Let S^1 be the unit circle in $\mathbb{R}^2$, and $\gamma\colon S^1 \longrightarrow \mathbb{R}^2$ be a smooth plane curve with $|\ \dot{\gamma}(t)\ \ \ddot{\gamma}(t)\ | \neq 0$, where $\dot{\gamma}(t) = \frac{d\gamma}{dt}(t)$.

By the similar arguments as those of in Euclidean differential geometry, we have the followings. *The affine arc-length parameter* of a curve γ, measured from $\gamma(t_0)$, $t_0 \in S^1$, is $s(t) = \int_{t_0}^{t} |\ \dot{\gamma}(t)\ \ \ddot{\gamma}(t)\ |^{\frac{1}{3}}\, dt$, then the curve satisfies that $|\ \gamma'(s)\ \ \gamma''(s)\ | = 1$, where $\gamma'(s) = \frac{d\gamma}{ds}(s)$. So we say that a curve γ is *parameterized by affine arc-length* if it satisfies that $|\ \gamma'(s)\ \ \gamma''(s)\ | = 1$. We call a new parameter s *the affine arc-length parameter* or *the affine parameter.* We call $\gamma'(s)$ the *affine tangent vector* and $\gamma''(s)$ the *affine normal vector* of γ. The *affine curvature* is defined to be $\kappa(s) = |\ \gamma''(s)\ \ \gamma'''(s)\ |$. Then we have the Frenet-Serret type formula $\gamma'''(s) = -\kappa(s)\gamma'(s)$.

Suppose that $\kappa(s_0) \neq 0$, then the point $\gamma(s_0) + \frac{1}{\kappa(s_0)}\gamma''(s_0)$ is called the *affine center of curvature* of γ at s_0. The *affine evolute* is defined to be the locus of affine center of curvature which has been classically known, see [2,13].

We define the notion of affine vertices as follows. We say that the point $\gamma(s_0)$ of curve γ is an *affine vertex of order* $n-1$ if $\kappa^{(p)}(s_0) = 0$ for all $1 \leq p \leq (n-1)$ and $\kappa^{(n)}(s_0) \neq 0$. We also say that *the order of the affine vertex is* $n-1$. In particular, the affine vertex of order 1 is called the *ordinary affine vertex* and the affine vertex of order greater than 1 is called *the higher affine vertex.*

We now define the affine parallel to a plane curve γ. Let $\gamma\colon S^1 \longrightarrow \mathbb{R}^2$ be a smooth plane curve parameterized by affine arc-length and $r \in \mathbb{R}$ be a fixed real number. Then the *affine parallel to* γ is given by

$$AP_\gamma(s,r) = \gamma(s) + r\gamma''(s).$$

See Figure 1 in Section 4.

In Euclidean differential geometry, we adopt the unit normal vector instead of the affine normal vector $\gamma''(s)$, so we can consider that the Euclidean parallel is the envelope of circles centered on $\gamma(S^1)$ and having a fixed radius $r > 0$. However, the affine normal vector $\gamma''(s)$ is not a unit vector in affine differential geometry, so we can not obtain the affine parallel as the envelope of circles.

An affine parallel to γ has the following property.

Lemma 2.1. *The affine parallel to γ is a regular curve except for values of s_0 where $\kappa(s_0) \neq 0$ and $r = \frac{1}{\kappa(s_0)}$.*

Lemma 2.1 means that the singular point of the affine parallel is the affine center of curvature of γ at s_0.

Here, we have the following question : If we alter the real number r, how will affine parallels and singular points of affine parallel change ?

We assume that γ has the following property, which is satisfied generically (cf., [6]).
(A) There is no conic having greater than six-point contact with $\gamma(S^1)$.

We remark that γ satisfies the condition (A) if and only if γ does not have higher affine vertices (cf., [10]).

Let $F_i\colon (\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$ be smooth functions germs and $(X_i,0)$ be set germs in $(\mathbb{R}^n,0)$, where $i=1,2$. We say that (F_1,X_1) and (F_2,X_2) are *$\mathcal{R}$-equivalent* if there exists a deffeomorphism germ $\Phi\colon (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that $\Phi(X_1)=X_2$ and $F_1=F_2\circ\Phi$.

For any $\gamma\colon S^1 \longrightarrow \mathbb{R}$, we define the set $\mathcal{AP}_\gamma$ as

$$\mathcal{AP}_\gamma = \{\ (AP_\gamma(s,r),r) \in \mathbb{R}^2\times\mathbb{R} \mid s\in S^1\ \}.$$

The main result in this paper is the following theorem.

Theorem 2.2. *Let $\pi_{\mathbb{R}}\colon \mathbb{R}^2\times\mathbb{R} \longrightarrow \mathbb{R}$ be the canonical projection. Then we have the following :*

(1) *If γ does not have an affine vertex at s_0 and $r_0=\frac{1}{\kappa(s_0)}$, then the pair of germs $(\pi_{\mathbb{R}},\mathcal{AP}_\gamma)$ at $(AP_\gamma(s_0,r_0),r_0)$ is $\mathcal{R}$-equivalent to the pair of germs $(\pi_1,\mathbf{C})$ at $(0,0)$.*

(2) *If γ has an ordinary affine vertex at s_0 and $r_0=\frac{1}{\kappa(s_0)}$, then the pair of germs $(\pi_{\mathbb{R}},\mathcal{AP}_\gamma)$ at $(AP_\gamma(s_0,r_0),r_0)$ is $\mathcal{R}$-equivalent to the pair of germs $(\pi_1,\mathbf{T})$ at $(0,0)$.*

where $\mathbf{C}=\{\ (t,x_1,x_2)\mid x_1^2=x_2^3\ \}$ $\mathbf{T}=\{\ (t,x_1,x_2)\mid x_1=3u^4+u^2v, x_2=4u^3+2uv, t=v\ \}$ and $\pi_1\colon\mathbb{R}^2\times\mathbb{R}\longrightarrow\mathbb{R}$ is the canonical projection given by $\pi_1(t,x_1,x_2)=t$.

We say that the $\mathcal{R}$-equivalence class represented by $(\pi_3,\mathbf{T})$ is the swallowtail bifurcations. We give an answer to the previous question as a corollary of Theorem 2.2.

Corollary 2.3. *Let $\gamma\colon S^1\longrightarrow\mathbb{R}^2$ be a smooth plane curve parameterized by affine arc-length satisfying the condition (A). Then the affine parallel through the affine center of curvature at s_0 has ordinary cusp there provided γ does not have an affine vertex. Moreover, if $\gamma(s_0)$ is the ordinary affine vertex of γ, the affine parallel has the swallowtail bifurcation at $AP_\gamma(s_0,r_0)$. (See Figure 2 in Section 4.)*

3. Proof of Main Theorem

In this section we give the proof of Theorem 2.2. In order to give a proof of Theorem 2.2, we prepare the following : Let $\gamma\colon S^1\longrightarrow\mathbb{R}^2$ be a smooth plane curve with $\mid \gamma'(s)\ \ \gamma''(s)\mid = 1$. We consider the affine distance-cubed function D of a plane curve. This function is quite useful for the study of generic properties of invariants of extrinsic affine differential geometry on convex plane curves (*cf.*, [7,8,10,12]). The affine distance-cubed function $D\colon S^1\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is defined by $D(s,u)=\mid\gamma'(s)\ \ \gamma(s)-u\mid$.

Using the affine distance-cubed function of γ, we define a three parameter family of smooth functions $F\colon S^1\times\mathbb{R}^2\times\mathbb{R}\longrightarrow\mathbb{R}$ by

$$F(s,u,r)=\mid u-\gamma(s)\ \ \gamma'(s)\mid - r.$$

We also define a smooth function $f=f_{u,r}\colon S^1\longrightarrow\mathbb{R}$ by $f(s)=f_{u,r}(s)=F(s,u,r)$, for each $(u,r)\in\mathbb{R}^2\times\mathbb{R}$.

Differentiating f with respect to s and applying the Frenet-Serret type formula, we have,

$$f'(s) = |\ u - \gamma(s)\ \ \gamma''(s)\ | \tag{3.1}$$

$$f''(s) = -\kappa(s)\,|\ u - \gamma(s)\ \ \gamma'(s)\ | - 1 \tag{3.2}$$

$$f'''(s) = -\kappa'(s)\,|\ u - \gamma(s)\ \ \gamma'(s)\ | - \kappa(s)\,|\ u - \gamma(s)\ \ \gamma''(s)\ | \tag{3.3}$$

$$\begin{aligned} f^{(4)}(s) = (\kappa(s)^2 - \kappa''(s))\,|\ u - \gamma(s)\ \ \gamma'(s)\ | \\ - 2\kappa'(s)\,|\ u - \gamma(s)\ \ \gamma''(s)\ | + \kappa(s) \end{aligned} \tag{3.4}$$

It follows from these formulae that we have the following proposition.

Proposition 3.1. *Let $\gamma\colon S^1 \longrightarrow \mathbb{R}^2$ be a convex plane curve parameterized by affine arc-length. Then*

(1) *$f'(s_0) = 0$ if and only if there exits $\lambda \in \mathbb{R}$ such that $u = \gamma(s_0) + \lambda\gamma''(s_0)$.*
(2) *$f'(s_0) = f''(s_0) = 0$ if and only if $\kappa(s_0) \neq 0$ and $u = \gamma(s_0) + \frac{1}{\kappa(s_0)}\gamma''(s_0)$.*
(3) *$f'(s_0) = f''(s_0) = f'''(s_0) = 0$ if and only if $\kappa(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\kappa(s_0)}\gamma''(s_0)$ and $\kappa'(s_0) = 0$.*
(4) *$f'(s_0) = f''(s_0) = f'''(s_0) = f^{(4)}(s_0) = 0$ if and only if $\kappa(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\kappa(s_0)}\gamma''(s_0)$ and $\kappa'(s_0) = \kappa''(s_0) = 0$.*

Proof.

(1) By the formula (3.1), $f'(s_0) = 0$ if and only if $\gamma''(s_0)$ and $u - \gamma(s_0)$ are parallel.
(2) By the formula (3.2), $f''(s_0) = 0$ if and only if $\kappa(s_0)\,|\ \gamma'''(s_0)\ \ u - \gamma(s_0)\ | = 1$. $f'(s_0) = f''(s_0) = 0$ if and only if $1 = \kappa(s_0)\,|\ \gamma'''(s_0)\ \ \lambda\gamma''(s_0)\ | = -\lambda\kappa(s_0)$ because of (1). The last condition is equivalent to the condition that $\kappa(s_0) \neq 0$ and $\lambda = -\frac{1}{\kappa(s_0)}$.
(3) We also assert $f'(s_0) = f''(s_0) = f'''(s_0) = 0$ if and only if $\kappa(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\kappa_0(s_0)}\gamma''(s_0)$ and $0 = f'''(s_0) = -\kappa'(s_0)\left|\ \gamma'(s_0)\ \ \frac{1}{\kappa(s_0)}\gamma''(s)\ \right| = -\frac{\kappa'(s_0)}{\kappa(s_0)}$. The last condition is equivalent to $\kappa'(s_0) = 0$.
(4) The assertion follows from the similar arguments as the proof of the assertion (3).

This completes the proof of Proposition 3.1. *Q.E.D.* ■

We now introduce some basic notions of singularity theory. For more details on the results, see [1,3,4,5,9].

Let $G\colon (\mathbb{R} \times \mathbb{R}^n, (0,0)) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We call G an unfolding of $g(t) = G(t, 0)$. We consider the extended unfolding $\widetilde{G}(t, u, r) = G(t, u) - r$, where $r \in \mathbb{R}$. The discriminant set $\mathfrak{D}_{\widetilde{G}}$ of $\widetilde{G}$ is defined by

$$\mathfrak{D}_{\widetilde{G}} = \left\{ (u, r) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; G(t, u) = r, \frac{\partial G}{\partial t}(t, u) = 0 \right\}.$$

The unfolding $\widetilde{G}$ gives rise to families of discriminant sets obtained by fixing the parameter r. We have natural projections $\pi\colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ and we want to consider the restriction to $\mathfrak{D}_{\widetilde{G}}$. We say that $g(t)$ has an A_k-singularity at t if $g^{(p)}(t) = 0$ for all $1 \leq p \leq k$ and $g^{(p+1)}(t) \neq 0$. For the definition of the versal unfolding of a function germ, see [5].

Theorem 3.2. (*cf.*, [3]) *Let $G(t,u)$ be the unfolding of the function $g(t)$ with the A_k-singularity at 0. If 1 and $\frac{\partial G}{\partial u_i}(t,0)$ $(1 \le i \le n)$ span $\mathbb{R}[t]/\langle t^k \rangle$ as the $\mathbb{R}$-vector space, then $\widetilde{G}$ is a versal unfolding of the function $g(t)$. In this case we have the following.*

(a) *If $\frac{\partial G}{\partial u_i}(t,0)$ span $\mathbb{R}[t]/\langle t^k \rangle$, $(\pi, \mathfrak{D}_{\widetilde{G}})$ is $\mathcal{R}$-equivalent to the trivial projection onto one factor of a product discriminant set.*

(b) *If G is of minimal dimension $k-1$ and $\frac{\partial G}{\partial u_i}(t,0)$ span $\mathbb{R}[t]/\langle t^k \rangle$ then $(\pi, \mathfrak{D}_{\widetilde{G}})$ is $\mathcal{R}$-equivalent to the projection of the standard discriminant set of H above onto the a_1-coordinate,*

where $H(t,a) = \pm t^{k+1} + a_1 t^{k-1} + \cdots + a_{k-1}t + a_k$.

We have the following simple corollary of Theorem 3.2.

Corollary 3.3. *Let G be as the above and assume that g has A_k-singularity at 0 $(k \ge 1)$. Write the $(k-1)$-jet with constant of $\frac{\partial G}{\partial u_i}(t,0)$ at 0 as $\alpha_{0i} + \sum_{j=1}^{k-1} \alpha_{ji} t^j$. If the $k \times l$-matrix of coefficients $(\alpha_{ji}, \mathbf{e})$ for $j = 0,1,\cdots,k-1$, $i = 1,2,\cdots,l-1$ has rank k, then G is the versal unfolding of the function $g(t)$, where $\mathbf{e} = (-1,0,\cdots,0)^t$.*

In this case we have the following :

(a) *If the $k \times (l-1)$ matrix of coefficients (α_{ji}) for $j = 0,1,\cdots,k-1$, $i = 1,2,\cdots,l-1$ has rank k, then $(\pi, \mathfrak{D}_{\widetilde{G}})$ is $\mathcal{R}$-equivalent to the trivial projection onto one factor of a product discriminant set.*

(b) *If G is of minimal dimension $k-1$ and the $k \times (l-1)$ matrix of coefficients (α_{ji}) for $j = 0,1,\cdots,k-1$, $i = 1,2,\cdots,l-1$ has rank $k-1$, then $(\pi, \mathfrak{D}_{\widetilde{G}})$ is $\mathcal{R}$-equivalent to the projection of the standard discriminant set of H above onto the a_1-coordinate, where $H(t,a) = \pm t^{k+1} + a_1 t^{k-1} + \cdots + a_{k-1}t + a_k$.*

By definition and Proposition 3.1, the discriminant set of F is

$$\mathfrak{D}_F = \{\ (\gamma(s) + r\gamma''(s), r) \in \mathbb{R}^2 \times \mathbb{R} \mid s \in S^1\}.$$

$\mathfrak{D}_F$ is precisely the family of affine parallels of γ.

Since $F(s,u) = \gamma_1'(s)(\gamma_2(s) - u_2) - \gamma_2'(s)(\gamma_1(s) - u_1) - r$, we should like to show that $F(s,u)$ is the versal unfolding of $f(s)$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, $u = (u_1, u_2)$. We calculate the 2-jets with constant of $\frac{\partial F}{\partial u_1}(s,u)$ and $\frac{\partial F}{\partial u_2}(s,u)$ at s_0 as follows :

$$\begin{aligned}
\frac{\partial F}{\partial u_1}(s_0,u_0) + j^2\Big(\frac{\partial F}{\partial u_1}(s,u_0)\Big)(s_0) &= \gamma_2'(s_0) + \gamma_2''(s_0)s + \frac{1}{2}\gamma_2'''(s_0)s^2, \\
\frac{\partial F}{\partial u_2}(s_0,u_0) + j^2\Big(\frac{\partial F}{\partial u_2}(s,u_0)\Big)(s_0) &= -\gamma_1'(s_0) - \gamma_1''(s_0)s - \frac{1}{2}\gamma_1'''(s_0)s^2.
\end{aligned}$$

We distinguish into two cases.

(a) When f has the A_2-singularity at s_0 ;

Clearly F is always the versal unfolding of f, when f has the A_2-singularity at s_0.

(b) When f has the A_3-singularity at s_0 ;

We put

$$N = \begin{pmatrix} \gamma_2'(s_0) & -\gamma_1'(s_0) & -1 \\ \gamma_2''(s_0) & -\gamma_1''(s_0) & 0 \\ \frac{1}{2}\gamma_2'''(s_0) & -\frac{1}{2}\gamma_1'''(s_0) & 0 \end{pmatrix}.$$

We want to prove that $rank(N) = 3$. So it is enough to prove that $det(N) \neq 0$. By the direct calculation, we have

$$det(N) = -(-\frac{1}{2}\gamma_2''(s_0)\gamma_3'''(s_0) + \frac{1}{2}\gamma_1''(s_0)\gamma_2'''(s_0)) = -\frac{1}{2}|\ \gamma''(s_0)\ \ \gamma'''(s_0)\ | = \frac{1}{2}\kappa(s_0).$$

Applying Proposition 3.1, if f has the A_3-singularity at s_0, then we have $\kappa(s_0) \neq 0$. So we have $det(N) \neq 0$. Therefor F is the versal unfolding of f.

Thus F is always the versal unfolding of f under the assumption.

We consider the projection $\pi_{\mathbb{R}} : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$. If f has the A_2-singularity at s_0, then the matrix

$$\begin{pmatrix} \gamma_2'(s_0) & -\gamma_1'(s_0) \\ \gamma_2''(s_0) & -\gamma_1''(s_0) \end{pmatrix}$$

has rank 2 by $|\ \gamma'(s)\ \ \gamma''(s)\ | = 1$. By Corollary 3.3, projection $\pi_{\mathbb{R}}$ is trivial one.

On the other hand, if f has the A_3-singularity at s_0, then the matrix

$$\begin{pmatrix} \gamma_2'(s_0) & -\gamma_1'(s_0) \\ \gamma_2''(s_0) & -\gamma_1''(s_0) \\ \gamma_2'''(s_0) & -\gamma_1'''(s_0) \end{pmatrix}$$

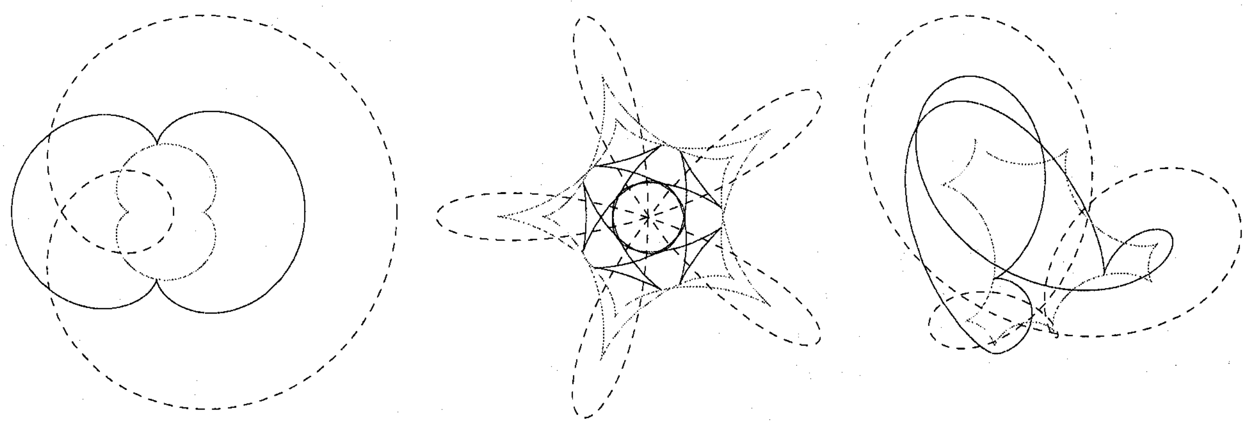
has rank 2 by $|\ \gamma'(s)\ \ \gamma''(s)\ | = 1$. The assumption for the assertion (b) in corollary 3.3 is automatically satisfied.

This completes the proof of Theorem 2.2. *Q.E.D.* ■

4. Examples

In this section we give some examples of affine parallels of plane curves and the bifurcation of singular points of affine parallels.

In Figure 1, we consider $\gamma_a(t) = (\cos t + \cos 2t + 1, \sin t + \sin 2t)$, $\gamma_b(t) = (\cos 3t - \cos 2t, \sin 3t + \sin 2t)$, $\gamma_c(t) = (\cos 2t - \cos(t + 3) - \sin 3t, \sin 2t + \sin(t - 3) + \cos 3t)$, and their affine parallels and affine evolutes. The original curve is drawn by the dotted line, the affine parallel is draw by the real line and the affine evolute is drawn by the real line of a light color.



(γ_a) (γ_b) (γ_c)

Figure 1

In Figure 2, we consider $\gamma_c(t)$ and the one parameter family of affine parallels of γ for the parameters $r = 0.70$, 0.65, 0.60, 0.55, 0.50 and 0.45. These pictures illustrate the situation described in Theorem 2.2. We find the swallowtail bifurcations.

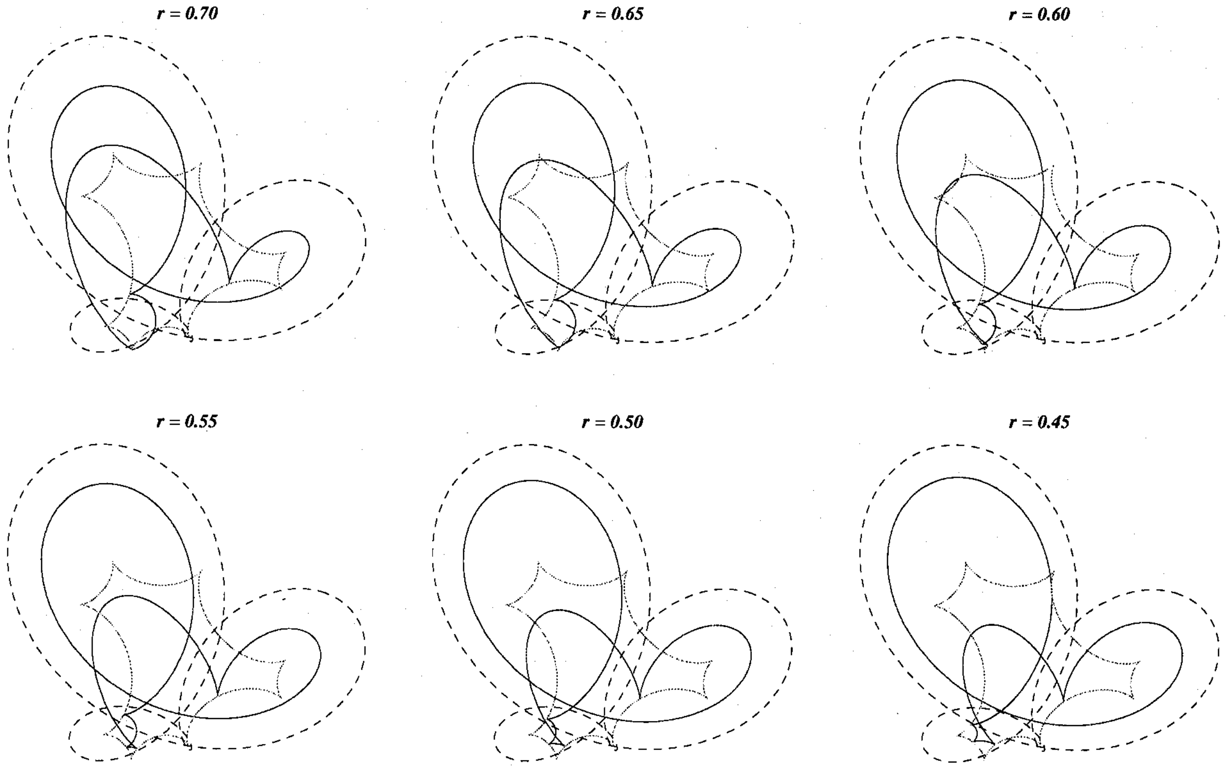


Figure 2

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