CRYSTALLINE AND LEVEL SET FLOW-CONVERGENCE OF A CRYSTALLINE ALGORITHM FOR A GENERAL ANISOTROPIC CURVATURE FLOW IN THE PLANE

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MI-HO GIGA AND YOSHIKAZU GIGA*

Abstract. Recently, a level set formulation is extended by the authors to handle evolution of curves driven by singular interfacial energy including crystalline energy. In this paper as an application of this theory a general convergence result is established for a crystalline algorithm for a general anisotropic curvature flow.

1 Introduction

We consider an evolution equation of a closed, simple curve Γ_t in the plane $\mathbf{R}^2$:

$$V = M(\mathbf{n})(\Lambda_\gamma(\mathbf{n}) + C) \tag{1.1}$$

or its generalization

$$V = g(\mathbf{n}, \Lambda_\gamma(\mathbf{n})). \tag{1.2}$$

Here V denotes the normal velocity of Γ_t in the direction of the outward unit normal $\mathbf{n}$. For a given smooth simple closed curve S in $\mathbf{R}^2$ the quantity $\Lambda_\gamma(\mathbf{n})$ on S is the first variation of the interfacial energy $\int_S \gamma(\mathbf{n})ds$ with respect to change of area enclosed by S. It is formally of form

$$\Lambda_\gamma(\mathbf{n}) = -\mathrm{div}\ \xi(\mathbf{n}), \tag{1.3}$$

where $\xi = \nabla\gamma$ and γ is a given positively homogeneous function of degree one in $\mathbf{R}^2$ called an *interfacial energy (density)*; div denotes the divergence on the curve S. If $\gamma(p) = |p|$ with then $\Lambda_\gamma(\mathbf{n})$ equals the usual curvature κ of S. So Λ_γ is called an (enegetically) *weighted curvature*. In (1.1) $M(\mathbf{n})$ denotes a given positive function called *mobility* and C is a given constant. In (1.2) we always assume that g is nondecreasing in the second variable. Clearly, (1.1) is an example of (1.2). We always assume that the interfacial

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energy density γ is convex. By these assumptions (1.2) is at least degenerate parabolic. The dependence of $\mathbf{n}$ in g gives a kinetic anisotropy. The reader is referred to an interesting book of [22] for derivation of (1.1) and (1.2) from thermomechanics and review articles [14], [15] for mathematical analysis on these equations. For example the well-posedness of the initial value problem is well-known when γ is at least C^2 (outside the origin) and g is continuous in the level set formulation; see [14] and papers cited there.

If γ is not C^1, the meaning of (1.1) may not be clear. The authors of [26] and [2] proposed a notion of a solution when the Frank diagram Frank $\gamma = \{p \in \mathbf{R}^2;\ \gamma(p) \leq 1\}$ of γ is a convex polygon containing the origin as an interior point. In this case γ is called a *crystalline energy*. They consider a special family of evolving polygonal curves to define a notion of solutions for (1.1) which is sometimes called a *crystalline flow*. An extension to (1.2) is given in §2 with various properties of solutions. In fact, we give a sufficient condition for (1.2) so that a crystalline flow is extended after disappearing of some facets before formation of a self-intersection or extinction. If (1.2) is $V = |\Lambda_\gamma|^{\alpha-1}\Lambda_\gamma$ $(\alpha > 0)$ and Frank γ is a rectangle, our condition $(\alpha \geq 1)$ is also necessary. This relates to the problem whether a convex polygon shrinks to a point or line. As shown in [31] all rectangle shrinks to a point if and only if $\alpha \geq 1$ when Frank γ is a square. We extend this result for more general equation (1.2) when g is not necessarily a power of curvatures.

Recently, a level set formulation is extended by the authors [10], [12] so that it applies to some classs of non C^1 interfacial energy γ including any crystalline energy. The theory is nontrivial since the evolution when γ is not C^1 has nonlocal feature. One has to build up fundamental calculus for nonlocal curvatures. In [10], [12] we gave a notion of solutions (called a *level set flow*) without a priori restriction of the shape. This approach is synthetic since it applies to both smooth and singular interfacial energy simultaneously. Typical results include:
(i) (**well-posedness**) the problem (1.2) admits a unique (up to fattening) global-in-time solution for *any* initial curve; (1.2) is solvable beyond pinching.
(ii) (**approximation property**) the solution of (1.2) can be approximated by solutions of approximate equation with approximate interfacial energy.

In this paper we prove the consistency of two notions of solutions (§3). In fact we prove that a crystalline flow is a level set flow under some assumptions on $g(p, 0)$. A general theory developed in [10], [12] provides several important properties on a crystalline flow. In particular, from the approximation property it follows that a crystalline flow can be approximated by solutions of approximate strictly parabolic equations with smooth γ. The approximation property also gives the convergence of crystalline algorithm for (1.2) which is one of main topics in this paper (§4). The convergence of a crystalline algorithm of graph-like solutions of $V = \kappa$ is proved by [18] with convergence rate. Independently, general convergence of graph-like solutions of (1.1) with $C = 0$ is proved in [4] via nonlinear semigroup theory. Later, the result in [18] is extended for closed convex curves in [17]; see also [19] for overview of these development. In [25] K. Ishii and H. M. Soner proved the convergence crystalline algorithm for $V = \kappa$ for any closed curve (not necessary convex) by using viscosity solutions. At the same time we proved the convergence for graph-like solutions for general equation (1.2) by extending the theory of viscosity solutions to motion by nonlocal curvature [6], [7], [9]. In [6] the consistency has been proved for graph-like solutions while in [7] well-posedness of the problem is studied. In [9] the convergence has been established. In the meanwhile the result in [17] is extended in [30] to $V = |\kappa|^{\alpha-1}\kappa$ with $\alpha > 0$ for closed convex curves, where time is also discreted. Our approach is synthetic and does not require convexity of curves. Also the equation that our theory applies is a general equation of form (1.2). However, our theory does not provide rate of convergence since the situation is too general.

There are several other topics related to crystalline flow but we do not mention them here. Instead, we refer review papers [27], [29], [15] and a recent paper [12] and references therein for these topics.

2 Crystalline flow

We consider a crystalline energy γ, i.e., Frank γ is a convex m-polygon. Let q_i ($i = 1, 2, \ldots, m$) be its verteces. Let $\mathcal{N} \subset S^1 = \{|p| = 1\}$ be the set of all unit vectors of form $q_i/|q_i|$ ($i = 1, 2, \ldots, m$). We say that a simple polygonal curve S in $\mathbf{R}^2$ is an *admissible crystal* if all outward normal (orientation) belongs to $\mathcal{N}$ and orientations of adjacent segment (facet) point to verteces adjacent in Frank γ. We say a family of polygon $\{S_t\}_{t\in J}$ is an *admissible evolving crystal* if S_t is an admissible crystal for all $t \in J$ and corners of S_t move at least C^1 in time, where J is a time interval. The last requirement implicitly assumes that the number of facets and the orientation at each facet are independent of time. In other words, S_t is of form $S_t = \bigcup_{j=1}^{r} S_j(t)$ and $S_j(t)$ is a maximal nontrivial closed segment (facet) of S_t and the orientation $\mathbf{n}_j$ of $S_j(t)$ is independent of time. For later convenience we number facets clockwise. Our definition is consistent with [16].

2.1 Regular flow

Let $\{S_t\}_{t\in J}$ be an admissible evolving crystal with $J = [0, T)$. We say that $\{S_t\}_{t\in J}$ is a *γ-regular flow* of (1.2) if

$$V = g(\mathbf{n}_j;\ \chi_j \Delta(\mathbf{n}_j)/L_j(t)) \quad \text{on} \quad S_j(t) \tag{2.1}$$

for $j = 1, 2, \ldots, r$. The quantity $\chi_j \Delta(\mathbf{n}_j)/L_j(t)$ is a nonlocal weighted curvature $\Lambda_\gamma(\mathbf{n}_j)$, where $L_j(t)$ is the length of $S_j(t)$ and $\Delta(\mathbf{m}_i) = \tilde{\gamma}'(\theta_i{+}0) - \tilde{\gamma}'(\theta_i{-}0)$, $\mathbf{m}_i = (\cos\,\theta_i,\, \sin\,\theta_i) \in \mathcal{N}$ with $\tilde{\gamma}(\theta) = \gamma(\cos\theta, \sin\theta)$. The quantity χ_j is a *transition number*. It takes $+1$ (resp.

-1) if S_t is concave (resp. convex) in the direction of $\mathbf{n}_j$; we use convention that $\chi_j = -1$ for all $j = 1, \ldots, r$ if $\{S_t\}$ is a convex polygon. Otherwise we set $\chi_j = 0$. The quantity $\Delta(\mathbf{m}_i)$ is the length of facet of Wulff shape

$$W_\gamma = \{x \in \mathbf{R}^2;\ x \cdot p \leq \gamma(p) \quad \text{for all} \quad p \in \mathbf{R}^2\} \tag{2.2}$$

with outward normal $\mathbf{m}_i \in \mathcal{N}$. By this convention the weighted curvature Λ_γ of Wulff shape is -1 independent of the facet. This explains that the quantity $\chi_j \Delta(\mathbf{n}_j)/L_j(t)$ is a substitute of Λ_γ in (1.3) provided we postulate that Λ_γ is constant on each facet. (This constancy hypotheses may not be appropriate when the equation (1.2) or (1.1) is not spatially homogeneous [8] or surface evolution [3], [32].)

Lemma 1(Local existence). *Assume that $\lambda \mapsto g(\mathbf{m}_i, \lambda)(\mathbf{m}_i \in \mathcal{N})$ is locally Lipschitz (continuous) on $\mathbf{R} \setminus \{0\}$. Let S_0 be an admissible crystal. Then there is a constant $T > 0$ and a unique γ-regular flow $\{S_t\}_{t \in J}$ of (1.2) with initial data S_0, where $J = [0, T)$.*

This follows from the local existence theorem of a system of ordinary differential equations for L_j's. Indeed, L_j always fulfills

$$\frac{dL_j}{dt}(t) = (\cot\ \vartheta_{j+1} + \cot\ \vartheta_j)V_j - (\sin\ \vartheta_j)^{-1}V_{j-1} - (\sin\ \vartheta_{j+1})^{-1}V_{j+1} \quad (j = 1, \ldots, r) \tag{2.3}$$

as in [2], [22]. Here $\vartheta_j = \theta_j - \theta_{j-1}$ for $\mathbf{n}_j = (\cos\ \theta_j, \sin\ \theta_j)$ and V_j denotes the normal velocity of $S_j(t)$; the index j is considered modulo r. Combining (2.1) and (2.3) we get an r-system of ordinary differential equations for L_j's.

For later citation we define a few classes of g. We say that g belongs to $\mathcal{D}_\gamma$ if $\lambda \mapsto g(\mathbf{m}_i, \lambda)$ is locally Lipschitz on $\mathbf{R} \setminus \{0\}$, nondecreasing on $\mathbf{R}$ and $\lim_{\lambda \to \pm\infty} g(\mathbf{m}_i, \lambda) = \pm\infty$ for all $\mathbf{m}_i \in \mathcal{N}$. If $g \in \mathcal{D}_\gamma$ satisfies a growth condition

$$\int_{I_\pm} g(\mathbf{m}_i, \lambda)\lambda^{-2} d\lambda = \pm\infty \quad \text{for all} \quad \mathbf{m}_i \in \mathcal{N} \tag{2.4}$$

with $I_+ = (1, \infty)$ and $I_- = (-\infty, -1)$, we say that g belongs to $\mathcal{D}_\gamma^o$.

Lemma 2(Facet disappearing). *For $g \in \mathcal{D}_\gamma$ let $\{S_t\}_{t \in J_0}$ be a γ-regular flow of (1.2) with $J_0 = [0, T_*)$, $T_* < \infty$. Let S_{tj} denote a facet of S_t disappearing at T_* i.e. the length ℓ_{tj} of S_{tj} tends to zero as $t \to T_*$; this follows from a weaker condition $\underline{\lim}_{t \to T_*} \ell_{tj} = 0$.*
(i) If the transition number χ_j of S_{tj} is not zero near T_*, then one of following two phenomena occurs exclusively.
(a) (Single point extinction). S_t becomes convex near T_* and all facets disappear at T_*.
(b) (Degenerate pinching). For t close to T_* there are two parallel facets S_{ti_0}, S_{ti_1} ($i_0 < j < i_1$) with opposite orientations such that all facet S_{ti} ($i_0 < i < i_1$) disappears at $t = T_*$ with $\chi_i = \chi_j$ and that S_{tk} ($k = i_0, i_1$) does not disappear at $t = T_*$ unless $\chi_k = 0$.

(ii) If (a) and (b) does not occur, all facet disappearing at $t = T_*$ always has zero transition number and at most two consecutive facets disappear. The limit S_* of $\{S_t\}$ as $t \to T_*$ is a polygonal curve satisfying conditions of admissible crystals except the embeddedness assumption.
(iii) If $g \in \mathcal{D}^o_\gamma$, then (b) does not occur.
(iv) If the Wulff γ has not parallel facets (i.e. $\mathcal{N} \cap (-\mathcal{N}) = \emptyset$), then (b) does not occur.

The part (i) and (ii) can be proved as in [28], [25, Lemma 3.4] where (1.1) with $C = 0$ and $M = \gamma$ is discussed. However, it is easy to generalize to our setting. There are no parallel facets for (iv) so (iv) is trivial. For graph-like solutions clearly (b) does not occur. In [6] all possible way of facet disappearing is discussed for (1.1) for graph-like solutions. The part (iii) is new in this generality. For (1.1) with $C = 0$ and $M = \gamma$ it is essentially proved in [28]; it corresponds to [25, Case 1, subcase 2] where no details are given. The growth condition (2.4) is actually necessary to exclude (b) when Wulff γ is a rectangle. We shall show it after the proof of (iii).

Proof of (iii). We shall derive a contradiction assuming (b). Let $\ell_i(t)$ be the length of S_{ti}. Let $w(t)$ be the distance between of S_{ti_0} and S_{ti_1}. Since the argument is symmetric, we may assyme that $\chi_i = 1$ for $i = i_0 + 1, \ldots, i_1 - 1$. By assumption and geometry V_{i_0} is bounded and $(\sin \vartheta_{i_0})^{-1} V_{i_0 - 1}$ is bounded from below near T_* so by (2.3) we see that

$$\dot{\ell} = d\ell/dt \leq C - aV_{i_0+1} \quad \text{with} \quad \ell = \ell_{i_0} \tag{2.5}$$

with some constant $C > 0$ independent of t, where $a = 1/\sin \vartheta_{i_0+1}$. By our convention and $\chi_{i_0+1} = 1$ the constant a is positive. Since $\ell_{i_0+1} \leq aw$, we see

$$V_{i_0+1} \geq g(\mathbf{n}_{i_0+1}, \Delta(\mathbf{n}_{i_0+1})/(aw)) \tag{2.6}$$

by monotonicity of g. Since $\lim_{t \to T_*} w = 0$ and $\lim_{\lambda \to \infty} g(\cdot, \lambda) = \infty$, (2.5) and (2.6) implies $\dot{\ell}_{i_0} < 0$ for t close to T_* and similarly $\dot{\ell}_{i_1} < 0$. Thus, monotonicity of g and (2.1) implies that V_{i_0} and V_{i_1} are nondecreasing near T_*. Thus $\dot{w} = -(V_{i_0} + V_{i_1}) \leq 0$ since $\lim_{t \to T_*} w = 0$. Since S_{i_0} does not disappear if $\chi_{i_0} \neq 0$ and the same for S_{i_1}, $\dot{w}$ is bounded near T_*. Thus

$$I = \int_{T_* - \delta}^{T_*} \dot{\ell} \dot{w} dt \leq C_0 \int_{T_* - \delta}^{T_*} (-\dot{\ell}) dt = C_0 \ell(T_* - \delta) < \infty$$

with some constant C_0 for small $\delta > 0$. On the other hand by (2.5) and (2.6)

$$I \geq \int_{T_* - \delta}^{T_*} (C - ag(\mathbf{n}_{i_0+1}, \Delta(\mathbf{n}_{i_0+1})/(aw))) \dot{w} dt.$$

Our assumption (2.4) implies $\int_0^1 g(\mathbf{m}_i, 1/s) ds = \infty$ so we get $I \geq \infty$ which is a contradiction. □

Example. Assume that Wulff γ is a rectangle. Assume that (2.4) is not fulfilled, say the integral for I_- is finite for $\mathbf{m}_1$ (orthogonal to $\mathbf{m}_2$). Then a small rectangle whose facet S_{t1} of orientation $\mathbf{m}_1$ is small compared with adjacent S_{t2} shrinks to a line and S_{t1} disappears. Indeed, let x denote the length of S_{t1} and y denote that of S_{t2}. Then by (2.1) and (2.3) $\dot{y} = -h_1(x)$, $\dot{x} = -h_2(y)$, with $h_i(x) = -g(\mathbf{m}_i, -\Delta(\mathbf{m}_i)/x)$, $i = 1, 2$. If initial data is taken small, say $x_0, y_0 \in (0, \varepsilon)$, then $\dot{x} \le 0$, $\dot{y} \le 0$ and at least one of x and y tends to zero in a finite time T_*. Since $\int_{I_-} g(\mathbf{m}_1, \lambda)\lambda^{-2} d\lambda > -\infty$ is equivalent to $\int_0^1 h_1(x)dx < \infty$, for small ξ there is η such that $\int_0^\eta h_2(y)dy > \int_0^\xi h_1(x)dx$. Let $\eta_0(\xi)$ denote infimum of $\eta > 0$. Then $\eta_0(\xi) \to 0$ as $\xi \to 0$. We take $x_0, y_0 \in (0, \varepsilon)$ so that $y_0 > \eta_0(x_0)$. Since $\dot{x}h_1(x) = \dot{y}h_2(y)$, integrating on $(0, t)$ yields $\int_{y(t)}^{y_0} h_2(y)dy = \int_{x(t)}^{x_0} h_1(x)dx$. When $x(t) \to 0$ as $t \to T_*$, $y(t)$ does not converge to zero because of the choice of y_0 since otherwise we get a contradiction.

Similarly, it is possible to construct an example of initial data so that it exhibits degenerate pinching before extinction time. For this purpose consider a big squre with orientations $\mathbf{m}_1$, $\mathbf{m}_2$ touching the above rectangle on a part of one of long facets of the rectangle. We remove the intersection (except end points) to get a desired admissible crystal S_0 of 8 facets.

In [31] for $g(\lambda) = |\lambda|^{\alpha-1}\lambda$, $\alpha > 0$ it is shown that all rectangle shrinks to a point if and only if $\alpha \ge 1$, which is equivalent to (2.4) for such a type of g. Our results generalize his result for general g. When Frank γ is $2k$-regular polygon, it is reasonable to conjecture that all convex polygon shrinks to a point if and only if $\alpha \ge \alpha_k = (1 + 2\cos\pi/k)^{-1}$; this is settled for $k = 2$ as mentioned above. The number α_k is a critical number such that regular $2k$-polygon is "linearly stable" discussed in [31], where it is shown that some convex polygon shrinks to a line for small $\alpha > 0$ for $k = 3, 4$, while all convex polygon shrinks to a point for $\alpha \ge 1$ for all k; the last result is included in Lemma 2.

2.2 Extension

By Lemma 1 it is rather clear that one can construct a γ-regular flow after some facet disappears provided that S_* is an admissible crystal. We say that $\{S_t\}_{t\in J}$ is a *crystalline flow* with initial data S_0 there is some $t_0 = 0 < t_1 < \cdots < t_\ell = T \le \infty$ such that $\{S_t\}_{t\in J_h}$ is a γ-regular flow for $J_h = [t_h, t_{h+1})$ with initial data S_{t_h} $(h = 0, \ldots, \ell - 1)$ and $S_t \to S_{t_{h+1}}$ in the sense of Hausdorff distance topology as $t \uparrow t_{h+1}$ and at t_{h+1} some facet disappears for $h = 0, 1, \ldots, \ell - 2$. By definition a crystalline flow S_t is continuous in time $t \in [0, T)$. Since the number of facets is finite, facet disappearing occurs at most finitely many epochs. At the time of extinction (i.e. enclosed open set becomes empty), at least one of facets with nonzero transition number disappears by geometry. So by Lemma 2 either (a) or (b) occurs. Lemma 2 yields:

Lemma 3. *Assume that (i) $g \in \mathcal{D}^o_\gamma$ or that (ii) $g \in \mathcal{D}_\gamma$ and Wulff γ has no parallel facets. At the maximal time T_* of existence of a crystalline flow $\{S_t\}_{t\in J}$ with $J = [0, T_*)$ for (1.2) S_t self-intersects or shrinks to a point at T_* provided that $T_* < \infty$.*

Remark 4. If S_0 is convex so that transition number of all facets is -1, then S_t cannot self-intersect from its geometry before the extinction time. By Lemma 2 we have:

Theorem 5. *Assume that $g \in \mathcal{D}_\gamma$. Let $T_* \in (0, \infty]$ be the maximal time of the existence of a crystalline flow $\{S_t\}_{t\in J}$ with $J = [0, T_*)$ for (1.2). If S_0 is convex, then S_t stays convex so that no facet disappears during evolutions. Moreover, if $T_* < \infty$, then S_t shrinks to a point or a line.*

This is interpreted as a discrete analog for (1.2) of the result of [5] when $V = \kappa$ is discussed. For (1.1) with $M = 1$ and $C = 1$ this is contained in [28, Theorem 3.2]. If the Wulff shape of γ has a center of symmetry and $g(p, \lambda) = -g(-p, -\lambda)$ for $p \in \mathbf{R}^2$, $|p| = 1$, $\lambda \in \mathbf{R}$, then the evolution law (1.2) is independent of the choice of orientation of $\mathbf{n}$. The condition of γ is for example fulfilled if we assume that γ is even, i.e., $\gamma(p) = \gamma(-p)$ for all $p \in \mathbf{R}^2$. A typical example is (1.1) with $C = 0$, $M(p) = M(-p)$, $\gamma(p) = \gamma(-p)$ for all $p \in \mathbf{R}^2$ with $|p| = 1$. For such an evolution law one can prove that self intersection does not occur even if S_0 is not convex. The proof is essentially the same as in [25, Lemma 3.3] (reflecting idea of [16]) and [28, Theorem 3.2]; one should also handle the case that self-intersection and facet disappearing occur simultaneously although it is not difficult. Lemma 2 now yields the next result, which is a discrete analog of the result of [21] for $V = \kappa$.

Proposition 6. *Assume that $g \in \mathcal{D}^o_\gamma$ and that $\gamma(p) = \gamma(-p)$ and $g(p, \lambda) = -g(-p, -\lambda)$ for all $p \in \mathcal{N}$, $\lambda \in \mathbf{R}$. Let $T_* \in (0, \infty]$ be as in Theorem 5. If $T_* < \infty$, then S_t shrinks to a point at T_*. In particular, S_t becomes convex for t close to T_*.*

3 Consistency with a level set flow

Our goal is to prove that a crystalline flow is a level set flow under some assumptions on $g(p, 0)$ for $p \notin \mathcal{N}$.

3.1 Level set flow

Let D (resp. E) be an open (a closed) set in $\overline{\mathbf{R}}^3_+ = \mathbf{R}^2 \times [0, \infty)$. Assume that $D \cap \mathcal{R}_T$ ($E \cap \mathcal{R}_T$) is bounded for every T, where $\mathcal{R}_T = \mathbf{R}^2 \times [0, T]$. We say that D (resp. E) is an *open* (a *closed*) *evolution* of (1.2) with initial data $D(0)$ (resp. $E(0)$) if there is a continuous function u in $\overline{\mathbf{R}}^3_+$ that is a continuous "solution" of the level set equation of (1.2) in $\mathbf{R}^3_+ = \mathbf{R}^2 \times (0, \infty)$ such that $D = \{(x, t) \in \overline{\mathbf{R}}^3_+;\ u(x, t) > 0\}$ (resp. $E =$

$\{(x,t) \in \overline{\mathbf{R}}^3_+;\ v(x,t) \geq 0\}$) and that $u(x,t)$ is negative constant outside some bounded set in $\mathcal{R}_T$ for every $T < \infty$. Here $A(t)$ denotes the cross-section of A at time t i.e., $A(t) = \{x \in \mathbf{R}^2; (x,t) \in A\}$ for a set A in $\overline{\mathbf{R}}^3_+$. If a closed set Γ is expressed as $E \setminus D$ with closed and open evolution E and D, we say that Γ is a *level set flow* with initial data $\Gamma_0 = E(0) \setminus D(0)$ of (1.2); the orientation is reflected in the level set equation.

Of course, the definition of solutions of the level set equation is not standard when Frank γ has corners. We refer [10], [12] for rigorous definitions. We list several properties of open and closed evolutions.

Lemma 7. *Assume that γ is crystalline. Assume that $\lambda \mapsto g(p,\lambda)$ is nondecreasing and that $g(p,\lambda)$ is continuous on $S^1 \times \mathbf{R}$. Assume that*

$$\overline{\lim}_{|\lambda|\to\infty} \sup_{|p|=1} |g(p,\lambda)/\lambda| < \infty. \tag{3.1}$$

(The totality of g satisfying these three conditions denotes $\mathcal{A}$).
(i) (Unique existence). For a given bounded open (closed) set D_0 (resp. E_0) there is a unique open (closed) evolution of (1.2) D(resp. E) in $\overline{\mathbf{R}}^3_+$ with initial data D_0(resp. E_0). In particular, there is a unique level set flow of (1.2) with initial data $\Gamma_0 = E_0 \setminus D_0$.
(ii) (Semigroup property). Let $\mathcal{M}(t)$ denote the mapping from Γ_0 to $\Gamma(t)$. Then $\mathcal{M}(t+s)\Gamma_0 = \mathcal{M}(t)(\mathcal{M}(s)\Gamma_0)$ for any $t \geq 0$, $s \geq 0$.
(iii) (Left continuity). $\Gamma(t-\delta) \to \Gamma(t)$ as $\delta \downarrow 0$ for $t > 0$ in the Hausdorff distance topology.

The part (i) is contained in one of main results of [10], [12], where more general γ is discussed. The part (ii) immediately follows from (i). Note that even if $\Gamma_0 = \partial D_0$ the set $\Gamma(t)$ may have interior. This phenomenon may occur even for $V = \kappa$ [14] and called fattening phenomena. The part (iii) needs more explanation. The upper semicontinuity of $\Gamma(t)$ is immediate since Γ is closed. To see left lower semicontinuity we observe that center of small ball survives for a while (depending on the size) when we flow the ball. (This is implicit in [12, Section 8].) The left lower semiconitinuity of $\Gamma(t)$ follows by comparing such evolutions with the aid of the comparison principle in [10], [12]. For a related argument see Remark 14(i). Similar phenomenon is found in [1] for the mean curvature flow equation.

Remark 8. The growth assumption (3.1) can be removed if we modify the notion of solution as in [24] or [20] although it needs more technical complexity.

3.2 Condition on preserving corners

We consider an evolution equation $V = g^0(\mathbf{n})$ with $g^0(p) = g(p,0)$. It is easy to see that the coners of $A_i = \bigcap_{j=0}^{1} H_{i+j}$ and $B_i = \bigcap_{j=0}^{1} H_{i+j}$ with $H_{i+j} = \{x \in \mathbf{R}^2;\ x \cdot \mathbf{m}_{i+j} \leq g^0(\mathbf{m}_{i+j})\}$

stay as coners by motion by $V = g^0(\mathbf{n})$ if and only if

$$A_i \subset \{x \in \mathbf{R}^2;\ x \cdot \mathbf{m} \leq g^0(\mathbf{m})\} \subset B_i \tag{3.2}$$

for all $\mathbf{m} = (\cos\theta, \sin\theta)$ such that $\theta_i \leq \theta \leq \theta_{i+1}$, where $\mathbf{m}_{i+j} = (\cos\theta_{i+j}, \sin\theta_{i+j}) \in \mathcal{N}$ $(j = 0, 1)$ and $\mathcal{N} = \{\mathbf{m}_i\}_{i=1}^m$; $\mathbf{m}_i$ is indexed (modulo m) counterclockwise i.e., $\theta_i < \theta_{i+1}$.

If evolving curve is given as the graph of a function $u = u(x_1, t)$, then $V = g^0(\mathbf{n})$ is of the form

$$u_t = a(u_{x_1}),\ a(\sigma) = \sqrt{1+\sigma^2}g(\frac{-\sigma}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+\sigma^2}}),\ u_{x_1} = \partial u/\partial x_1,$$

where $\mathbf{n}$ is taken upward. The condition (3.2) is equivalent to

$$a(p) = \delta a(p_i) + (1-\delta)a(p_{i+1}) \tag{3.3}$$

for $p = \delta p_i + (1-\delta)p_{i+1}$, $0 \leq \delta \leq 1$, where $\mathbf{m}_{i+j} = (-p_{i+j}, 1)/\sqrt{1+p_{i+j}^2}$ with $j = 0, 1$. (If the corner perserving condition (3.2) is not fulfilled, the coner may be cut and rounded.) This condition is well-noticed for example in [23], [13]. The form (3.3) is found in [6].

3.3 Consistency with γ-regular flow

Theorem 9. *Assume that γ is crystalline and $g \in \mathcal{A} \cap \mathcal{D}_\gamma$. Assume that g satisfies (3.2) for all i. Let S_0 be an admissible crystal. Let D_0 be the bounded open set enclosed by S_0. Then a γ-regular flow $\{S_t\}_{t\in J_0}$ of (1.2) with $S_t|_{t=0} = S_0$ agree with a level set flow of (1.2) with initial data $S_0 = \partial D_0$ in $\mathbf{R}^2 \times J$, where J_0 is a time interval $[0, T_0)$.*

Proof. By the semigroup perperty (Lemma 7 (ii)) it suffices to prove that γ-regular flow $\{S_t\}$ agrees with the level set flow $\{\Gamma(t)\}$ for a short time $[0, T)$. Indeed, let $T'(\geq T)$ be the maximal time such that S_t agree with $\Gamma(t)$ for $t \in [0, T')$. If $T' < T$, then $\Gamma(T') = S_{T'}$ follows from by left continuity of $\Gamma(t)$ (Lemma 7 (iii)) and continuity of S_t. By a short time consistency $\mathcal{M}(s)\Gamma(T') = S_{T'+s}$ for small s. The semigroup property yields $\Gamma(T'+s) = S_{T'+s}$. This contradicts the maximality of T'.

We shall prove a short time consistency. We set

$$u_0(x) = \max\{-\delta,\ \min\{\delta,\ \mathrm{dist}_\gamma(x, S_0)\}\},\quad x \in \mathbf{R}^2$$

for $\delta > 0$. Here dist_γ denotes the signed distance function with respect to the Minkowski metric such that W_γ is a unit ball. (We use the convention that $\mathrm{dist}_\gamma(x, S_0) > 0$ for $x \in D_0$.) The value of dist_γ is defined by $\mathrm{dist}_\gamma(x, S_0) = \inf_{y\in S_0}\{\mu \geq 0;\ x - y \in \mu W_\gamma\}$ for $x \in D_0$. If δ is taken sufficiently small, then every c-level set S_0^c $(-\delta < c < \delta)$ and the boundary of $\pm\delta$-level set $S_0^{\pm\delta}$ is an admissible crystal. One can construct γ-regular flow S_t^c starting from $S_0^c(-\delta \leq c \leq \delta)$ for some time interval $[0, T_c)$. By continuous dependence of solution of ODE (2.1), (2.3) in c we see that T_c for $c \in [-\delta, \delta]$ is bounded from below by some positive constant T. For $t \in [0, T)$ we set $u(x,t) = c$ if $x \in S_t^c$ for $c \in [-\delta, \delta]$; $u(x,t) = \delta$ if x is a point enclosed by S_t^δ; $u(x,t) = -\delta$ if x is a point not enclosed by $S_t^{-\delta}$. By continuous dependence of solution of ODE and the comparison principle stated at the end of §3.3 (Lemma 10) u is well-defined in $Q_0 = \mathbf{R}^2 \times [0, T)$. Moreover, by nonincreasing of distance of S_t^c and $S_t^{c'}$ in Lemma 10 we see that u is Lipschitz continuous in x locally uniformly in time $t \in [0, T)$. It is not difficult to see that $u(x, \cdot)$ is also continuous in time for fixed x. Thus the function u is continuous in Q_0 and $u(x,t) = -\delta$ for x, $|x| \geq R_0$, $t \in [0, T)$ with some large R_0. By construction S_t is the zero level set of u for $t \in [0, T)$.

It suffices to prove that u is a sub and supersolution of the level set equation of (1.2). We present a proof only for a subsolution since the proof for a supersolution is symmetric. Let φ be a test function of u at $(\hat{x}, \hat{t}) \in Q = \mathbf{R}^2 \times (0, T)$, i.e.,

$$\max_Q (u - \varphi) = (u - \varphi)(\hat{x}, \hat{t}) = 0, \tag{3.4}$$

with $\varphi(x,t) = h(x) + k(t)$, $k \in C^1(0,T)$, $h \in C_\gamma^2(U)$, where U is a neighborhood of $(\hat{x}, \hat{t})$. The class C_γ^2 is defined in [10], [12]. If $\nabla\varphi(\hat{x}, \hat{t}) = 0$ and $\nabla\nabla\varphi(\hat{x}, \hat{t}) = 0$, then by construction $u(\hat{x}, \hat{t}) = \pm\delta$. If $\hat{x} \notin S_{\hat{t}}^{\pm\delta}$, then by (3.4) clearly $\varphi_t(\hat{x}, \hat{t}) = k'(\hat{t}) = 0$. If $\hat{x} \in S_{\hat{t}}^\delta$, $\varphi_t(\hat{x}, \hat{t}) \leq 0$ since otherwise near $\hat{x}$, $u(x,t) < \delta$ for $t (> \hat{t})$ close to $\hat{t}$ which contradicts continuity of S_t^δ near $\hat{t}$. Similarly, $\hat{x} \in S_{\hat{t}}^{-\delta}$ implies $\varphi_t(\hat{x}, \hat{t}) \leq 0$. In any cases we have $\varphi_t(\hat{x}, \hat{t}) \leq 0$ which is the desired inequality.

It remains to handle the case $\nabla\varphi(\hat{x}, \hat{t}) \neq 0$. By construction $\hat{x} \in S_{\hat{t}}^c$ for some $c \in [-\delta, \delta]$. There are two cases.

Case A. $-\nabla h(\hat{x})/|\nabla h(\hat{x})| \in \mathcal{N}$. If $\hat{x}$ is not a corner point of $S_{\hat{t}}^c$, i.e., $\hat{x} \in S_{\hat{t}j}^c$ is not an end point for some j, then by (3.4) and comparison of the curvature as in [16] we have

$$\Lambda_\gamma(\mathbf{n}_j) \leq \Lambda_\gamma(h)(\hat{x}), \tag{3.5}$$

where $\Lambda_\gamma(h)$ is defined as in [10], [12]; it is a nonlocal curvature of the $h(\hat{x})$-level set of h at $\hat{x}$. Since $\{S_{\hat{t}}^c\}$ is a γ-regular flow, the normal velocity V_j of $\{S_{\hat{t}j}^c\}$ satisfies $V_j = g(\mathbf{n}_j, \Lambda_\gamma(\mathbf{n}_j))$. By (3.4) we see $V_j = \varphi_t(\hat{x}, \hat{t})/|\nabla\varphi(\hat{x}, \hat{t})|$. Since g is nondecreasing in the last variable, (3.5) now yields

$$\varphi_t(\hat{x}, \hat{t})/|\nabla\varphi(\hat{x}, \hat{t})| - g(-\nabla h(\hat{x})/|\nabla h(\hat{x})|,\ \Lambda_\gamma(h)(\hat{x})) \leq 0. \tag{3.6}$$

Suppose that a facet containing $\hat{x}$ of $h(\hat{x})$-level set of h touches a facet $S_{\hat{t}j}^c$ other than the end points. Then we still get (3.6) since the left hand side of (3.6) is unchanged on the intersection of $S_{\hat{t}j}^c$ and $h(\hat{x})$-level set of h and the necessary inequality is derived at some intersection point other than end points in the same way.

Thus it remains to handle the case that $h(\hat{x})$-level set of h touches S_{ij}^c only at a corner point $\hat{x}$ of two facets of S_{ij}^c. By (3.4) and geometry we have (3.5) for one of facet S_{ij}^c containing $\hat{x}$ and obtain (3.6). The proof of Case A is now complete.

Case B. $\mathbf{m} = -\nabla h(\hat{x})/|\nabla h(\hat{x})| \notin \mathcal{N}$. By (3.4) and geometry $\hat{x}$ must be a corner of S_{ij}^c. Moreover, $\Lambda_\gamma(\mathbf{n}_j) \leq 0$ if a facet S_{ij}^c containing $\hat{x}$. Since γ is crystalline, $\Lambda_\gamma(h)(\hat{x}) = 0$. By corner preserving condition (3.2) and (3.4) we observe that $\varphi_t(\hat{x},\hat{t})/|\nabla\varphi(\hat{x},\hat{t})| - g(\mathbf{m},0) \leq V_j - g(\mathbf{n}_j, 0)$ for at least one of facet S_{ij}^c containing $\hat{x}$. Since $\Lambda_\gamma(\mathbf{n}_j) \leq 0$ and since g is nondecreasing in the last variable, we have $V_j - g(\mathbf{n}_j, 0) \leq 0$. We have thus proved (3.6) for the Case B. □

Lemma 10. *Assume that γ is crystalline and $g \in \mathcal{D}_\gamma$. Let S_0' be an admissible crystal enclosing another admissible crystal S_0. Let $\{S_t'\}_{t\in J}$ and $\{S_t\}_{t\in J}$ be a crystalline flow for (1.2) with initial S_0' and S_0, respectively. Then S_t' always encloses S_t for $t \in J$ and the distance of S_t' is nonincreasing in t.*

In [16] this statement is proved for (1.1). However, it is straightforward to extend their result to (1.2) based on the comparison of nonlocal curvature Λ_γ.

3.4 Consistency with a crystalline flow

Theorem 11. *Assume that γ is crystalline and $g \in \mathcal{A} \cap \mathcal{D}_\gamma$ satisfies (3.2) for all i. Let S_0 be an admissible crystal. Let D_0 be the bounded open set enclosed by S_0. Then a crystalline $\{S_t\}_{t\in J}$ of (1.2) with $S_t|_{t=0} = S_0$ agrees with a level set flow of (1.2) with initial data $S_0 = \partial D_0$ in $\mathbf{R}^2 \times J$, where J is a time interval $[0, T)$.*

By Theorem 9 one can prove Theorem 11 by using semigroup property and left semicontinuity (Lemma 7) as described in the begining of the proof of Theorem 9 since $\{S_t\}$ is continuous in time t (even if some facet disappear at that time.)

Remark 12. A level set flow exists after the time self-intersection occurs. Thus it gives a unique way to extend a crystalline flow after pinching up to fattening. We do not discuss this topic in this paper. A heuristic argument is found in [28].

4 Convergence of a crystalline algorithm

In [12] we proved a general convergence of a level set flow. We give here a special version of [12, Corollary 8.3, Remark 8.5 (i), (iii)].

Theorem 13. *Assume that γ_ε is crystalline for $\varepsilon > 0$ and γ_ε converges to γ_0 locally uniformly on $\mathbf{R}^2$ as $\varepsilon \to 0$. Assume that $\gamma_0 > 0$ and γ_0 is C^2 outside the origin. Assume* *that* $g_\varepsilon \in \mathcal{A}$ *for* $\varepsilon \geq 0$ *and that*

$$\overline{\lim}_{|\lambda|\to\infty} \sup_{0<\varepsilon<1} \sup_{|p|=1} |g_\varepsilon(p,\lambda)/\lambda| < \infty. \tag{4.1}$$

Assume moreover that g_ε *converges to* g_0 *locally uniformly on* $S^1 \times \mathbf{R}$ *as* $\varepsilon \to 0$. *Let* $D_0^\varepsilon(\varepsilon \geq 0)$ *be a bounded open set in* $\mathbf{R}^2$ *and let* $E_0^\varepsilon = \overline{D_0^\varepsilon}$. *Let* E^ε *and* D^ε *be a closed (an open) evolution in* $\overline{\mathbf{R}}_+^3$ *of*

$$V = g_\varepsilon(\mathbf{n}, \Lambda_{\gamma_\varepsilon}(\mathbf{n})) \tag{I_ε}$$

with initial data E_0^ε *and* D_0^ε, *respectively. Assume that* $\lim_{\varepsilon\to 0} d_H(E_0^\varepsilon, E_0^0) = 0$, *where* d_H *denotes the Hausdorff distance in* $\mathbf{R}^2$. *Then the following two properties hold.*
(i) For $T < \infty$

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq T} d_H(E^\varepsilon(t), E^0(t)) = 0 \tag{4.2}$$

provided that E^0 *is strongly regular in* $[0,T]$ *in the sense that*

$$E^0(t) = \overline{D^0(t)} \quad \text{for all } \; t \in [0,T]. \tag{4.3}$$

(ii) Let T^ε *be the extinction time of* E^ε $(\varepsilon \geq 0)$ *i.e.,* $T_\varepsilon = \sup\{t \geq 0; E^\varepsilon(t) = \emptyset\}$. *If (4.3) holds for all* $T < T_0$ *(so that* $E^0 = \overline{D^0}$ *in* $\overline{\mathbf{R}}_+^3$*), then* E^ε *converges to* E^0 *in the Hausdorff distance topology in* $\overline{\mathbf{R}}_+^3$. *Moreover,* $T_\varepsilon \to T_0$ *as* $\varepsilon \to 0$.

Remark 14. (i) The condition (4.3) is stronger than usual nonfattening condition $E = \overline{D}$ in $\mathbf{R}^2 \times [0,T]$ as remarked in [12, Remark 8.5 (iii)]. It also guarantees that $E(t)$ is continuous in $t \in [0,T]$ so Theorem 13 follows from [12, Corollary 8.3]. Indeed, by Lemma 7 (iii) and closedness of E it suffices to prove that $E(t)$ is right lower semicontinuous at each $t_0 \in [0,T]$. If not, there is an open ball $B \subset \mathbf{R}^2 \setminus E(t_j)$ and $t_j \downarrow t_0$ such that $B \cap E(t_0) \neq \emptyset$. Since $E(t_0) = \overline{D(t_0)}$ this implies there is another small open ball $B_1 \subset B$ such that $B_1 \subset E(t_0)$. If we flow B_1 by (I_0) its center $\hat{x}$ stays in an evolution for a while. By the comparison we see $B_1 \subset E(t_j)$ which is a contradiction.
(ii) Under the hypotheses of Theorem 13 it follows that $\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T} d_H(\Gamma^\varepsilon(t), \Gamma^0(t)) = 0$, where $\Gamma^\varepsilon = E^\varepsilon \setminus D^\varepsilon$ is the level set flow of (I_ε) with initial data $\Gamma_0^\varepsilon = \partial D_0^\varepsilon$. To prove convergence of crystalline algorithm it is convient to notice the following elementary property.

Lemma 15(Interpolation). *Assume the same hypotheses of Theorem 13 concerning* $\gamma_\varepsilon(\varepsilon > 0)$. *Assume that* $g_0 \in \mathcal{A}$. *Then there is a sequence* $\{g_\varepsilon\} \subset \mathcal{A}$ *satisfying the following properties.*
(i) $g_\varepsilon \to g_0$ *as* $\varepsilon \to 0$ *locally uniformly on* $S^1 \times \mathbf{R}$.
(ii) The uniform growth condition (4.1) is fulfilled.
(iii) g_ε *satisfies the corner perserving condition (3.2) for all* $\mathbf{m}_i \in \mathcal{N}_\varepsilon$, *where* $\mathcal{N}_\varepsilon$ *is the set of unit vectors pointing to verteces of Frank* γ_ε.

(iv) $g_\varepsilon(\mathbf{m}_i, \lambda) = g_0(\mathbf{m}_i, \lambda)$ *for all* $\mathbf{m}_i \in \mathcal{N}_\varepsilon$.

It is also convenient to define a notion of a kind of simplicity of curves for the boundary of $E(t)$. For a closed set E in $\overline{\mathbf{R}}^3_+$ and $\delta > 0$ we set $D_\delta(t) = \{x \in E(t);\ \mathrm{dist}(x, \partial(E(t))) > \delta\}$ and $U_\delta(t) = \{x \notin E(t);\ \mathrm{dist}\ (x, \partial(E(t))) > \delta\}$. For $\eta > 0$ if both $D_\delta(t)$ and $U_\delta(t)$ is connected for all $t \in [0, T]$ and all $\delta < \eta$, we say that $\partial(E(t))$ is *η-simple* on $[0, T]$. By definition a smoothly evolving smooth simple curve on $[0, T]$ is η-simple for some $\eta > 0$.

Main Theorem. *Assume that γ_ε is crystalline for $\varepsilon > 0$ and γ_ε converges to γ_0 locally uniformly on $\mathbf{R}^2$ as $\varepsilon \to 0$. Assume that $\gamma_0 > 0$ and γ_0 is C^2 outside the origin. Assume that $g_0 \in \mathcal{A}$. Assume that $g_0 \in \mathcal{D}_{\gamma_\varepsilon}$ for $\varepsilon > 0$. Let E_0 be the closure of a bounded open set D_0. Let E be a closed evolution in $\overline{\mathbf{R}}^3_+$ of*

$$V = g_0(\mathbf{n},\ \Lambda_\gamma(\mathbf{n})) \tag{4.4}$$

with initial data E_0. For $T < \infty$ assume that E is strongly regular in $[0, T]$. Let S_0^ε be an admissible crystal with respect to $\gamma_\varepsilon(\varepsilon > 0)$. Assume that $d_H(E_0^\varepsilon, E_0) \to 0$ as $\varepsilon \to 0$, where $E_0^\varepsilon = \overline{D_0^\varepsilon}$ and D_0^ε is a bounded open set enclosed by S_0^ε. Assume one of following three conditions.
(a) (Convexity) S_0^ε is convex for small $\varepsilon > 0$ (so that $E_0^\varepsilon(t)$ is convex.)
(b) (Symmetry) $\gamma_\varepsilon(p) = \gamma_\varepsilon(-p)$ for $\varepsilon \geq 0$ and $g_0(p, \lambda) = -g_0(-p, -\lambda)$ for all $p \in S^1$, $\lambda \in \mathbf{R}$ and $g_0 \in \mathcal{D}^o_{\gamma_\varepsilon}$ for $\varepsilon > 0$.
(c) $\partial(E(T))$ is η-simple on $[0, T]$. Moreover, either Wulff γ_ε does not have parallel facets or $g_0 \in \mathcal{D}^o_{\gamma_\varepsilon}$ for $\varepsilon > 0$.
Then we have the following properties
(i) (Existence of crystalline flow up to T for small $\varepsilon > 0$) For sufficiently small $\varepsilon > 0$ there is a crystalline flow $\{S_t^\varepsilon\}_{t \in J}$ of

$$V = g_0(\mathbf{n}, \Lambda_{\gamma_\varepsilon}(\mathbf{n})) \tag{4.5}$$

for some $J = [0, T'_\varepsilon)$ with $T'_\varepsilon > T$.
(ii) (Convergence) Let E_t^ε be the compact set enclosed by $\{S_t^\varepsilon\}$. Then

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} d_H(E_t^\varepsilon, E(t)) = 0,\ \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} d_H(S_t^\varepsilon, \partial(E(t))) = 0.$$

Proof. (i) We take g_ε approximating g_0 as in Lemma 15. Let E^ε be a closed evolution of (I_ε) in $\overline{\mathbf{R}}^3_+$ with $E^\varepsilon(0) = E_0^\varepsilon$. Then by the consistency (Theorem 11) $\partial(E^\varepsilon(t))$ is a crystalline flow of (4.5) with initial data S_0^ε on $[0, T'_\varepsilon)$, where $T'_\varepsilon > 0$ is the maximal time of the existence of the crystalline flow. By the convergence (4.2) and strong regularity (4.3) for sufficiently small ε the set $E^\varepsilon(t)$ cannot be a singleton nor a line for $t \in [0, T]$. In the case (a) Theorem 5 yields $T'_\varepsilon > T$. Assume that $T'_\varepsilon \leq T$. Then by Lemma 3 $\partial(E^\varepsilon(T'_\varepsilon))$

must self-interest and satisfies conditions of admissible crystals except embededness; degenerate pinch does not occur. This does not happen in the case (b) by Proposition 6. In the case (c) by (4.2) and η-simplicity of $\partial(E^0(t))$ on $[0,T]$, we see for small $\varepsilon > 0$ $\partial(E^\varepsilon(T'_\varepsilon))$ is δ-simple for small $\delta > 0$, which yields a contradiction. We thus have $T'_\varepsilon > T$.
(ii) Since $E^\varepsilon_t = E^\varepsilon(t)$ for $t < T'_\varepsilon$, this follows from (4.2). □

Remark 16. (i) In [12, Corollary 8.3] a more general interfacial energy γ_ε is considered in Theorem 13. For example γ_0 is allowed to be some non C^1 function in Theorem 13 and also in Main Theorem. For $g_0 \in \mathcal{A}$ there is $g_\varepsilon \in \mathcal{D}^o_{\gamma_\varepsilon}$ satisfying (i)-(iii) of Lemma 15 and $g_\varepsilon(-p,-\lambda) = -g_\varepsilon(p,\lambda)$ if g_0 has such a property. Thus, if we consider a crystalline flow of $V = g_\varepsilon(\mathbf{n}, \Lambda_{\gamma_\varepsilon}(\mathbf{n}))$ instead of (4.5) we need not assume $g_0 \in \mathcal{D}^o_{\gamma_\varepsilon}$ nor $g_0 \in \mathcal{D}_{\gamma_\varepsilon}$ to get convergence results in the Main Theorem.
(ii) Our main theorem includes the convergence result of [25] for $V = \kappa$ as a very special case of (b). In their situation $E(t) \setminus D(t)$ is a smooth solution until it shrinks to a point at $t = T_0$. So strong regularity (4.3) is automatically fulfilled.
(iii) If $E_0 \setminus D_0$ is a C^1-curve, it is not difficult to construct a sequence $E^\varepsilon_0 = \overline{D^\varepsilon_0}$ such that $S^\varepsilon_0 = E^\varepsilon_0 \setminus D^\varepsilon_0$ is an admissible crystal with respect to γ_ε and that $d_H(E^\varepsilon_0, E_0) \to 0$ as $\varepsilon \to 0$.
(iv) The convergence of normals or curvature of S^ε_t is not known although such a result is available for graph-like solutions [11].
(v) (Approximation of extinction time) If E^0 is strongly regular for every $T < T_0$, where T_0 is the extinction time of E^0, then $T'_\varepsilon \to T_0$ provided that one of (a), (b) holds. Here T'_ε is the maximal time of existence of a crystalline flow of (I_ε) with initial data S^ε_0. This follows from Theorem 13 (ii), Theorem 5 and Proposition 6. Note that we need to prove that $\partial(E^\varepsilon(t))$ agrees with crystalline flow up to $t = T'_\varepsilon$ although it is not difficult.

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