Two Dimensional Q-Algebras

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Two Dimensional Q-Algebras

By

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Abstract. Two dimensional commutative Banach algebra $\mathcal{B}$ with a unit has a simple form : $a + bB$ for some fixed B in $\mathcal{B}$ and for a, b in $\mathbf{C}$. When $\mathcal{B}$ is an operator algebra on a Hilbert space, we show that the norm on $\mathcal{B}$ is explicitly determined and then $\mathcal{B}$ is a Q-algebra. Moreover, we describe completely two dimensional Q-algebras with their norms.

§1. Introduction

Let $\mathcal{B}$ be a two dimensional commutative Banach algebra with a unit I. Hence $\mathcal{B} = \{aI + bB \; ; \; a, b \in \mathbf{C}\}$. The following proposition shows that we can choose good ones as B in $\mathcal{B}$.

Proposition 1. *Suppose $\mathcal{B} = \{aI + bB \; ; \; a, b \in \mathbf{C}\}$ is a commutative Banach algebra. Then $B^2 = \alpha I + \beta B$ for some $\alpha, \beta \in \mathbf{C}$. If $\beta^2 + 4\alpha \neq 0$ then $\mathcal{B} = \{aI + bB_0 \; ; \; a, b \in \mathbf{C}\}$ where $B_0^2 = I$. If $\beta^2 + 4\alpha = 0$ then $\mathcal{B} = \{aI + bB_0 \; ; \; a, b \in \mathbf{C}\}$ where $B_0^2 = 0$.*

Proof. Suppose $B^2 = \alpha I + \beta B$. If $\beta^2 + 4\alpha \neq 0$, put $B_0 = a_0 I + b_0 B$ where

$$a_0 = -\frac{\beta}{\sqrt{\beta^2 + 4\alpha}} \quad \text{and} \quad b_0 = \frac{2}{\sqrt{\beta^2 + 4\alpha}},$$

then $B_0^2 = (a_0^2 + \alpha b_0^2)I + (2a_0 b_0 + b_0^2 \beta)B = I$. If $\beta^2 + 4\alpha = 0$, put $B_0 = a_0 I + b_0 B$ where

$$a_0 = -\frac{\beta}{2} \quad \text{and} \quad b_0 = 1,$$

then $B_0^2 = (a_0^2 + \alpha b_0^2)I + (2a_0 b_0 + b_0^2 \beta)B = 0$.

In Proposition 1, if $\beta^2 + 4\alpha \neq 0$ then $\mathcal{B} = \{aP + bQ \; ; \; a, b \in \mathbf{C}\}$ where $P + Q = I$, $P^2 = P$ and $Q^2 = Q$. In fact $P = (I + B_0)/2$ and $Q = (I - B_0)/2$.

Two dimensional semi-simple Banach algebras have been studied by Drury [3], and Cole, Lewis and Wermer [2].

In Section 2, when $\mathcal{B}$ is an operator algebra on a Hilbert space H, the norm of $\alpha I + \beta B_0$ is given explicitly using α, β and $\|B_0\|$. When $B_0^2 = I$, this is a theorem of Feldman, Krupnik and Markus [4]. In Section 3, as a result in Section 2 we give a formula for a norm on two dimensional Q-algebra using a part metric and a norm of a bounded point derivation. In Section 4, we consider the algebras $\mathbf{C}^2$ under a few multiplication.

§2. Norm of $\alpha I + \beta B$

In this section, assuming that $B^2 = I$ or $B^2 = 0$ we give explicit formulae for norms of $\alpha I + \beta B$ using α, β and $\|B\|$. (1) of Theorem 1 is a theorem of Feldman, Krupnik and Marcus [4] and (2) of Theorem 1 can be proved by [4]. We give elementary proofs of them.

Theorem 1. *Let B be a bounded linear operator on a Hilbert space H.*

(1) If $B^2 = I$ and B is not a scalor multiple of I then

$$\begin{aligned}\|\alpha I + \beta B\| &= \sqrt{\left|\frac{\beta}{2}\right|^2 \left(\|B\| - \frac{1}{\|B\|}\right)^2 + \left(\frac{|\alpha+\beta| + |\alpha-\beta|}{2}\right)^2} \\ &+ \sqrt{\left|\frac{\beta}{2}\right|^2 \left(\|B\| - \frac{1}{\|B\|}\right)^2 + \left(\frac{|\alpha+\beta| - |\alpha-\beta|}{2}\right)^2}.\end{aligned}$$

(2) If $B^2 = 0$ then

$$\|\alpha I + \beta B\| = \sqrt{\left|\frac{\beta}{2}\right|^2 \|B\|^2 + |\alpha|^2} + \left|\frac{\beta}{2}\right| \|B\|.$$

Proof. (1) Set $P = (I-B)/2$ and $Q = (I+B)/2$ then $P+Q = I$, $P^2 = P$, $PQ = 0$ and $\alpha I + \beta B = (\alpha+\beta)P + (\alpha-\beta)Q = aP + bQ$. Put $H_1 = PH$ and $H_2 = QH$ then

$$P = \begin{pmatrix} I_1 & C \\ 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & -C \\ 0 & I_2 \end{pmatrix}$$

on $H = H_1 \oplus H_2$ where I_j is an identity operator on H_j $(j = 1,2)$ and C is a bounded linear operator from H_2 to H_1. Hence

$$\alpha I + \beta B = \begin{pmatrix} aI_1 & (a-b)C \\ 0 & bI_2 \end{pmatrix}.$$

Suppose $\|\alpha I + \beta B\| = \gamma$, then

$$\begin{pmatrix} \bar{a}I_1 & 0 \\ (\bar{a}-\bar{b})C^* & \bar{b}I_2 \end{pmatrix} \begin{pmatrix} aI_1 & (a-b)C \\ 0 & bI_2 \end{pmatrix} \le \gamma^2$$

and hence

$$\begin{pmatrix} (\gamma^2 - |a|^2)I_1 & (\bar{a}b - |a|^2)C \\ (a\bar{b} - |a|^2)C^* & (\gamma^2 - |b|^2)I_2 - |a-b|^2 C^*C \end{pmatrix} \ge 0.$$

Hence

$$(\gamma^2 - |a|^2)\|f\|^2 + 2Re(\bar{a}b - |a|^2)\langle Cg, f\rangle + (\gamma^2 - |b|^2)\|g\|^2 - |a-b|^2\|Cg\|^2 \ge 0$$

for any $f \oplus g \in H_1 \oplus H_2$ where $\langle\ ,\ \rangle$ denotes the inner product on H. Therefore it is easy to see that

$$(\gamma^2 - |a|^2)^{1/2}\left[\gamma^2 - (|b|^2 + |a-b|^2\frac{\|Cg\|^2}{\|g\|^2})\right]^{1/2} \ge |\bar{a}b - |a|^2| \cdot |\langle Cg, f\rangle| / \|f\| \cdot \|g\|.$$

Taking the supremum over f and g with $\|f\| \leq 1$ and $\|g\| \leq 1$, we get that

$$(\gamma^2 - |a|^2)^{1/2}[\gamma^2 - (|b|^2 + |a-b|^2\|C\|^2)]^{1/2} \geq |\bar{a}b - |a|^2| \cdot \|C\|$$

and hence

$$\gamma^4 - (|a|^2 + |b|^2 + |a-b|^2\|C\|^2)\gamma^2 + |a|^2|b|^2 \geq 0.$$

Since $\gamma^2 \geq \max(|a|^2, |b|^2) \geq |ab|$,

$$\begin{aligned}\gamma^2 &\geq \frac{1}{2}\{(|a|^2 + |b|^2 + |a-b|^2\|C\|^2) \\ &+ \sqrt{(|a|^2 + |b|^2 + |a-b|^2\|C\|^2)^2 - 4|a|^2|b|^2}\}.\end{aligned}$$

Since $\sqrt{t + \sqrt{t^2 - |a|^2|b|^2}} = \sqrt{\frac{t+|ab|}{2}} + \sqrt{\frac{t-|ab|}{2}}$,

$$\begin{aligned}\|aP + bQ\| &= \gamma \\ &\geq \sqrt{\left|\frac{a-b}{2}\right|^2 \|C\|^2 + \left(\frac{|a|+|b|}{2}\right)^2} + \sqrt{\left|\frac{a-b}{2}\right|^2 \|C\|^2 + \left(\frac{|a|-|b|}{2}\right)^2}.\end{aligned}$$

Conversely if

$$\gamma = \sqrt{\left|\frac{a-b}{2}\right|^2 \|C\|^2 + \left(\frac{|a|+|b|}{2}\right)^2} + \sqrt{\left|\frac{a-b}{2}\right|^2 \|C\|^2 + \left(\frac{|a|-|b|}{2}\right)^2},$$

then $(\gamma^2 - |a|^2)^{1/2}\{\gamma^2 - (|b|^2 + |a-b|^2) \cdot \|C\|^2)^{1/2} = |\bar{a}b - |a|^2|\ \|C\|$ and hence

$$(\gamma^2 - |a|^2)^{1/2}\left[\gamma^2 - (|b|^2 + |a-b|^2\frac{\|Cg\|^2}{\|g\|^2})\right]^{1/2} \geq |\bar{a}b - |a|^2| \cdot |\langle Cg, f\rangle| \ / \ \|f\| \cdot \|g\|.$$

By the first half of the proof, this implies that $\|aP + bQ\| \leq \gamma$. Thus

$$\begin{aligned}\|\alpha I + \beta B\| &= \sqrt{|\beta|^2\|C\|^2 + \left(\frac{|\alpha+\beta| + |\alpha-\beta|}{2}\right)^2} \\ &+ \sqrt{|\beta|^2\|C\|^2 + \left(\frac{|\alpha+\beta| - |\alpha-\beta|}{2}\right)^2}.\end{aligned}$$

As $\alpha = 0$ and $\beta = 1$, $\|B\| = \sqrt{\|C\|^2 + 1} + \|C\|$ and hence $\|C\| = (\|B\|^2 - 1)/2\|B\|$. This implies (1).

(2) Put $H_1 = BH$ and $H_2 = H \ominus H_1$ then

$$B = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \text{ on } H = H_1 \oplus H_2$$

where C is a bounded linear operator from H_2 to H_1. Hence

$$\alpha I + \beta B = \begin{pmatrix} \alpha I_1 & \beta C \\ 0 & \alpha I_2 \end{pmatrix}$$

where I_j is an identity operator on H_j $(j = 1, 2)$. Suppose $\|\alpha I + \beta B\| = \gamma$, then

$$\begin{pmatrix} \bar{\alpha} I_1 & 0 \\ \bar{\beta} C^* & \bar{\alpha} I_2 \end{pmatrix} \begin{pmatrix} \alpha I_1 & \beta C \\ 0 & \alpha I_2 \end{pmatrix} \leq \gamma^2$$

and hence

$$\begin{pmatrix} (\gamma^2 - |\alpha|^2) I_1 & -\bar{\alpha}\beta C \\ -\alpha\bar{\beta} C^* & \{\gamma^2 - (|\beta|^2 C^* C + |\alpha|^2)\} I_2 \end{pmatrix} \geq 0.$$

Hence

$$(\gamma^2 - |\alpha|^2)\|f\|^2 - 2Re\bar{\alpha}\beta\langle Cg, f\rangle + (\gamma^2 - |\alpha|^2)\|g\|^2 - |\beta|^2\|Cg\|^2 \geq 0$$

for any $f \oplus g \in H_1 \oplus H^2$. Therefore it is easy to see that

$$(\gamma^2 - |\alpha|^2)^{1/2} \left[(\gamma^2 - |\alpha|^2) - |\beta|^2 \frac{\|Cg\|^2}{\|g\|^2} \right]^{1/2} \geq |\alpha\beta| \frac{|\langle Cg, f\rangle|}{\|f\|\|g\|}.$$

and so

$$(\gamma^2 - |\alpha|^2)(\gamma^2 - |\alpha|^2 - |\beta|^2\|C\|^2) \geq |\alpha\beta|^2\|C\|^2.$$

Then $\gamma^2 \geq |\alpha|^2 + |\beta|^2\|C\|^2$ because $\gamma^2 \geq |\alpha|^2$. Since $\gamma^4 - (2|\alpha|^2 + |\beta|^2\|C\|^2)\gamma^2 + |\alpha|^4 \geq 0$ and $\gamma^2 \geq |\alpha|^2 + |\beta|^2\|C\|^2$,

$$\gamma^2 \geq \frac{1}{2}\left\{(2|\alpha|^2 + |\beta|^2\|C\|^2) + \sqrt{(2|\alpha|^2 + |\beta|^2\|C\|^2)^2 - 4|\alpha|^4}\right\}$$

and hence

$$\gamma \geq \sqrt{\left|\frac{\beta}{2}\right|^2 \|C\|^2 + |\alpha|^2} + \left|\frac{\beta}{2}\right| \|C\|.$$

As in the proof of (1),

$$\|\alpha I + \beta B\| = \sqrt{\left|\frac{\beta}{2}\right|^2 \|C\|^2 + |\alpha|^2} + \left|\frac{\beta}{2}\right| \|C\|$$

and so $\|B\| = \|C\|$. This implies (2).

§3. Two dimensional Q-algebra

Let A be a uniform algebra and J a closed ideal of A. The quotient algebra A/J is called a Q-algebra. It is known that if A/J is of two dimension then J has one of the following two forms :

$$J = \{f \in A \ ; \ f(x) = f(y) = 0\}$$

where x and y are two points in the maximal ideal space $M(A)$ of A, or

$$J = \{f \in A \ ; \ f(x) = \delta(f) = 0\}$$

where $x \in M(A)$ and δ is a bounded point derivation at x, that is , δ is a bounded linear functional on A such that $\delta(fg) = f(x)\delta(g) + g(x)\delta(f)$ for $f, g \in A$. Put

$$\sigma(x, y) = \sigma_A(x, y) = \sup\{|f(y)| \ ; \ f \in A, \ f(x) = 0, \ \|f\|_\infty \leq 1\}$$

and

$$\omega(x, \delta) = \omega_A(x, \delta) = \sup\{|\delta(f)| \ ; \ f \in A, \ f(x) = 0, \ \|f\|_\infty \leq 1\}.$$

In Corollary 2, if the Hilbert space is of two dimensional and $\mathcal{B}$ is a semi-simple then it was proved by Drury [3] (see [2]).

Lemma. *Let x, y be two points in $M(A)$ and δ a bounded point derivation at x.*

(1) Suppose $J = \{f \in A \ ; \ f(x) = f(y) = 0\}$. If $f_1(x) = 1$ and $f_1(y) = 0$, then $\|f_1 + J\| = 1/\sigma(x, y)$.

(2) Suppose $J = \{f \in A \ ; \ f(x) = \delta(f) = 0\}$. If $f_0(x) = 0$ and $\delta(f_0) = 1$, then $\|f_0 + J\| = 1/\omega(x, \delta)$.

Proof. (1) $(\sigma f_1)(y) = 0$ and $(\sigma f_1)(x) = \sigma$ where $\sigma = \sigma(x, y)$. By the definition of $\sigma = \sigma(x, y)$

$$\|\sigma f_1\|_\infty \geq 1 \text{ and } \|\sigma f_1 + J\| \geq 1.$$

There exists $\{g_n\}$ in A with $\|g_n\|_\infty = 1$ such that

$$g_n(y) = 0 \text{ and } \sigma - \frac{1}{n} \leq g_n(x) \leq \sigma$$

for $n = 1, 2, \cdots$. Put $f_n = g_n/g_n(x)$ then

$$\frac{1}{\sigma} \leq \|f_n\|_\infty \leq \frac{1}{|g_n(x)|} \leq \frac{1}{\sigma - \frac{1}{n}}.$$

Then $f_n(x) = 1, \ f_n(y) = 0$ and $f_n \in f_1 + J$. Hence $\|f_1 + J\| \leq \frac{1}{\sigma} \frac{1}{n}$ and so $\|\sigma f_1 + J\| \leq 1$. Thus $\|\sigma f_1 + J\| = 1$. (2) follows from the same argument to (1)

Theorem 2. *Let x, y be two points in $M(A)$ and δ a bounded point derivation at x.*

(1) If $J = \{f \in A \,;\, f(x) = f(y) = 0\}$ then

$$\begin{aligned}\|f+J\| &= \sqrt{\left|\frac{f(x)-f(y)}{2}\right|^2\left(\frac{1}{\sigma^2}-1\right)+\left(\frac{|f(x)|+|f(y)|}{2}\right)^2} \\ &+ \sqrt{\left|\frac{f(x)-f(y)}{2}\right|^2\left(\frac{1}{\sigma^2}-1\right)+\left(\frac{|f(x)|-|f(y)|}{2}\right)^2}\end{aligned}$$

where $\sigma = \sigma(x, y)$.

(2) If $J = \{f \in A \,;\, f(x) = \delta(f) = 0\}$ then

$$\|f+J\| = \sqrt{\left|\frac{\delta(f)}{2}\right|^2\frac{1}{\omega^2}+|f(x)|^2} + \left|\frac{\delta(f)}{2}\right|\frac{1}{\omega}$$

where $\omega = \omega(x, \delta)$.

Proof. By a theorem of Cole [2], A/J is isometrically isomorphic to an algebra of bounded operators on a Hilbert space H. Hence there exists a unital homomorphism from A to $L(H)$ such that $\|\Phi(f)\| = \|f + J\|$ for all $f \in A$, where $\|f + J\|$ is the quotient norm of the coset $f + J$ of f in A/J. Then $J = \ker \Phi$.

(1) By [5, Lemma 1],

$$\Phi(f) = \begin{pmatrix} f(x)I_1 & (f(x)-f(y))C \\ 0 & f(y)I_2 \end{pmatrix}$$

on $H = H_1 \oplus H_2$ for all $f \in A$, where I_j is an identity operator on H_j $(j = 1, 2)$ and C is a bounded linear operator from H_2 to H_1. Then

$$\Phi(f) = f(x)I + f(y)B$$

where $I = I_1 \oplus I_2$, $B^2 = I$ and

$$B = \begin{pmatrix} I_1 & 2C \\ 0 & -I_2 \end{pmatrix}.$$

By (1) of Theorem 1, we can give the norm of $\Phi(f)$ using $f(x)$, $f(y)$ and $\|B\|$. Suppose f_1 and f_2 in A such that $f_1(x) = f_2(y) = 1$ and $f_1(y) = f_2(x) = 0$. Then

$$\Phi(f_1) = \begin{pmatrix} I_1 & C \\ 0 & 0 \end{pmatrix} \text{ and } \Phi(f_2) = \begin{pmatrix} 0 & -C \\ 0 & I_2 \end{pmatrix}.$$

By Lemma, $\|\Phi(f_1)\| = \|f_1 + J\| = 1/\sigma(x, y)$. Since $\|C\|^2 = \|\Phi(f_1)\|^2 - 1$ and $\|B\| = \sqrt{\|C\|^2 + 1} + \|C\|$, (1) of Theorem 1 implies (1).

(2) By [5, Lemma 1],

$$\Phi(f) = \begin{pmatrix} f(x)I_1 & \delta(f)C \\ 0 & f(y)I_2 \end{pmatrix}$$

on $H = H_1 \oplus H_2$ for all $f \in A$. Then

$$\Phi(f) = f(x)I + \delta(f)B$$

where $B^2 = 0$ and

$$B = \begin{pmatrix} 0 & \delta(f)C \\ 0 & 0 \end{pmatrix},$$

and so $\|B\| = |\delta(f)|\|C\|$. By (2) of Theorem 1, we can give the norm of $\Phi(f)$ using $f(x)$, $\delta(f)$ and $\|B\|$. Suppose f_0 in A such that $f_0(x) = 0$ and $\delta(f_0) = 1$. Then

$$\Phi(f_0) = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.$$

By Lemma, $\|\Phi(f_0)\| = \|f_0 + J\| = 1/\omega(x, \delta)$ and so (2) of Theorem 1 implies (2).

Corollary 1. *Let A be an arbitrary uniform algebra and J a closed ideal of A. If $\dim A/J = 2$ then A/J is isometrically isomorphic to $\mathcal{A}/\mathcal{J}$ where $\mathcal{A}$ is the disc algebra and $\mathcal{J}$ is a closed ideal of $\mathcal{A}$*

Proof. By the remark above Lemma, if $\dim A/J = 2$ then $J = \{f \in A\ ;\ f(x) = f(y) = 0\}$ or $J = \{f \in A\ ;\ f(x) = \delta(f) = 0\}$. There exist two points x' and y' in $M(\mathcal{A})$ and a bounded point derivation δ' at x' such that

$$\sigma_A(x,y) = \sigma_{\mathcal{A}}(x', y') \text{ and } \omega_A(x, \delta) = \omega_{\mathcal{A}}(x', \delta').$$

Put $\mathcal{J} = \{f \in \mathcal{A}\ ;\ f(x') = f(y') = 0\}$ or $\mathcal{J} = \{f \in \mathcal{A}\ ;\ f(x') = \delta'(f) = 0\}$. Then by Theorem 2 $\mathcal{A}/\mathcal{J}$ is isometrically isomorphic to A/I.

Corollary 2. *If a two dimensional commutative Banach algebra $\mathcal{B}$ with a unit is an operator algebra on a Hilbert space then $\mathcal{B}$ is a Q-algebra.*

Proof. By Proposition 1 $\mathcal{B} = \{\alpha I + \beta B\ ;\ \alpha, \beta \in \mathbf{C}\}$ with $B^2 = I$ or $B^2 = 0$. By Theorems 1 and 2, $\mathcal{B}$ is a Q-algebra.

§4. Q-algebra $\mathbf{C}^2$

$\mathbf{C}^2$ is an algebra under coordinate-wise multiplication and if $\|(\alpha,\beta)\| = \max(|\alpha|,|\beta|)$ then $\mathbf{C}^2$ is a Q-algebra. In this section we consider the converse. For elements (α,β) and (α',β') in $\mathbf{C}^2$, we introduce the following three kinds of product.

(1) $(\alpha,\beta)\circ(\alpha',\beta') = (\alpha\alpha',\beta\beta')$,

(2) $(\alpha,\beta)\bullet(\alpha',\beta') = (\alpha\alpha'+\beta\beta',\beta\alpha'+\alpha\beta')$,

(3) $(\alpha,\beta)\times(\alpha',\beta') = (\alpha\alpha',\beta\alpha'+\alpha\beta')$.

Theorem 3. *Let $\mathbf{C}^2$ be an algebra under one of the three kinds of products $\circ,\bullet$ and $\times$.*

(1) $\{\mathbf{C}^2,\ \|(\alpha,\beta)\|\}$ is a Q-algebra under a product $\{\circ\}$ if and only if for some constant $\rho \geq 0$

$$\|(\alpha,\beta)\| = \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2\rho^2 + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2\rho^2 + \left(\frac{|\alpha|-|\beta|}{2}\right)^2}.$$

(2) $\{\mathbf{C}^2,\ \|(\alpha,\beta)\|\}$ is a Q-algebra under a product $\{\bullet\}$if and only if for some constant $\rho \geq 0$

$$\|(\alpha,\beta)\| = \sqrt{|\beta|^2\rho^2 + \left(\frac{|\alpha+\beta|+|\alpha-\beta|}{2}\right)^2} + \sqrt{|\beta|^2\rho^2 + \left(\frac{|\alpha+\beta|-|\alpha-\beta|}{2}\right)^2}.$$

(3) $\{\mathbf{C}^2,\ \|(\alpha,\beta)\|\}$ is a Q-algebra under a product $\{\times\}$ if and only if for some constant $\rho \gneq 0$

$$\|(\alpha,\beta)\| = \sqrt{\left|\frac{\beta}{2}\right|^2\rho^2 + |\alpha|^2} + \left|\frac{\beta}{2}\right|\rho.$$

Proof. (1) Suppose

$$\|(\alpha,\beta)\| = \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2\rho^2 + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2\rho^2 + \left(\frac{|\alpha|-|\beta|}{2}\right)^2}$$

for some $\rho \geq 0$. Then $(1,0)\circ(1,0) = (1,0)$, $(0,1)\circ(0,1) = (0,1), (1,0)+(0,1) = (1,1)$ and $\|(1,0)\| = \|(0,1)\| = \sqrt{\rho^2+1}$. If P and Q are projections on a Hilbert space with $P+Q = I$, and $\|P\| = \|Q\| = \sqrt{\rho^2+1}$, then $\{\mathbf{C}^2,\ \circ,\ \|(\alpha,\beta)\|\}$ is isometrically isomorphic to $\{\alpha P+\beta Q\ ;\ \alpha,\beta\in\mathbf{C}\}$. By Theorems 1 and 2, $\{\mathbf{C}^2,\ \circ,\ \|(\alpha,\beta)\|\}$ is a Q-algebra. Conversely if $\{\mathbf{C}^2, \|(\alpha,\beta)\|\}$ is a Q-algebra under a product $\{\circ\}$, by a theorem of Cole [2] it is isometrically isomorphic to $\{\alpha P+\beta Q\ ;\ \alpha,\beta\in\mathbf{C}\}$ where P and Q are projections on a Hilbert space, and $P + Q = I$. Theorem 1 determines the norm $\|(\alpha, \beta)\|$. (2) and (3) are can be shown similarly to (1).

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