A Class of Deformations of the Schrödinger Representation of the Heisenberg Commutation Relation and Exact Solutions to a Heisenberg Equation and a Schrödinger Equation

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A Class of Deformations of the Schrödinger Representation of the Heisenberg Commutation Relation and Exact Solutions to a Heisenberg Equation and a Schrödinger Equation

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Abstract

We consider a class of representations $\{(H_V, F_V)\}_V$ of the Heisenberg commutation relation ($H_V F_V - F_V H_V = -i$) which are deformations of the Schrödinger representation, where the index parameter V is in a class of real-valued C^∞-functions on $\mathbf{R}$. The Schrödinger representation is given by the case $V \equiv 1$. We classify the representations in terms of properties of V and show that they are divided into three subclasses. All elements of one of these subclasses are unitarily equivalent to the Schrödinger representation. But the others contain no elements unitarily equivalent to the Schrödinger representation. In particular a subclass consists of representations in each of which H_V has no self-adjoint extension. Moreover, we obtain exact solutions to the Heisenberg equation and the Schrödinger equation associated with H_V. We show that, even in the case where H_V has no self-adjoint extension, exact solutions *local in time* can be constructed.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. Let P and Q be symmetric operators on $\mathcal{H}$ satisfying (i) $\mathcal{D} \subset D(PQ) \cap D(QP)$ ($D(\cdot)$ denotes operator domain)

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and (ii) the Heisenberg commutation relation (HCR)

$$[P, Q] := PQ - QP = -i \tag{1.1}$$

on $\mathcal{D}$. Then the triple $\{\mathcal{H}, \mathcal{D}, (P, Q)\}$ is called a representation of the HCR. A standard representation of the HCR is the Schrödinger representation $\{L^2(\mathbf{R}), C_0^\infty(\mathbf{R}), (p, q)\}$, where

$$p = -iD_x, \quad q = x, \tag{1.2}$$

with D_x being the generalized differential operator in the variable $x \in \mathbf{R}$. The operators p and q are called the momentum operator and the position operator respectively.

In this paper we present a class of representations $\pi_V := \{L^2(\mathbf{R}), C_0^\infty(\mathbf{R}), (H_V, F_V)\}$ of the HCR which are deformations of the Schrödinger representation, where the index parameter V is a real-valued C^∞-function on $\mathbf{R}$ and F_V is a multiplication operator essentially self-adjoint on $C_0^\infty(\mathbf{R})$. The Schrödinger representation is given by the case $V \equiv 1$. We classify the representations in terms of properties of V. We show that the family $\{\pi_V\}_V$ contains three kinds of represnetations. One of them consists of representations unitarily equivalent to the Schrödinger representation. But the others contain no elements unitarily equivalent to the Schrödinger representation. In particular, a subclass of representations appear in each of which H_V has no self-adjoint extension. In this sense the family $\{\pi_V\}_V$ contains a class of singular perturbations of the Schrödinger representation. The family $\{\pi_V\}_V$ is not so complicated and rather tractable. But it is interesting to see that, even in such simple deformations of the Schrödinger representation, a variety of representations appear.

One of the motivations for this work comes from a paper [1], which formally discusses a method using representations of the HCR to construct exact operator solutions to the Heisenberg equation associated with a given quantum mechanical Hamiltonian. To explain the idea of the method briefly, let $H = H(p, q)$ be a symmetric operator on $L^2(\mathbf{R})$, which may be a quantum mechanical Hamiltonian. Then one can consider the Heisenberg equation

$$\frac{dA(t)}{dt} = i[H, A(t)] \tag{1.3}$$

associated with H, where $A(t)$ is an unknown operator to be sought for ($t \in \mathcal{I}$; $\mathcal{I}$ is an interval of $\mathbf{R}$) and the equation should be taken as an operator equation on a suitable domain $\mathcal{D}$ of $L^2(\mathbf{R})$ [$\mathcal{D} \subset D(HA(t)) \cap D(A(t)H)$ and, for all $\psi \in \mathcal{D}$, $A(t)\psi$ is strongly differentiable in t]. Suppose that H is self-adjoint. Then we can define

$$q_H(t) := e^{itH} q e^{-itH}, \quad p_H(t) := e^{itH} p e^{-itH}, \quad t \in \mathbf{R},$$

the Heisenberg operators of the position and the momentum operators. Under suitable conditions, $q_H(t)$ and $p_H(t)$ satisfy (1.3). Moreover, suppose that there exists an operator $F := F(q,p)$ such that (H,F) is a representation of the HCR:$[H,F] = -i$ on a dense domain. Then, formally,

$$\frac{dF(q_H(t), p_H(t))}{dt} = ie^{itH}[H,F]e^{-itH} = 1.$$

Hence we have a formal solution $F(q_H(t), p_H(t)) = F(q,p) + t$. On the other hand, $H(q_H(t), p_H(t)) = H(p,q)$. Solving these two equations, one may obtain exact forms for $q_H(t)$ and $p_H(t)$. Of course, this discussion is only heuristic. The present paper examine this method with the class $\{\pi_V\}_V$ and construct exact solutions to (1.3) with $H = H_V$. In particular, we show that, even in the case where H_V has no self-adjoint extension, exact solutions *local in time* to (1.3) with $H = H_V$ can be constructed. To our best knowledge, the concept of a solution *local in time* to Heisenberg equations is new (usually only solutions global in time are considered). We also present exact solutions to the Schrödinger equation associated with H_V.

The outline of the present paper is as follows. In Section 2, we define the class $\{\pi_V\}_V$ and prove basic properties of it. Section 3 concerns a classification and a structure of $\{\pi_V\}_V$. In Section 4 we discuss self-adjointness of H_V, giving a complete characterization of its essential self-adjointness on $C_0^\infty(\mathbf{R})$ (resp. existence of self-adjoint extensions, non-existence of self-adjoint extensions) in terms of properties of V. In Section 5 we introduce a concept of regularity (resp. semi-regularity, singularity) of representations of the HCR as well as a continuity of π_V in V and classify π_V by these concepts. We see by some examples that regularity (resp. semi-regularity, singularity) does not necessarily continue to hold under deformations of V. In Section 6 we construct exact solutions to the Heisenberg equation (1.3) with $H = H_V$. In the last section we give exact solutions to the Schrödinger equation associated with H_V.

2 A Class of Representations of the HCR

We denote by $C^\infty_{\mathrm{real}}(\mathbf{R})$ the set of real-valued infinitely differentiable functions on $\mathbf{R}$. For each $V \in C^\infty_{\mathrm{real}}(\mathbf{R})$, we define an operator

$$H_V := \frac{pV + Vp}{2} \tag{2.1}$$

acting on $L^2(\mathbf{R})$. This operator is symmetric with

$$H_V C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R}). \tag{2.2}$$

If V is the constant function

$$V_c(x) = 1, \quad x \in \mathbf{R}, \tag{2.3}$$

then $H_{V_c} = p$. Hence H_V is a perturbation of the momentum operator p.

Lemma 2.1 *Let F be an absolutely continuous function on $\mathbf{R}$ such that its generalized derivative F' is in $L^2_{\mathrm{loc}}(\mathbf{R})$. Then, for all $\psi \in C_0^\infty(\mathbf{R})$, $[F, H_V]\psi = i\psi$ if and only if $V(x) \neq 0$ for a.e. $x \in \mathbf{R}$ and $F' = 1/V$.*

Proof. It is easy to see that $C_0^\infty(\mathbf{R}) \subset D(H_V F) \cap D(F H_V)$ and, for all $\psi \in C_0^\infty(\mathbf{R})$, $[F, H_V]\psi = iF'V\psi$, from which the assertion immediately follows. ■

In view of Lemma 2.1, we introduce a class of C^∞-functions on $\mathbf{R}$:

$$\mathsf{V} := \left\{ V \in C^\infty_{\mathrm{real}}(\mathbf{R}) |\ 1/V \in L^2_{\mathrm{loc}}(\mathbf{R}) \right\}. \tag{2.4}$$

and, for each $V \in \mathsf{V}$, we define a function

$$F_V(x) := \int_0^x \frac{1}{V(y)} dy \tag{2.5}$$

on $\mathbf{R}$. This function is real and continuous on $\mathbf{R}$. Hence it defines uniquely a self-adjoint multiplication operator on $L^2(\mathbf{R})$. It is obvious that

$$C_0^\infty(\mathbf{R}) \subset D(F_V). \tag{2.6}$$

Note that

$$F_{V_c} = q \tag{2.7}$$

Hence F_V is a perturbaiton of q.

Proposition 2.2 *Let $V \in \mathsf{V}$. Then:*

(i) *$C_0^\infty(\mathbf{R}) \subset D(F_V H_V) \cap D(H_V F_V)$ and*

$$[F_V, H_V] = i \quad \textit{on } C_0^\infty(\mathbf{R}). \tag{2.8}$$

(ii) *For all $\psi \in C_0^\infty(\mathbf{R})$ and $t \in \mathbf{R}$, $e^{itF_V}\psi \in D(H_V)$ and*

$$H_V e^{itF_V}\psi = e^{itF_V}(H_V + t)\psi. \tag{2.9}$$

Proof. (i) This follows from Lemma 2.1.
(ii) Let $\psi \in C_0^\infty(\mathbf{R})$. It is easy to see that $e^{itF_V}\psi$ is differentiable a.e. and

$$D_x(e^{itF_V}\psi) = \frac{it}{V} e^{itF_V}\psi + e^{itF_V} D_x\psi \in L^2(\mathbf{R}),$$

from which it follows that $e^{itF_V}\psi \in D(Vp)$ with

$$Vpe^{itF_V}\psi = te^{itF_V}\psi + e^{itF_V} Vp\psi.$$

Similarly we see that $e^{itF_V}\psi \in D(pV)$ with

$$pVe^{itF_V}\psi = te^{itF_V}\psi + e^{itF_V} pV\psi.$$

Thus $e^{itF_V}\psi \in D(H_V)$ and (2.9) follows. ■

Proposition 2.2-(i) shows that, for all $V \in \mathsf{V}$, the triple

$$\pi_V := \{L^2(\mathbf{R}), C_0^\infty(\mathbf{R}), (H_V, F_V)\} \tag{2.10}$$

is a representation of the HCR with π_{V_c} being the Schrödinger representation. Hence $\{\pi_V\}_{V\in\mathsf{V}}$ gives a family of representations of the HCR as perturbations of the Schrödinger representation.

For a linear operator T on a Hilbert space $\mathcal{H}$, its numerical range $\Theta(T)$ is defined by

$$\Theta(T) := \{(u, Tu)_{\mathcal{H}} | u \in D(T), \|u\|_{\mathcal{H}} = 1\}, \tag{2.11}$$

where $(\cdot, \cdot)_{\mathcal{H}}$ (resp. $\|\cdot\|_{\mathcal{H}}$) denotes the inner product (resp. norm) of $\mathcal{H}$ (we often omit the subscript $\mathcal{H}$ in them).

Proposition 2.3 *Let $V \in C^\infty_{\mathrm{real}}(\mathbf{R})$. Then*

$$\Theta(H_V) = \mathbf{R}. \tag{2.12}$$

In particular, H_V is not semi-bounded (i.e., neither bounded from above nor bounded from below).

Proof. For a real-valued function $f \in C_0^\infty(\mathbf{R})$ $(f \neq 0)$, $s \in \mathbf{R}$, and $\varepsilon > 0$, we define

$$\psi_{f,s,\varepsilon}(x) := (V(x)^2 + \varepsilon^2)^{-1/4} e^{isF_V(x)} f(x).$$

Then $\psi_{f,s,\varepsilon} \in D(H_V)$ and

$$H_V\psi_{f,s,\varepsilon} = s\psi_{f,s,\varepsilon} - \frac{i}{2}\frac{\varepsilon^2}{V^2+\varepsilon^2}V'\psi_{f,s,\varepsilon} - iV(V^2+\varepsilon^2)^{-1/4}e^{isF_V}f'.$$

Using this fact and the reality of f, we obtain

$$(\psi_{f,s,\varepsilon}, H_V\psi_{f,s,\varepsilon})_{L^2(\mathbf{R})} = s\|\psi_{f,s,\varepsilon}\|^2_{L^2(\mathbf{R})}.$$

Since $s \in \mathbf{R}$ is arbitrary, (2.12) follows. ■

We want to make a remark on a property of H_V which may be related to Proposition 2.3. Let

$$\mathsf{V}_+ := \{V \in C^\infty(\mathbf{R}) | V(x) > 0, \ \forall x \in \mathbf{R}\}. \tag{2.13}$$

Then $\mathsf{V}_+ \subset \mathsf{V}$. For each $z \in \mathbf{C}$ and $V \in \mathsf{V}_+$, we define a function

$$\psi_z(x) := \frac{e^{izF_V(x)}}{\sqrt{V(x)}}, \tag{2.14}$$

which is in $C^\infty(\mathbf{R})$.

Let

$$H_{V,0} := H_V | C_0^\infty(\mathbf{R}). \tag{2.15}$$

Proposition 2.4 *Let $z \in \mathbf{C}$ and $V \in \mathsf{V}_+$. Then:*

(i) *For all $f \in C_0^\infty(\mathbf{R})$,*

$$\int_{\mathbf{R}} \overline{\psi_z(x)}(H_V f)(x)dx = z\int_{\mathbf{R}} \overline{\psi_z(x)} f(x)dx. \tag{2.16}$$

(ii) *If $e^{-2\Im z F_V}/V \in L^1(\mathbf{R})$, then $\psi_z \in D(H_{V,0}^*)$ and*

$$H_{V,0}^* \psi_z = \bar{z}\psi_z. \tag{2.17}$$

Proof. (i) This follows from direct computation using by integration by parts.

(ii) If $e^{-2\Im z F_V}/V \in L^1(\mathbf{R})$, then $\psi_z \in L^2(\mathbf{R})$. Hence (2.16) can be written as $(\psi_z, H_{V,0}f)_{L^2(\mathbf{R})} = (\bar{z}\psi_z, f)_{L^2(\mathbf{R})}$ for all $f \in D(H_{V,0})$. This implies that $\psi_z \in D(H_{V,0}^*)$ and (2.17). ∎

Schmüdgen introduced a class of representations of the HCR:

Definition 2.5 [5, Definition 1 in §2] Let P be a symmetric operator defined on a dense domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ and Q be a self-adjoint operator on $\mathcal{H}$ such that $\mathcal{D} \subset D(Q)$. The triple $\{\mathcal{H}, \mathcal{D}, (P, Q)\}$ is said to be in the class $\mathcal{K}(\mathcal{H})$ if the following conditions are satisfied:

(i) $Q\mathcal{D} \subset \mathcal{D}$, $P\mathcal{D} \subset \mathcal{D}$.

(ii) The Op$*$-algebra generated by $Q|\mathcal{D}, P$ and identity I is closed on $\mathcal{D}$.

(iii) $e^{itQ}\psi \in \mathcal{D}$ and $Pe^{itQ}\psi = e^{itQ}(P+t)\psi$ for all $\psi \in \mathcal{D}$ and $t \in \mathbf{R}$.

For a closable operator A, we denote its closure by $\bar{A}$.

Proposition 2.6 *Let $V \in \mathsf{V}_+$ and $\mathcal{A}_V$ be the Op$*$-algebra generated by $H_{V,0}$, $F_V|C_0^\infty(\mathbf{R})$ and I. Let*

$$\mathcal{D}_V := \cap_{A\in\mathcal{A}_V} D(\bar{A}). \tag{2.18}$$

Then $\{L^2(\mathbf{R}), \mathcal{D}_V, (\overline{H}_{V,0}|\mathcal{D}_V, F_V)\}$ is in $\mathcal{K}(L^2(\mathbf{R}))$.

Proof By (2.2), $C_0^\infty(\mathbf{R})$ is a dense invariant domain of H_V. For $V \in \mathsf{V}_+$, F_V is $C^\infty(\mathbf{R})$, so that $F_V C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R})$ and $e^{itF_V}C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R})$. By these facts and (2.9), we can apply [5, Lemma 2 in §2] (cf. also [6, p.671]) to establish the desired results. ∎

Remark 2.1 By Proposition 2.6, all the abstract results established in [5, 6] can be applied to to the representation $(\overline{H}_{V,0}|\mathcal{D}_V, F_V)$. But here we do not go into the details. In the present paper we want to discuss aspects the abstract theory does not cover.

3 A Classification and a Structure of the Representation π_V

We may classify the family $\{\pi_V\}_{V \in \mathsf{V}_+}$ in the following manner:

$$\mathcal{C}_1 := \{\pi_V | V \in \mathsf{V}_+, F_V \text{ is not semi-bounded}\}, \tag{3.1}$$
$$\mathcal{C}_2 := \{\pi_V | V \in \mathsf{V}_+, F_V \text{ is bounded from below,} \tag{3.2}$$
$$\text{but not bounded from above}\}, \tag{3.3}$$
$$\mathcal{C}_3 := \{\pi_V | V \in \mathsf{V}_+, F_V \text{ is bounded from above,} \tag{3.4}$$
$$\text{but not bounded from below}\}, \tag{3.5}$$
$$\mathcal{C}_4 := \{\pi_V | V \in \mathsf{V}_+, F_V \text{ is bounded}\}. \tag{3.6}$$

We want to identify these subclasses more concretely. For each $V \in \mathsf{V}_+$, we can define

$$a_V := \lim_{x \to -\infty} \int_0^x \frac{1}{V(y)} dy, \quad b_V := \lim_{x \to \infty} \int_0^x \frac{1}{V(y)} dy, \tag{3.7}$$

with $0 < b_V \leq \infty$ and $-\infty \leq a_V < 0$. For a linear operator A, we denote its spectrum by $\sigma(A)$.

Proposition 3.1 *Let $V \in \mathsf{V}_+$. Then:*

(i) *The function F_V is strictly monotone increasing and*

$$\overline{\{F_V(x) | x \in \mathbf{R}\}} = [a_V, b_V], \tag{3.8}$$

where $[a_V, b_V] := (-\infty, b_V]$ for $a_V = -\infty, b_V < \infty$, $[a_V, b_V] := [a_V, \infty)$ for $a_V > -\infty, b_V = \infty$, and $[a_V, b_V] := \mathbf{R}$ for $a_V = -\infty, b_V = \infty$. In particular,

$$\sigma(F_V) = [a_V, b_V]. \tag{3.9}$$

(ii) *The operator F_V is bounded if and only if $1/V \in L^1(\mathbf{R})$.*

(iii) *The operator F_V is bounded from below if and only if $\int_{-\infty}^0 V(x)^{-1} dx < \infty$.*

(iv) *The operator F_V is bounded from above if and only if $\int_0^\infty V(x)^{-1} dx < \infty$.*

Proof. An easy exercise. ∎

By Proposition 3.1, we have the following characterization of each subclass $\mathcal{C}_j$:

$$\mathcal{C}_1 := \{\pi_V | V \in \mathsf{V}_+, \int_{-\infty}^0 V(x)^{-1} dx = \infty, \int_0^\infty V(x)^{-1} dx = \infty\}, \tag{3.10}$$
$$\mathcal{C}_2 := \{\pi_V | V \in \mathsf{V}_+, \int_{-\infty}^0 V(x)^{-1} dx < \infty \int_0^\infty V(x)^{-1} dx = \infty\}, \tag{3.11}$$
$$\mathcal{C}_3 := \{\pi_V | V \in \mathsf{V}_+, \int_{-\infty}^0 V(x)^{-1} dx = \infty, \int_0^\infty V(x)^{-1} dx < \infty\}, \tag{3.12}$$
$$\mathcal{C}_4 := \{\pi_V | V \in \mathsf{V}_+, 1/V \in L^1(\mathbf{R})\}. \tag{3.13}$$

In particular, each subclass $\mathcal{C}_j$ $(j = 1, 2, 3, 4)$ is not empty. It is obvious that each representation $\pi_V \in \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ is not unitarily equivalent to the Schrödinger representation (p, q).

In connection with this classification, we recall a definition:

Definition 3.2 [2, Definition 1.1] Let P, Q be symmetric operators on a Hilbert space $\mathcal{H}$ and $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. The pair (P, Q) is said to be in the class $\mathcal{K}_{\rm DD}(\mathcal{H})$ if the following conditions are satisfied:

(D.1) $\mathcal{D} \subset D(P) \cap D(Q)$ and $P\mathcal{D} \subset \mathcal{D}, Q\mathcal{D} \subset \mathcal{D}$.

(D.2) $PQ - QP = -i$ on $\mathcal{D}$.

(D.3) $P = \overline{P|\mathcal{D}}, Q = \overline{Q|\mathcal{D}}$.

(D.4) Q is bounded.

Remark 3.1 The class $\mathcal{K}_{\rm DD}(\mathcal{H})$ is included in $\mathcal{K}(\mathcal{H})$ [5, §2.5].

The following proposition follows from Proposition 2.2-(i), (3.6) and (3.13):

Proposition 3.3 *Let $V \in \mathsf{V}_+$ and*

$$\widehat{H}_V := \overline{H}_{V,0}. \tag{3.14}$$

Then the pair $(\widehat{H}_V, F_V)$ is in $\mathcal{K}_{\rm DD}(L^2(\mathbf{R}))$ with $\mathcal{D} = C_0^\infty(\mathbf{R})$ if and only if $1/V \in L^2(\mathbf{R})$.

Schmüdgen established a general structure theorem for each representation of the HCR in the class $\mathcal{K}(\mathcal{H})$ (Definition 2.5) [5, Theorem 1 in §4]. It may be instructive to see what it is like in the present representation π_V.

Let $V \in \mathsf{V}_+$ and define a linear operator $U_V : L^2(\mathbf{R}) \to L^2(a_V, b_V)$ by

$$(U_V\psi)(y) := \sqrt{V(F_V^{-1}(y))}\psi(F_V^{-1}(y)), \quad \text{a.e.} y \in (a_V, b_V), \tag{3.15}$$

where $F_V^{-1}(\cdot)$ is the inverse function of F_V. Then it is easy to see that U_V is unitary with

$$U_V^{-1}\phi = \frac{\phi(F_V(\cdot))}{\sqrt{V}}, \quad \phi \in L^2(a_V, b_V). \tag{3.16}$$

and

$$U_V C_0^\infty(\mathbf{R}) = C_0^\infty(a_V, b_V). \tag{3.17}$$

Proposition 3.4 *Let $V \in \mathsf{V}_+$. Then, for all $\phi \in C_0^\infty(a_V, b_V)$,*

$$(U_V F_V U_V^{-1}\phi)(y) = y\phi(y), \tag{3.18}$$

$$(U_V H_V U_V^{-1}\phi)(y) = -i\phi'(y), \quad y \in \mathbf{R}. \tag{3.19}$$

Proof. It is straightforward to show (3.18). For all $\phi \in U_V C_0^\infty(\mathbf{R})$, we can show that

$$(U_V p U_V^{-1}\phi)(y) = \frac{1}{V(F_V^{-1}(y))}\left\{-i\phi'(y) + \frac{i}{2}V'(F_V^{-1}(y))\phi(y)\right\},$$
$$(U_V V U_V^{-1}\phi)(y) = V(F_V^{-1}(y)), \quad y \in (a_V, b_V).$$

Using these relations, we obtain (3.19). ∎

Proposition 3.4 shows that the unitary operator U_V transforms (H_V, F_V) to $(-iD_y, y)$ such that the V-dependence of the representation is with the Hilbert space $L^2((a_V, b_V))$ and boundary conditions of $-iD_y$ in the interval (a_V, b_V).

4 Self-Adjointness of H_V

In this section we consider self-adjointness of H_V. For each $V \in \mathsf{V}_+$, we define functions

$$G_V^\pm := \frac{1}{\sqrt{V}} e^{\pm F_V}. \tag{4.1}$$

We introduce subsets of V_+:

$$\begin{aligned}
\mathsf{V}_1 &:= \{V \in \mathsf{V}_+ | G_V^\pm \notin L^2(\mathbf{R})\}, &(4.2)\\
\mathsf{V}_2 &:= \{V \in \mathsf{V}_+ | G_V^+ \notin L^2(\mathbf{R}), G_V^- \in L^2(\mathbf{R})\}, &(4.3)\\
\mathsf{V}_3 &:= \{V \in \mathsf{V}_+ | G_V^+ \in L^2(\mathbf{R}), G_V^- \notin L^2(\mathbf{R})\}, &(4.4)\\
\mathsf{V}_4 &:= \{V \in \mathsf{V}_+ | G_V^\pm \in L^2(\mathbf{R})\}. &(4.5)
\end{aligned}$$

Theorem 4.1 *Let $V \in \mathsf{V}_+$. Then:*

(i) *H_V is essentially self-adjoint on $C_0^\infty(\mathbf{R})$ if and only $V \in \mathsf{V}_1$.*

(ii) *H_V is not essentially self-adjoint on $C_0^\infty(\mathbf{R})$ and has self-adjoint extensions if and only $V \in \mathsf{V}_4$.*

(iii) *$\widehat{H}_V$ is a non-self-adjoint maximal symmetric operator if and only $V \in \mathsf{V}_2 \cup \mathsf{V}_3$.*

To prove this theorem, we establish a lemma.

Lemma 4.2 *Let $V \in \mathsf{V}_+$. Then:*

(i) $\ker(H_{V,0}{}^* + i) \neq \{0\}$ *if and only if $G_V^+ \in L^2(\mathbf{R})$. In that case,*

$$\ker(H_{V,0}{}^* + i) = \{\alpha G_V^+ | \alpha \in \mathbf{C}\}. \tag{4.6}$$

(ii) $\ker(H_{V,0}{}^* - i) \neq \{0\}$ *if and only if $G_V^- \in L^2(\mathbf{R})$. In that case,*

$$\ker(H_{V,0}{}^* - i) = \{\alpha G_V^- | \alpha \in \mathbf{C}\}. \tag{4.7}$$

Proof. Let $\ker(H_{V,0}{}^* \pm i) \neq \{0\}$ and $\psi_\pm \in \ker(H_{V,0}{}^* \pm i) \setminus \{0\}$. Then we have

$$D_x \psi_\pm = \frac{-\frac{V'}{2} \pm 1}{V} \psi_\pm \tag{4.8}$$

in $\mathcal{D}'(\mathbf{R})$ (the space of distributions on $\mathbf{R}$). Note that

$$\frac{-\frac{V'}{2} \pm 1}{V} \psi_\pm \in L^2_{\mathrm{loc}}(\mathbf{R}).$$

which implies that $D_x \psi_\pm \in L^1_{\mathrm{loc}}(\mathbf{R})$. Hence $\psi_\pm$ can be indetified with continuous functions on $\mathbf{R}$. It is not difficult to show that every solution of (4.8) in $\mathcal{D}'$ has the form $\psi_\pm = c_\pm G_V^\pm$ with $c_\pm \in \mathbf{C}$ being constants. Hence $G_V^\pm \in L^2(\mathbf{R})$.

Conversely, let $G_V^\pm \in L^2(\mathbf{R})$. Then it is easy to see that $G_V^\pm \in \ker(H_{V,0}{}^* \pm i)$. Hence $\ker(H_{V,0}{}^* \pm i) \neq \{0\}$.

The proof just given above implies (4.6) and (4.7) too. ■

Proof of Theorem 4.1

(i) By a general criterion (e.g., [3, p.257, Corollary]), H_V is essentially self-adjoint on $C_0^\infty(\mathbf{R})$ if and only $\ker(H_{V,0}{}^* \pm i) = \{0\}$. By Lemma 4.2, the latter condition is equivalent to $G_V^\pm \notin L^2(\mathbf{R})$.

(ii) By a general criterion (e.g., [4, p.141, Corollary]), H_V is not essentially self-adjoint on $C_0^\infty(\mathbf{R})$ and has self-adjoint extensions if and only $\dim\ker(H_{V,0}{}^* + i) = \dim\ker(H_{V,0}{}^* - i) \neq 0$. By Lemma 4.2, the latter condition is equivalent to $G_V^\pm \in L^2(\mathbf{R})$.

(iii) By a general criterion (e.g., [4, p.141, Corollary]), $\widehat{H}_V$ is a non-self-adjoint maximal symmetric operator if and only either $\dim\ker(H_{V,0}{}^* + i) = 0, \dim\ker(H_{V,0}{}^* - i) \neq 0$ or $\dim\ker(H_{V,0}{}^* + i) \neq 0, \dim\ker(H_{V,0}{}^* - i) = 0$. By Lemma 4.2, the latter condition is equivalent to either $G_V^+ \notin L^2(\mathbf{R}), G_V^- \in L^2(\mathbf{R})$ or $G_V^+ \in L^2(\mathbf{R}), G_V^- \notin L^2(\mathbf{R})$.
■

Remark 4.1 *Theorem 4.1 can be proved also by using Proposition 3.4.*

We can identify the set V_j $(j = 1, 2, 3)$ more explcitly.

Proposition 4.3

$$\mathsf{V}_1 = \{V \in \mathsf{V}_+ | \int_{-\infty}^0 V(x)^{-1} dx = \infty, \int_0^\infty V(x)^{-1} dx = \infty\}, \tag{4.9}$$

$$\mathsf{V}_2 = \{V \in \mathsf{V}_+ | \int_{-\infty}^0 V(x)^{-1} dx < \infty, \int_0^\infty V(x)^{-1} dx = \infty\}, \tag{4.10}$$

$$\mathsf{V}_3 = \{V \in \mathsf{V}_+ | \int_{-\infty}^0 V(x)^{-1} dx = \infty, \int_0^\infty V(x)^{-1} dx < \infty\}, \tag{4.11}$$

$$\mathsf{V}_4 = \{V \in \mathsf{V}_+ | 1/V \in L^1(\mathbf{R})\}. \tag{4.12}$$

Proof. By the change of variable $y = F_V(x)$, we have

$$\int_{\mathbf{R}} |G_V^{\pm}(x)|^2 dx = \int_{a_V}^{b_V} e^{\pm 2y} dy,$$

from which the assertion immediately follows. ■

5 Regularity

We introduce a definition on representation of the HCR.

Definition 5.1 Let $\pi := \{\mathcal{H}, \mathcal{D}, (P, Q)\}$ be a representation of the HCR.

(i) We say that π is regular if P and Q are essentially self-adjoint on $\mathcal{D}$.

(ii) We say that π is semi-regular if both $P|\mathcal{D}$ and $Q|\mathcal{D}$ have self-adjoint extensions.

(iii) We say that π is singular if one of $P|\mathcal{D}$ and $Q|\mathcal{D}$ has no self-adjoint extension.

The following classification of the representation π_V immediately follows from Theorem 4.1.

Proposition 5.2 *Let $V \in \mathsf{V}_+$. Then:*

(i) *π_V is regular if and only $V \in \mathsf{V}_1$.*

(ii) *π_V is not regular, but, semi-regular if and only $V \in \mathsf{V}_4$.*

(iii) *π_V is singular if and only $V \in \mathsf{V}_2 \cup \mathsf{V}_3$.*

The following proposition shows that the representation π_V with $V \in \mathsf{V}_1$ is unitarily equivalent to the Schrödinger representation.

Proposition 5.3 *Let $V \in \mathsf{V}_1$. Then U_V defined by (3.15) is a unitary operator on $L^2(\mathbf{R})$ such that $U_V C_0^\infty(\mathbf{R}) = C_0^\infty(\mathbf{R})$ and*

$$U_V \widehat{H}_V U_V^{-1} = p, \quad U_V F_V U_V^{-1} = q. \tag{5.1}$$

Proof. Since $V \in \mathsf{V}_1$, we have $a_V = -\infty, b_V = \infty$ so that U_V is a unitary operator on $L^2(\mathbf{R})$. It is easy to see that $U_V C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R})$ and $U_V^{-1} C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R})$. Hence $U_V C_0^\infty(\mathbf{R}) = C_0^\infty(\mathbf{R})$. By Proposition 3.4, (5.1) holds on $C_0^\infty(\mathbf{R})$. Since $C_0^\infty(\mathbf{R})$ is a core of q and p, (5.1) follows. ■

We consider continuity of π_V in V.

Proposition 5.4 *Let $V_n, V \in \mathsf{V}_+$ $(n \in \mathbf{N})$ and, for all $R > 0$,*

$$\lim_{n\to\infty} \sup_{|x|\le R} |V_n(x) - V(x)| = 0, \quad \lim_{n\to\infty} \sup_{|x|\le R} |V_n'(x) - V'(x)| = 0. \tag{5.2}$$

Then, for all $f \in C_0^\infty(\mathbf{R})$,

$$\lim_{n\to\infty} \|H_{V_n} f - H_V f\|_{L^2(\mathbf{R})} = 0, \tag{5.3}$$

$$\lim_{n\to\infty} \|F_{V_n} f - F_V f\|_{L^2(\mathbf{R})} = 0,. \tag{5.4}$$

Proof. Let $f \in C_0^\infty(\mathbf{R})$. Then

$$\|H_{V_n} f - H_V f\|_{L^2(\mathbf{R})} \le \frac{1}{2}\|V_n' f - V' f\|_{L^2(\mathbf{R})} + \|(V_n - V)pf\|_{L^2(\mathbf{R})}.$$

Hence (5.3) follows.

We have

$$\|F_{V_n} f - F_V f\|_{L^2(\mathbf{R})}^2 = \int_{-R}^{R} G_n(x)|f(x)|^2 dx,$$

where $R > 0$ is a constant such that $\operatorname{supp} f \subset [-R, R]$ and

$$G_n(x) := \left| \int_0^x \frac{V_n(y) - V(y)}{V_n(y)V(y)} dy \right|^2.$$

We have for all $x \in [-R, R]$

$$G_n(x) \le (\sup_{|y|\le R} |V_n(y) - V(y)|)^2 \left| \int_0^x \frac{1}{V_n(y)V(y)} dy \right|^2.$$

Let $V_0 := \min_{x\in[-R,R]} V(x)$. Then $V_0 > 0$. For each $\varepsilon \in (0, V_0/2)$, there exists an $n_0 \in \mathbf{N}$ such that, for all $n \ge n_0$,

$$|V_n(x) - V(x)| < \varepsilon, \quad x \in [-R, R].$$

Hence

$$V_n(x) > V(x) - \varepsilon > \frac{V_0}{2},$$

from which it follows that

$$G_n(x) \le (\sup_{|y|\le R} |V_n(y) - V(y)|)^2 \frac{4}{V_0^4} R^2, \quad n \ge n_0, \ x \in [-R, R].$$

Thus (5.4) follows. ∎

We mean by $\pi_{V_n} \to \pi_V$ $(n \to \infty)$ that (5.3) and (5.4) hold. This defines a continuity of π_V. As the following examples show, this continuity does not necessarily preserve regularity of π_V.

Example 5.1 Let

$$V(x) = e^x, \quad V_n(x) = \frac{ne^x}{n + e^{2x}}, \quad n \in \mathbf{N}, \ x \in \mathbf{R}.$$

Then $V \in \mathsf{V}_3$ and $V_n \in \mathsf{V}_1$ for all n. Hence, by Proposition 5.2, π_{V_n} is regular and π_V is singular. It is easy to see that (5.2) holds. Hence $\pi_{V_n} \to \pi_V$ $(n \to \infty)$. This shows that a regular representation of the HCR can change to a singular one under a continuous deformation.

Example 5.2 Let

$$V_n(x) = e^{-x/n}, \quad n \in \mathbf{N}, \ x \in \mathbf{R}.$$

Then $V_n \in \mathsf{V}_2$. Hence, by Proposition 5.2, π_{V_n} is singular. Let V_c be the constant function defined by (2.3), so that π_{V_c} is the Schrödinger representation. It is easy to see that (5.2) holds. Hence $\pi_{V_n} \to \pi_{V_c}$ $(n \to \infty)$. This shows that a singular representation of the HCR can change to a regular one under a continuous deformation.

Example 5.3 Let

$$V(x) = 1 + x^2, \quad V_n(x) = \frac{1 + x^2}{\cosh(x/n)} \quad n \in \mathbf{N}, \ x \in \mathbf{R}.$$

Then $V \in \mathsf{V}_4, V_n \in \mathsf{V}_1$. Hence, by Proposition 5.2, π_V is semi-regular and π_{V_n} is regular. It is easy to see that (5.2) holds. Hence $\pi_{V_n} \to \pi_V \, (n \to \infty)$. This shows that a regular representation of the HCR can change to a semi-regular one under a continuous deformation.

Example 5.4 Let

$$V(x) = e^{-x}, \quad V_n(x) = e^{-x} e^{x^2/n}, \quad n \in \mathbf{N}, \ x \in \mathbf{R}.$$

Then $V \in \mathsf{V}_2, V_n \in \mathsf{V}_4$. Hence, by Proposition 5.2, π_V is singular and π_{V_n} is semi-regular. It is easy to see that (5.2) holds. Hence $\pi_{V_n} \to \pi_V \, (n \to \infty)$. This shows that a semi-regular representation of the HCR can change to a singular one under a continuous deformation.

6 Exact Solutions to a Heisenberg Equation

In this section, by applying the results in the previous sections, we solve the Heisenberg equation

$$\frac{dA(t)}{dt} = i[H_V, A(t)] \tag{6.1}$$

associated with H_V in the cases $A(0) = q, p$. We do this in two cases: (i) $V \in \mathsf{V}_1$; (ii) $V \in \mathsf{V}_2 \cup \mathsf{V}_3$.

6.1 The case $V \in \mathsf{V}_1$

In this subsection we assume that $V \in \mathsf{V}_1$. Then, by Theorem 4.1-(i), $\widehat{H}_V$ is self-adjoint. Hence the Heisenberg operators

$$q(t) := e^{it\widehat{H}_V} q e^{-it\widehat{H}_V}, \quad p(t) := e^{it\widehat{H}_V} p e^{-it\widehat{H}_V}, \tag{6.2}$$

of q and p with the operator $\widehat{H}_V$ can be defined for all $t \in \mathbf{R}$. Obviously we have

$$q(0) = q, \quad p(0) = p. \tag{6.3}$$

In the present case, we can define for all $t \in \mathbf{R}$ the following function on $\mathbf{R}$:

$$\Phi_t(x) := F_V^{-1}(F_V(x) + t), \quad x \in \mathbf{R}. \tag{6.4}$$

We set

$$W_t(x) := \frac{V(x)}{V(\Phi_t(x))}, \quad x \in \mathbf{R}. \tag{6.5}$$

Lemma 6.1 *Let $V \in \mathsf{V}_1$. Then, for all $t \in \mathbf{R}$, $W_t \in \mathsf{V}_1$. In particular, H_{W_t} is essentially self-adjoint on $C_0^\infty(\mathbf{R})$.*

Proof. It is easy to see that $W_t \in \mathsf{V}_+$. We have

$$\frac{d\Phi_t(x)}{dx} = \frac{1}{W_t(x)}. \tag{6.6}$$

Hecne

$$\int_{-\infty}^0 \frac{1}{W_t(x)} dx = \Phi_t(0) - \lim_{R\to\infty} \Phi_t(-R) = \infty,$$
$$\int_0^\infty \frac{1}{W_t(x)} dx = \lim_{R\to\infty} \Phi_t(R) - \Phi_t(0) = \infty.$$

Thus, by Proposition 4.3, $W_t \in \mathsf{V}_1$. By this fact and Theorem 4.1, H_{W_t} is essentially self-adjoint on $C_0^\infty(\mathbf{R})$. ∎

Theorem 6.2 *Let $V \in \mathsf{V}_1$. Then*

$$q(t) = \Phi_t, \tag{6.7}$$
$$p(t) = \widehat{H}_{W_t} \tag{6.8}$$

In particular, $q(t)C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R}), p(t)C_0^\infty(\mathbf{R}) \subset C_0^\infty(\mathbf{R})$ for all $t \in \mathbf{R}$.

Proof. By Proposition 5.3, we have

$$q(t) = U_V^{-1} e^{itp} U_V q U_V^{-1} e^{-itp} U_V.$$

It is easy to see that

$$U_V q U_V^{-1} = F_V^{-1}(q).$$

Hence, by the functional calculus and the well known fact that

$$e^{itp} q e^{-itp} = q + t,$$

we have

$$\begin{aligned} q(t) &= U_V^{-1} F_V^{-1}(q+t) U_V \\ &= F_V^{-1}(U_V^{-1} q U_V + t) \\ &= F_V^{-1}(F_V(q) + t). \end{aligned}$$

Thus (6.7) follows.

By direct computation, we have

$$p = \frac{V^{-1} H_V + H_V V^{-1}}{2}$$

on $C_0^\infty(\mathbf{R})$. Since $C_0^\infty(\mathbf{R})$ is a core of p, we obtain

$$p = \frac{\overline{V^{-1} H_V + H_V V^{-1}}}{2}.$$

Hence

$$\begin{aligned} p(t) &\supset \frac{1}{2}\{e^{itH_V} V^{-1} H_V e^{-itH_V} + e^{itH_V} H_V V^{-1} e^{-itH_V}\} \\ &= \frac{1}{2}\{V(q(t))^{-1} H_V + H_V V(q(t))^{-1}\} \\ &\supset H_{W_t} | C_0^\infty(\mathbf{R}). \end{aligned}$$

By this fact and Lemma 6.1, we obtain (6.8). ∎

We next show that $q(t)$ and $p(t)$ satisfy the Heisenberg equation (6.1).

Theorem 6.3 *Let $V \in \mathsf{V}_1$. Then,*

$$C_0^\infty(\mathbf{R}) \subset D(H_V q(t)) \cap D(q(t) H_V) \cap D(H_V p(t)) \cap D(p(t) H_V) \tag{6.9}$$

and, for all $f \in C_0^\infty(\mathbf{R})$, $q(t)f$ and $p(t)f$ are strongly differentiable in $t \in \mathbf{R}$ satisfying

$$\frac{dq(t)f}{dt} = i[H_V, q(t)]f, \tag{6.10}$$

$$\frac{dp(t)f}{dt} = i[H_V, p(t)]f. \tag{6.11}$$

To prove this theorem, we establish lemmas.

Lemma 6.4 *Let $V \in \mathsf{V}_1$ and $f \in C_0^\infty(\mathbf{R})$. Then:*

(i) *$\Phi_t f$ is strongly differentiable in t and*

$$\frac{d\Phi_t f}{dt} = V(\Phi_t)f, \quad t \in \mathbf{R}. \tag{6.12}$$

(ii) *For all $t \in \mathbf{R}$,*

$$i[H_V, \Phi_t]f = V(\Phi_t)f. \tag{6.13}$$

Proof. (i) Let $\operatorname{supp} f \subset [-R, R]$ $(R > 0)$. Then we have for all $t \in \mathbf{R}$ and $h \in \mathbf{R}$ with $0 < |h| < 1$

$$\begin{aligned}&\left\|\frac{\Phi_{t+h}f - \Phi_t f}{h} - V(\Phi_t)f\right\|_{L^2(\mathbf{R})}^2 \\ &= \int_{-R}^{R}\left|\frac{\Phi_{t+h}(x) - \Phi_t(x)}{h} - V(\Phi_t(x))\right|^2 |f(x)|^2 dx.\end{aligned}$$

It is easy to see that

$$\frac{d\Phi_t(x)}{dt} = V(\Phi_t(x)). \tag{6.14}$$

Hence

$$\left|\frac{\Phi_{t+h}(x) - \Phi_t(x)}{h}\right| \le \max_{t-1\le s\le t+1, x\in[-R,R]} |V(\Phi_s(x))| < \infty.$$

Therefore

$$\left|\frac{\Phi_{t+h}(x) - \Phi_t(x)}{h} - V(\Phi_t(x))\right| \le C_{t,R}, \quad x \in [-R, R],$$

where $C_{t,R}$ is a constant independent of h. Hence, by the Lebesgue dominated convergence theorem, we obtain the desired result.

(ii) We have

$$\begin{aligned} i[H_V, \Phi_t]f &= \frac{i}{2}\{[p, \Phi_t]V + V[p, \Phi_t]\}f \\ &= \frac{1}{2}\left\{\frac{d\Phi_t}{dx}V + V\frac{d\Phi_t}{dx}\right\}f, \end{aligned}$$

which, together with (6.6), yields (6.13). ∎

Lemma 6.5 *Let $V \in \mathsf{V}_1$ and $f \in C_0^\infty(\mathbf{R})$. Then:*

(i) *$H_{W_t}f$ is strongly differentiable in t and*

$$\frac{dH_{W_t}f}{dt} = -\frac{1}{2}\{pW_tV'(\Phi_t) + W_tV'(\Phi_t)p\}f, \quad t \in \mathbf{R}. \tag{6.15}$$

(ii) *For all $t \in \mathbf{R}$,*

$$i[H_V, H_{W_t}]f = -\frac{1}{2}\{pW_tV'(\Phi_t) + W_tV'(\Phi_t)p\}f. \tag{6.16}$$

Proof. (i) We have

$$H_{W_t}f = -\frac{i}{2}W_t'f + W_tpf.$$

In the same way as in the proof of Lemma 6.4-(i), we can show that each term on the right hand side is strongly differentiable in t with

$$\begin{aligned} \frac{dW_t'f}{dt} &= \left(\frac{dW_t}{dt}\right)' f, \\ \frac{dW_tpf}{dt} &= \frac{dW_t}{dt}pf. \end{aligned}$$

Hence $H_{W_t}f$ is strongly differentiable and

$$\frac{dH_{W_t}f}{dt} = \frac{1}{2}\left\{p\frac{dW_t}{dt} + \frac{dW_t}{dt}p\right\}f.$$

It is easy to see that

$$\frac{dW_t}{dt} = -W_tV'(\Phi_t). \tag{6.17}$$

Thus (6.15) follows.

(ii) We have

$$\begin{aligned} i[H_V, H_{W_t}]f &= \frac{i}{4}\{[pV, pW_t] + [pV, W_tp] + [Vp, pW_t] + [Vp, W_tp]\}f \\ &= -\frac{1}{2}\{p(V'W_t - W_t'V) + (W_tV' - W_t'V)p\}f. \end{aligned}$$

It is straightforward to see that

$$V'W_t - W_t'V = W_tV'(\Phi_t). \tag{6.18}$$

Thus (6.16) follows. ■

Proof of Theorem 6.3

The domain property (6.9) is due to Lemma 6.1 and Theorem 6.2. The other statements immediately follows from Lemmas 6.4 and 6.5. ■

Example 6.1 Consider the case where $V(x) = 1/\cosh(ax)$, $x \in \mathbf{R}$ with $a > 0$ a constant. Then $V \in \mathsf{V}_1$ and

$$F_V^{-1}(x) = \frac{1}{a}\text{arcsinh}(ax).$$

Hence

$$q(t) = \frac{1}{a}\text{arcsinh}\left(\sinh(aq) + at\right),$$
$$p(t) = \frac{1}{2}\left\{p\frac{\cosh\left(\text{arcsinh}(\sinh(aq) + at)\right)}{\cosh(aq)} + \frac{\cosh\left(\text{arcsinh}(\sinh(aq) + at)\right)}{\cosh(aq)}p\right\}$$

on $C_0^\infty(\mathbf{R})$.

6.2 The case $V \in \mathsf{V}_2 \cup \mathsf{V}_3$

In this subsection we assume that $V \in \mathsf{V}_2 \cup \mathsf{V}_3$. In this case $\widehat{H}_V$ is not self-adjoint [Theorem 4.1-(iii)]. Hence we can not define the Heisenberg operators $q(t)$ and $p(t)$ in the manner as in (6.2). Nevertheless, as is shown below, we can construct solutions to the Heisenberg equation (6.1) with $A(0) = q, p$ *locally* in time t.

We first consider the case $V \in \mathsf{V}_3$ so that $a_V = -\infty, b_V < \infty$, and

$$F_V(\mathbf{R}) = (-\infty, b_V). \tag{6.19}$$

Hence, if $t \leq 0$, then $F_V(x) + t \in (-\infty, b_V]$ for all $x \in \mathbf{R}$. Therefore we can define the operators

$$q_-(t) := \Phi_t, \tag{6.20}$$
$$p_-(t) := \widehat{H}_{W_t}. \tag{6.21}$$

for all $t \leq 0$. We have

$$q_-(0) = q, \quad p_-(0) = p. \tag{6.22}$$

Proposition 6.6 *Let $V \in \mathsf{V}_3$ and $f \in C_0^\infty(\mathbf{R})$. Then:*

(i) *$q_-(t)f$ and $p_-(t)f$ are strongly differentiable in $t \leq 0$ with*

$$\frac{dq_-(t)f}{dt} = i[H_V, q_-(t)]f, \tag{6.23}$$
$$\frac{dp_-(t)f}{dt} = i[H_V, p_-(t)]f. \tag{6.24}$$

(ii) *For all $t \leq 0$, $\{L^2(\mathbf{R}), C_0^\infty(\mathbf{R}), (p_-(t), q_-(t))\}$ is a representation of the HCR:*

$$[p_-(t), q_-(t)]f = -if. \tag{6.25}$$

For all $t < 0$, this representation is singular.

Proof. (i) The proofs of Lemmas 6.4 and 6.5 work in the present case too yielding (6.12), (6.13), (6.15) and (6.16) with $t \leq 0$. Hence the desired result follows.

(ii) By direct computation, we have

$$[p_-(t), q_-(t)]f = -i\Phi_t' W_t f,$$

which, together with (6.6), yields (6.25). It is easy to see that, in the present case, $W_t \in \mathsf{V}_3$. Hence, by Proposition 5.2, the representation $(p_-(t), q_-(t))$ is singular for all $t < 0$. ■

Remark 6.1 It is easy to see that, for all $f \in C_0^\infty(\mathbf{R})$,

$$\lim_{t\downarrow 0} \|q_-(t)f - qf\|_{L^2(\mathbf{R})} = 0, \quad \lim_{t\downarrow 0} \|p_-(t)f - pf\|_{L^2(\mathbf{R})} = 0.$$

Hence $(p_-(t), q_-(t))$ is a continuous deformation (perturbation) of the Schrödinger representation (q,p). Proposition 6.6-(ii) shows that $(p_-(t), q_-(t))$ with $t < 0$ is an example which drastically changes the property of the unperturbed representation of the HCR.

Example 6.2 Consider the case where $V(x) = e^{ax}$ with a constant $a > 0$. Then $V \in \mathsf{V}_3$ and

$$F_V(x) = \frac{1}{a}\left(1 - e^{-ax}\right),$$

so that $F_V(\mathbf{R}) = (-\infty, 1/a]$. In this case we have by (6.20) and (6.21)

$$q_-(t) = -\frac{1}{a}\log\left(e^{-aq} - at\right),$$
$$p_-(t) = p - \frac{at}{2}\left(pe^{aq} + e^{aq}p\right), \quad t \leq 0.$$

This example with $a = 1$ was formally discussed in [1, Example 1].

We next consider the case $V \in \mathsf{V}_2$ so that $a_V > -\infty, b_V = \infty$, and

$$F_V(\mathbf{R}) = (a_V, \infty). \tag{6.26}$$

Hence, if $t \geq 0$, then $F_V(x) + t \in [a_V, \infty)$ for all $x \in \mathbf{R}$. Therefore we can define the operators

$$q_+(t) := \Phi_t, \tag{6.27}$$
$$p_+(t) := \widehat{H}_{W_t}. \tag{6.28}$$

for all $t \geq 0$. We have

$$q_+(0) = q, \quad p_+(0) = p. \tag{6.29}$$

In the same way as in Proposition 6.6, we can prove the following:

Proposition 6.7 *Let $V \in \mathsf{V}_2$ and $f \in C_0^\infty(\mathbf{R})$. Then:*

(i) *$q_+(t)f$ and $p_+(t)f$ are strongly differentiable in $t \geq 0$ with*

$$\frac{dq_+(t)f}{dt} = i[H_V, q_+(t)]f, \tag{6.30}$$

$$\frac{dp_+(t)f}{dt} = i[H_V, p_+(t)]f. \tag{6.31}$$

(ii) *For all $t \geq 0$, $\{L^2(\mathbf{R}), C_0^\infty(\mathbf{R}), (p_+(t), q_+(t))\}$ is a representation of the HCR:*

$$[p_+(t), q_+(t)]f = -if. \tag{6.32}$$

For all $t > 0$, this representation is singular.

Remark 6.2 Properties similar to those stated in Remark 6.1 hold for $(p_+(t), q_+(t))$ $(t \geq 0)$.

Example 6.3 Consider the case where $V(x) = e^{-ax}$ with a constant $a > 0$. Then $V \in \mathsf{V}_2$ and

$$F_V(x) = \frac{1}{a}(e^{ax} - 1),$$

so that $F_V(\mathbf{R}) = (-1/a, \infty)$. In this case we have for all $t \geq 0$

$$q_+(t) = \frac{1}{a}\log\left(e^{aq} + at\right),$$

$$p_+(t) = p + \frac{at}{2}\left(pe^{-aq} + e^{-aq}p\right).$$

7 Exact Solutions to the Schrödinger Equation with the Operator $\widehat{H}_V$

In this section we show that the Schrödinger equation

$$i\frac{d\psi(t)}{dt} = \widehat{H}_V\psi(t) \tag{7.1}$$

with the operator $\widehat{H}_V$ has exact solutions if $V \in \mathsf{V}_1 \cup \mathsf{V}_2 \cup \mathsf{V}_3$ ($\psi(t) \in D(\widehat{H}_V)$), where the derivative $d\psi(t)/dt$ is taken in strong sense in $L^2(\mathbf{R})$.

7.1 The case $V \in \mathsf{V}_1$

Let $V \in \mathsf{V}_1$ and ψ be a Borel measurable function on $\mathbf{R}$. Then, for each $t \geq 0$, we can define a function $\psi(t)$ on $\mathbf{R}$ by

$$\psi(t)(x) := \sqrt{\frac{V(F_V^{-1}(F_V(x) - t))}{V(x)}}\,\psi(F_V^{-1}(F_V(x) - t)), \quad \text{a.e. } x \in \mathbf{R}. \tag{7.2}$$

Theorem 7.1 *Let $V \in \mathsf{V}_1$ and $\psi \in D(pU_V)$. Then, for all $t \in \mathbf{R}$, $\psi(t)$ is in $D(\widehat{H}_V)$ and a solution to (7.1) with $t \in \mathbf{R}$. Moreover, if $\psi \in C_0^\infty(\mathbf{R})$, then $\psi(t) \in C_0^\infty(\mathbf{R})$ for all $t \in \mathbf{R}$.*

Proof. Let $\phi \in D(\widehat{H}_V)$. Then, by Theorem 4.1-(i), the vector $\phi(t) := e^{-it\widehat{H}_V}\phi$ is a solution to (7.1) with $t \in \mathbf{R}$. By Proposition 5.3, $\psi \in D(\widehat{H}_V)$ if and only if $\psi \in D(pU_V)$ and we have for all $t \in \mathbf{R}$ and $\psi \in D(pU_V)$

$$e^{-it\widehat{H}_V}\psi = U_V^{-1}e^{-itp}U_V\psi.$$

By the well known formula $(e^{-itp}f)(x) = f(x-t)$, a.e. $x, f \in L^2(\mathbf{R})$, one sees that

$$(U_V^{-1}e^{-itp}U_V\psi)(x) = \sqrt{\frac{V(F_V^{-1}(F_V(x)-t))}{V(x)}}\psi(F_V^{-1}(F_V(x)-t)).$$

Hence the first one of the desired results follows. The assertion on the C_0^∞-regularity of $\psi(t)$ easily follows from (7.2). ■

7.2 The case $V \in \mathsf{V}_2 \cup \mathsf{V}_3$

In this case we have exact solutions to (7.1) local in time t.

We first consider the case $V \in \mathsf{V}_3$, so that $a_V = -\infty, b_V < \infty$, and (6.19) holds.

Theorem 7.2 *Let $V \in \mathsf{V}_3$ and $\psi \in C_0^k(\mathbf{R})$ $(k \geq 1)$. We define for all $t \in [0, \infty)$*

$$\psi_+(t)(x) := \sqrt{\frac{V(F_V^{-1}(F_V(x)-t))}{V(x)}}\psi(F_V^{-1}(F_V(x)-t)), \quad x \in \mathbf{R}. \tag{7.3}$$

Then $\psi_+(t)$ is strongly differentiable in $t \geq 0$ and is a solution to (7.1). Moreover, for all $t \in (0, \infty)$, $\psi(t)$ is in $C^k(\mathbf{R})$.

Proof. For $t \geq 0$, we define an operator S_t on $L^2(-\infty, b_V)$ by

$$(S_tf)(x) := f(x-t), \quad \text{a.e. } x \in (-\infty, b_V), f \in L^2(-\infty, b_V).$$

Then S_t is bounded. We can write $\psi_+(t) = U_V^{-1}S_tU_V\psi$. It is easy to see that, for all $f \in C_0^k(\mathbf{R})$ $(k \geq 1)$, S_tf is strongly differentiable in $t > 0$ and $dS_tf/dt = -S_tf' = -(S_tf)'$. Thus the disired result follows. ■

We next consider the case $V \in \mathsf{V}_2$, so that $a_V > -\infty, b_V = \infty$, and (6.26) holds.

Theorem 7.3 *Let $V \in \mathsf{V}_2$ and $\psi \in C_0^k(\mathbf{R})$ $(k \geq 1)$. We define for all $t \in (-\infty, 0]$*

$$\psi_-(t)(x) := \sqrt{\frac{V(F_V^{-1}(F_V(x)-t))}{V(x)}}\psi(F_V^{-1}(F_V(x)-t)), \quad x \in \mathbf{R}. \tag{7.4}$$

Then $\psi_-(t)$ is strongly differentiable in $t \leq 0$ and is a solution to (7.1). Moreover, for all $t \in (-\infty, 0)$, $\psi_-(t)$ is in $C^k(\mathbf{R})$.

Proof. For $t \leq 0$, we define an operator T_t on $L^2(a_V, \infty)$ by

$$(T_t f)(x) := f(x-t), \quad \text{a.e. } x \in (a_V, \infty), f \in L^2(a_V, \infty).$$

Then T_t is bounded. We can write $\psi_-(t) = U_V^{-1} T_t U_V \psi$. It is easy to see that, for all $f \in C_0^k(\mathbf{R})$ $(k \geq 1)$, $T_t f$ is strongly differentiable in $t < 0$ and $dT_t f/dt = -T_t f' = -(T_t f)'$. Thus the disired result follows. ■

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